Approximate P-wave Ray Tracing and Dynamic Ray Tracing in Inhomogeneous Weakly Orthorhombic Media of Varying Symmetry Orientation

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June 23, 2014

Motivation

- Ray tracing and dynamic ray tracing are useful in seismic modeling and imaging
- Orthorhombic symmetry is a realistic approximation of the Earth's subsurface
- Need for ray tracing, which:
  $\rightarrow$ preserves the orthorhombic symmetry throughout the model
  $\rightarrow$ requires minimum parameters to describe the model
  $\rightarrow$ is sufficiently accurate

Outline

1. Theoretical Background
   - First-Order Ray Tracing and Dynamic Ray Tracing Equations
   - Specification for the Orthorhombic Symmetry
2. Traveltime Computations
   - TORTHO Models
   - BP TORTHO Model
3. Conclusion

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Weak-Anisotropy Approximation

- **Weak-anisotropy: deviations of anisotropy from isotropy assumed small; perturbations are with respect to these deviations**

- **Basic idea of the first-order RT and DRT: replacement of exact P-wave eigenvalue G of the Christoffel matrix and its derivatives by their first-order counterparts**

  $\Rightarrow$ **simple and transparent equations**

Coordinate Systems

- $\mathbf{x_i}$ - **global Cartesian coordinates**
- $\xi_\mathbf{i}$ - **curvilinear orthogonal coordinates; coordinate lines perpendicular to the symmetry planes of the orthorhombic medium**
- **H - rotation matrix from curvilinear coordinates** $\xi_\mathbf{i}$ **to Cartesian coordinates** $\mathbf{x_i}$**:**

$$\mathbf{H} = \begin{bmatrix} \cos\phi\cos\theta\cos\nu - \sin\phi\sin\nu & \cos\phi\cos\theta\sin\nu + \sin\phi\cos\nu & \cos\phi\sin\theta \\ -\sin\phi\cos\theta\cos\nu - \cos\phi\sin\nu & -\sin\phi\cos\theta\sin\nu + \cos\phi\cos\nu & -\sin\phi\sin\theta \\ -\sin\theta\cos\nu & -\sin\theta\sin\nu & \cos\theta \end{bmatrix}$$

θ, ϕ **and** ν **are three Euler angles**

Ray Tracing Equations

- **First-order ray tracing for anisotropy of arbitrary symmetry (Iversen & Pšenčík, 2008):**

$$\frac{d}{d\tau}\begin{bmatrix}\mathbf{x}\\ \mathbf{p}\end{bmatrix} = \mathbf{C}\begin{bmatrix}\upsilon^{(\xi)}\\ \eta^{(\xi)}\end{bmatrix}$$

LHS: in Cartesian coordinates
RHS: in curvilinear coordinates (ξ)

- τ **- first-order traveltime**
- $\mathbf{x}$ **- position vector of the first-order ray**
- $\mathbf{p}$ **- first-order slowness vector in Cartesian coordinates**

$$\mathbf{p} = \frac{\partial\tau}{\partial\mathbf{x}}$$

Ray Tracing Equations

- First-order ray tracing for anisotropy of arbitrary symmetry:

$$\frac{d}{d\tau}\begin{bmatrix}\mathbf{x}\\ \mathbf{p}\end{bmatrix} = \mathbf{C}\begin{bmatrix}\upsilon^{(\xi)}\\ \eta^{(\xi)}\end{bmatrix}$$

C - transformation matrix:

$$\mathbf{C} = \begin{bmatrix}\mathbf{H} & \mathbf{0}\\ -\mathbf{K} & \mathbf{I}\end{bmatrix}$$

  - **H** - 3×3 rotation matrix from ξ_m to x_k
  - **K** - 3×3 matrix with elements:

$$K_{mk} = \frac{\partial H_{ik}}{\partial x_m} p_i$$

  - **I** - 3×3 Identity matrix

Ray Tracing Equations

- **First-order ray tracing for anisotropy of arbitrary symmetry:**

$$\frac{d}{d\tau}\begin{bmatrix}\mathbf{x}\\ \mathbf{p}\end{bmatrix} = \mathbf{C}\begin{bmatrix}\upsilon^{(\xi)}\\ \eta^{(\xi)}\end{bmatrix}$$

  - $\upsilon^{(\xi)}$ - **first-order ray velocity vector**
  - $\eta^{(\xi)}$ - **first-order η vector**

$$\upsilon_i^{(\xi)} = \frac{1}{2}\frac{\partial G^{(\mathcal{M})}}{\partial p_i^{\xi}} \qquad \eta_i^{(\xi)} = -\frac{1}{2}\frac{\partial G^{(\mathcal{M})}}{\partial x_i}$$

- $\mathbf{p}^{\xi}$ - **first order slowness vector in curvilinear coordinates**
- $G^{(\mathcal{M})}$ - **first-order P-wave eigenvalue of the Christoffel matrix in mixed phase-space coordinates** $(\mathcal{M}) = (x_m, p_n^{\xi})$

Dynamic Ray Tracing Equations

- **First-order dynamic ray tracing for anisotropy of arbitrary symmetry:**

$$\frac{d}{d\tau}\begin{bmatrix}\mathbf{Q}\\ \mathbf{P}\end{bmatrix} = \mathbf{C}\begin{bmatrix}\mathbf{S}^\top & \mathbf{T}\\ -\mathbf{R} & -\mathbf{S}\end{bmatrix}\mathbf{D}\begin{bmatrix}\mathbf{Q}\\ \mathbf{P}\end{bmatrix} + \begin{bmatrix}\mathbf{V}^\top & \mathbf{0}\\ -\mathbf{U} & -\mathbf{V}\end{bmatrix}\begin{bmatrix}\mathbf{Q}\\ \mathbf{P}\end{bmatrix}$$

  - $\mathcal{T}$**-first-order traveltime**
  - $\mathbf{Q}$ **- 3×2 matrix;** $\mathbf{P}$ **- 3×2 matrix**

$$Q_{iJ} = \left(\frac{\partial x_i}{\partial \gamma_J}\right)_{\tau=const} \qquad P_{iJ} = \left(\frac{\partial p_i}{\partial \gamma_J}\right)_{\tau=const}$$

  - Q_{iJ}, P_{iJ} **- relative change of the position and slowness vectors along the wavefront**
  - γ_J **(J=1, 2) - ray parameters**

Dynamic Ray Tracing Equations

- **First-order dynamic ray tracing for anisotropy of arbitrary symmetry:**

$$\frac{d}{d\tau}\begin{bmatrix}\mathbf{Q}\\ \mathbf{P}\end{bmatrix} = \mathbf{C}\begin{bmatrix}\mathbf{S}^\top & \mathbf{T}\\ -\mathbf{R} & -\mathbf{S}\end{bmatrix}\mathbf{D}\begin{bmatrix}\mathbf{Q}\\ \mathbf{P}\end{bmatrix} + \begin{bmatrix}\mathbf{V}^\top & \mathbf{0}\\ -\mathbf{U} & -\mathbf{V}\end{bmatrix}\begin{bmatrix}\mathbf{Q}\\ \mathbf{P}\end{bmatrix}$$

- **R, S, T, U, V - 3×3 matrices:**

$$R_{ij} = \frac{1}{2}\frac{\partial^2 G^{(\mathcal{M})}}{\partial x_i \partial x_j} \qquad S_{ij} = \frac{1}{2}\frac{\partial^2 G^{(\mathcal{M})}}{\partial x_i \partial p_j^\xi} \qquad T_{ij} = \frac{1}{2}\frac{\partial^2 G^{(\mathcal{M})}}{\partial p_i^\xi \partial p_j^\xi}$$

$$U_{ij} = \frac{1}{2}\frac{\partial G^{(\mathcal{M})}}{\partial p_k^\xi}\frac{\partial^2 p_k^\xi}{\partial x_i \partial x_j} \qquad V_{ij} = \frac{1}{2}\frac{\partial G^{(\mathcal{M})}}{\partial p_k^\xi}\frac{\partial^2 p_k^\xi}{\partial x_i \partial p_j}$$

- $G^{(\mathcal{M})}$ **- first-order P-wave eigenvalue of the Christoffel matrix in mixed phase-space coordinates** $(\mathcal{M}) = (x_m, p_n^\xi)$

Dynamic Ray Tracing Equations

- **First-order dynamic ray tracing for anisotropy of arbitrary symmetry:**

$$\frac{d}{d\tau}\begin{bmatrix}\mathbf{Q}\\\mathbf{P}\end{bmatrix} = \mathbf{C}\begin{bmatrix}\mathbf{S}^{\top} & \mathbf{T}\\-\mathbf{R} & -\mathbf{S}\end{bmatrix}\mathbf{D}\begin{bmatrix}\mathbf{Q}\\\mathbf{P}\end{bmatrix} + \begin{bmatrix}\mathbf{V}^{\top} & \mathbf{0}\\-\mathbf{U} & -\mathbf{V}\end{bmatrix}\begin{bmatrix}\mathbf{Q}\\\mathbf{P}\end{bmatrix}$$

- **C and D - transformation matrices:**

$$\mathbf{C} = \begin{bmatrix}\mathbf{H} & \mathbf{0}\\-\mathbf{K} & \mathbf{I}\end{bmatrix} \qquad \mathbf{D} = \begin{bmatrix}\mathbf{I} & \mathbf{0}\\\mathbf{K}^{\top} & \mathbf{H}^{\top}\end{bmatrix}$$

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Specification for the Orthorhombic Symmetry

- **First-order-P-wave eigenvalue of the Christoffel matrix for an orthorhombic medium in mixed coordinates (x_m, p_n^ξ):**

$$G^{(\mathcal{M})} = \alpha^2 \left\{ p_k^\xi p_k^\xi + 2 \left[\epsilon_x (p_1^\xi)^2 + \epsilon_y (p_2^\xi)^2 + \epsilon_z (p_3^\xi)^2 \right] + 2 (p_k^\xi p_k^\xi)^{-1} \left[\eta_x (p_2^\xi)^2 (p_3^\xi)^2 + \eta_y (p_1^\xi)^2 (p_3^\xi)^2 + \eta_z (p_1^\xi)^2 (p_2^\xi)^2 \right] \right\}$$

$$\eta_x = \delta_y - \epsilon_y - \epsilon_z, \quad \eta_y = \delta_x - \epsilon_x - \epsilon_z, \quad \eta_z = \delta_z - \epsilon_x - \epsilon_y$$

- $\mathbf{p}^\xi$ **- first-order slowness vector in curvilinear coordinates**
- α **reference velocity**
- $\epsilon_\mathbf{x}$, $\epsilon_\mathbf{y}$, $\epsilon_\mathbf{z}$, δ_x, δ_y, δ_z : **6 P-wave weak anisotropy parameters**

Specification for the Orthorhombic Symmetry

- **The P-wave weak anisotropy (WA) parameters for an orthorhombic medium:**

$$\epsilon_x = \frac{A_{11}-\alpha^2}{2\alpha^2}, \qquad \epsilon_y = \frac{A_{22}-\alpha^2}{2\alpha^2}, \qquad \epsilon_z = \frac{A_{33}-\alpha^2}{2\alpha^2},$$

$$\delta_x = \frac{A_{13}+2A_{55}-\alpha^2}{\alpha^2}, \quad \delta_y = \frac{A_{23}+2A_{44}-\alpha^2}{\alpha^2}, \quad \delta_z = \frac{A_{12}+2A_{66}-\alpha^2}{\alpha^2}$$

- **A_{ij} - the density-normalized elastic moduli in the Voigt notation**

Second-order traveltime correction

- **First-order RT $\Rightarrow$ first-order traveltime**
- **Second-order traveltime correction:**

$$\Delta\tau = -\frac{1}{2}\int_{\Omega}\left[c(x_m, n_m^{\xi})\right]^{-2}\frac{B_{13}^2(x_m, n_m^{\xi}) + B_{23}^2(x_m, n_m^{\xi})}{V_P^2 - V_S^2}\,d\tau$$

  - Ω **- first-order ray**
  - τ **- first-order traveltime**
  - $c(x_m, n_m^{\xi})$ **phase velocity:**

$$c^2(x_m, n_m^{\xi}) = \alpha^2\left\{1 + 2\left[\epsilon_x(n_1^{\xi})^2 + \epsilon_y(n_2^{\xi})^2 + \epsilon_z(n_3^{\xi})^2 + \eta_x(n_2^{\xi})^2(n_3^{\xi})^2 + \eta_y(n_3^{\xi})^2(n_1^{\xi})^2 + \eta_z(n_1^{\xi})^2(n_2^{\xi})^2\right]\right\}$$

Second-order traveltime correction

- **First-order RT $\Rightarrow$ first-order traveltime**
- **Second-order traveltime correction:**

$$\Delta\tau = -\frac{1}{2}\int_{\Omega}\left[c(x_m, n_m^{\xi})\right]^{-2}\frac{B_{13}^2(x_m, n_m^{\xi}) + B_{23}^2(x_m, n_m^{\xi})}{V_p^2 - V_S^2}\,d\tau$$

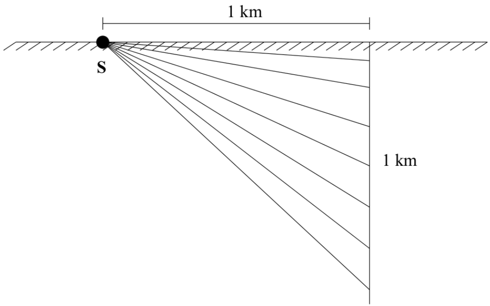
  - (x_m, n_m^{ξ}) - **mixed coordinates**
  - B_{13}, B_{23} - **projections of elements of the Christoffel matrix**
  - V_p **and** V_s - **P-wave and S-wave velocity of a closest reference isotropic medium; optimum choice:**

$$V_p = 1/|\mathbf{p}^{\xi}|, \qquad V_s = V_p/\sqrt{3}.$$

Outline

TORTHO Models

- **Configuration of the experiment: VSP(vertical seismic profiling)**
  $\rightarrow$ source on the surface and the receivers in a borehole



$\rightarrow$ 24 receivers placed on the borehole with 40m spacing

TORTHO Models

- **Synthetic models based on vertical linear interpolation of WA parameters between the top and bottom surfaces of the model**
- **WA model parameters:**

	ϵ_x	ϵ_y	ϵ_z	δ_x	δ_y	δ_z
Top surface	-0.15	-0.12	-0.25	-0.55	-0.5	-0.35
Bottom surface	0.1	0.15	-0.11	-0.28	-0.16	0.05

TORTHO models

- **Traveltimes computation for 3 orthorhombic models:**

	Description of the symmetry planes rotation
Example 1	symmetry planes parallel to coordinate planes
Example 2	symmetry planes inclined constantly
Example 3	linear rotation of symmetry planes with depth

- **Exact traveltimes are computed by a standard ray tracer for anisotropic media (ANRAY)**

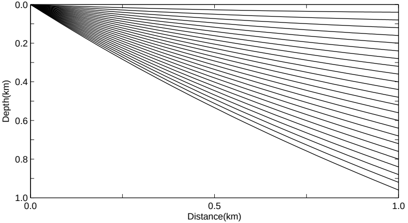
- **Relative traveltime difference $\Delta\tau$:**

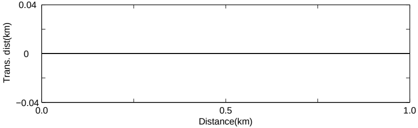
$$\Delta\tau = \frac{\tau - \tau_{exact}}{\tau_{exact}} \times 100\,\%$$

TORTHO models

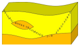
- **Example 1**

	Description of the symmetry planes rotation
Example 1	symmetry planes parallel to coordinate planes $\theta = \phi = \nu = 0^\circ$





Rays projected into the vertical plane (left) and the horizontal plane (right)



TORTHO models

- **Example 1**

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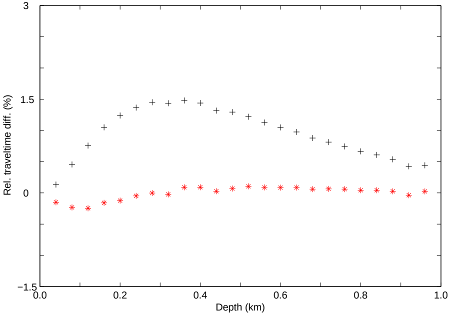
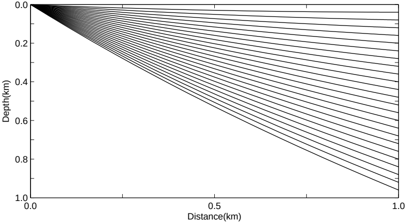


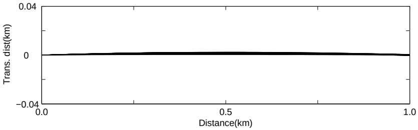
Relative traveltime difference of the first-order traveltimes (+) and of the second-order approximations (*)

TORTHO models

- **Example 2**

	Description of the symmetry planes rotation
Example 2	symmetry planes inclined constantly $\theta = 45^\circ$, $\phi = \nu = 0^\circ$

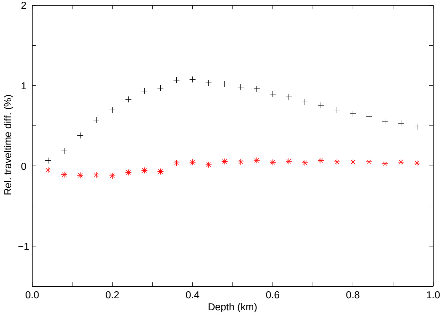




Rays projected into the vertical plane (left) and the horizontal plane (right)

- **Example 2**

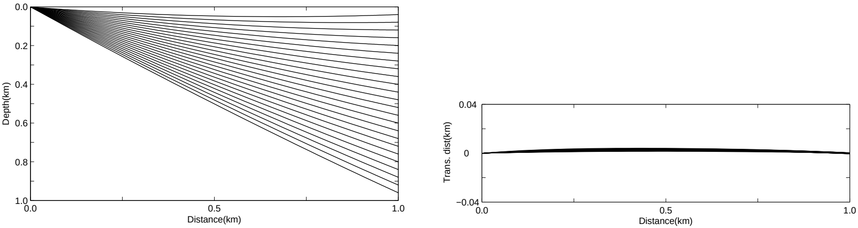
	Description of the symmetry planes rotation
Example 2	symmetry planes inclined constantly $\theta = 45°$, $\phi = \nu = 0°$



Relative traveltime difference of the first-order traveltimes (+) and of the second-order approximations (*)

- **Example 3**

	Description of the symmetry planes rotation			
	linear rotation of symmetry planes with depth			
		θ	ϕ	ν
Example 3	Top surface	45°	45°	0°
	Bottom surface	0°	0°	0°

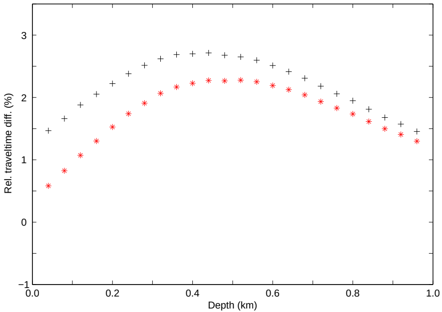


Rays projected into the vertical plane (left) and the horizontal plane (right)

TORTHO models

- **Example 3**

	Description of the symmetry planes rotation
Example 3	linear rotation of symmetry planes with depth



Relative traveltime difference of the first-order traveltimes (+) and of the second-order approximations (*)

TORTHO models

- **Traveltime deviations in Examples 1 and 2 due to inaccuracy of FORT and FODRT**
- **The deviations in Examples 1 and 2 considerably reduced by the use of 2nd-order traveltime approximation**

Relative error $<$ 0.2 %

- **The deviations in Example 3 due to ANRAY which does not conserve anisotropic symmetry**

Improved accuracy

Outline

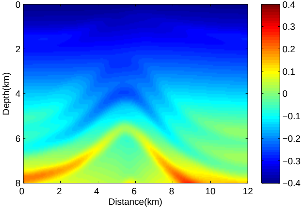
BP TORTHO model

- **BP model (TTI model)**
  $\rightarrow$ extended to tilted orthorhombic model
  $\rightarrow$ appropriate choices of additional WA parameters
  $\rightarrow$ structural features of the model do not vary in the direction perpendicular to the plane of the model $\rightarrow$ 2D model

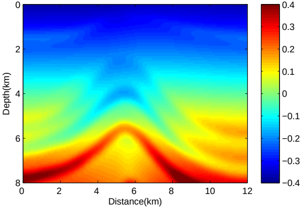
- **Model specification: 6 WA parameters in ξ_i coordinates:**
  (ϵ_x, ϵ_y, ϵ_z, δ_x, δ_y, δ_z) + 3 Euler angles (θ, ϕ, ν)

- **50m grid spacing in the three dimensions**

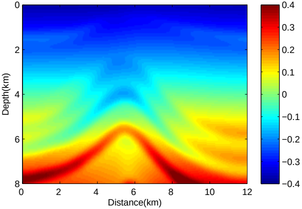
- **Distribution of WA parameters in the plane (x_1, x_3):**



ϵ_x

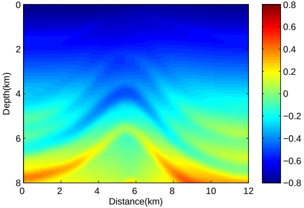


ϵ_y

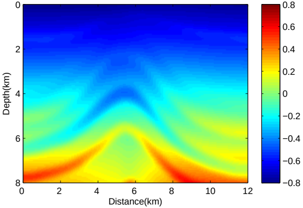


ϵ_z

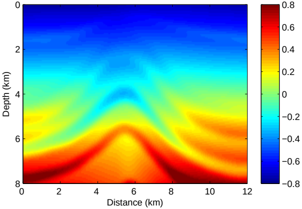
- **Distribution of WA parameters in the plane (x_1, x_3):**



δ_x



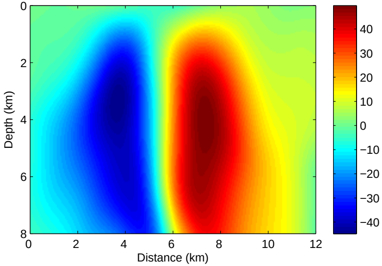
δ_y



δ_z

BP TORTHO model

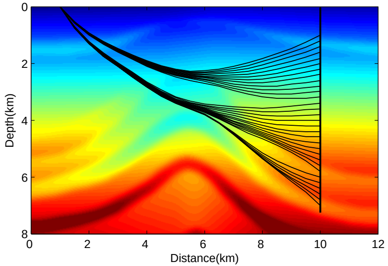
Specification of Euler angles		
θ	ϕ	ν
varying	constant	0°

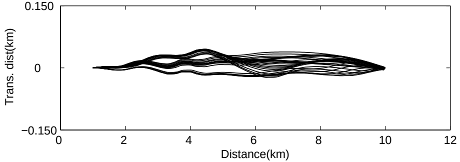


Variation of angle θ in the plane (x_1, x_3)

BP TORTHO model

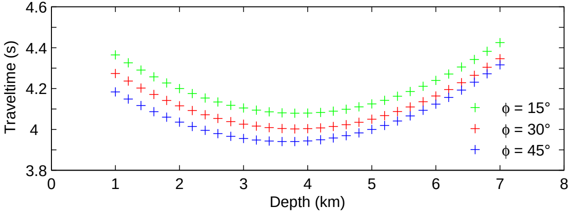
- **Configuration of the experiment: VSP (vertical seismic profiling), 200m spacing between the receivers in the borehole**



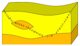


Projections of rays into the vertical plane with the distribution of δ_z on the background (left), projections of rays into the horizontal plane (right); results for the case $\phi = 15^\circ$

BP TORTHO model



Traveltime curves for three values of angle ϕ,
($\phi = 15^\circ, 30^\circ, 45^\circ$)



Outline

Conclusion

- Calculations in curvilinear coordinates, output in global Cartesian coordinates

Conservation of ORTHO symmetry

- Only 6 P-wave WA parameters and 3 Euler angles instead of 21 elastic parameters

Minimum number of parameters

- Sufficient accuracy guaranteed by the use of second-order traveltime correction

Accuracy guaranteed

Conclusion

- **Calculation of geometrical spreading and ray amplitudes**
- **Introduction of interfaces; layered models**
- **Generalization for coupled S waves**

Acknowledgments

- **KAUST for financial support**
- **Sponsors of SW3D consortium**
- **Research project P210/11/0117 of the Grant Agency of the Czech Republic**
- **All members of Seismic Wave Analysis Group at KAUST**

Thank you for your attention