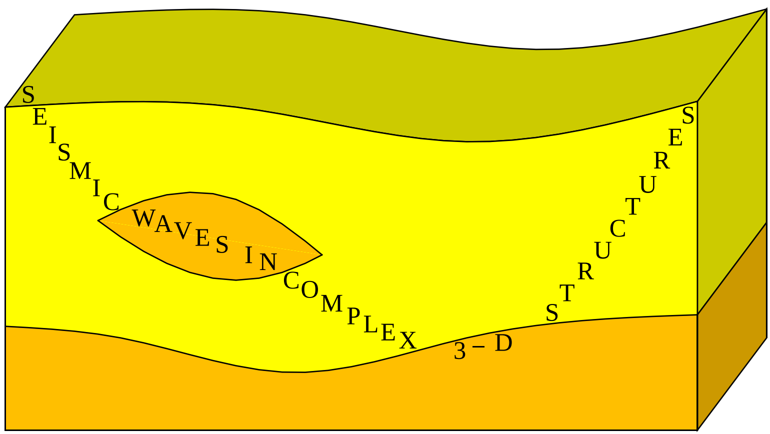
Calculation of the amplitudes of elastic waves in anisotropic media in Cartesian or ray-centred coordinates

Luděk Klimeš

Department of Geophysics
Faculty of Mathematics and Physics
Charles University in Prague



http://sw3d.cz

Transport equation for the scalar amplitude

Multivalued zero-order ray-theory scalar amplitude A of a general elastic wavefield satisfies transport equation

$$\frac{\partial}{\partial x_i}\left(A^2 \varrho V^i\right) = 0 \quad , \tag{21}$$

where V^i is the ray velocity vector. Function ϱ is a function parametrizing the transport equation. If A is the amplitude of the displacement of an elastic wavefield, ϱ is the density.

The transport equation is a partial differential equation for the square A^2 of the amplitude, not for the amplitude itself. Even if the solution A^2 of the transport equation is real-valued, amplitude A becomes complex-valued if its square A^2 becomes negative. Amplitude A is thus complex-valued. Since the complex-valued square root has two branches, it is difficult to determine amplitude A from its square A^2. We must determine which branch of the amplitude is correct.

Complex modulus of the scalar amplitude and phase shift due to caustics

We thus separate square root $A=\sqrt{A^2}$ into its complex modulus $|A|$ and complex argument φ:

$$A = |A| \exp(\mathrm{i}\varphi) \quad , \tag{22}$$

where φ is the phase shift due to caustics.

For the rules of determining the phase shift of a wavefield with general initial conditions due to caustics refer to Klimeš (2014a).

For the rules of determining the phase shift of the Green tensor due to caustics refer to Klimeš (2010).

In this contribution, we summarize various expressions for the complex modulus of the scalar amplitude of a wavefield with general initial conditions, and for the complex modulus of the scalar amplitude of the elastic Green tensor. The expressions are related to various solutions of the equations of geodesic deviation either in Cartesian coordinates or ray-centred coordinates. The equations are numbered according to Klimeš (2014b).

Ray-centred coordinates

Along a particular ray, we define ray-centred coordinates q^a (Klimeš, 2006).

We parametrize the points along the ray by an arbitrary monotonic variable q^3. At each point $x^i(q^3)$ of the ray, we choose two contravariant basis vectors $h_1^i(q^3)$ and $h_2^i(q^3)$ perpendicular to slowness vector p_i,

$$h^i_A(q^3)\, p_i = 0 \quad . \tag{6}$$

Contravariant basis vectors h^i_A should vary smoothly along the ray.

The transformation from the ray-centred coordinates q^a to Cartesian coordinates x^i is defined by relation

$$x^i = x^i(q^3) + h^i_A(q^3)\, q^A \quad . \tag{7}$$

Three contravariant basis vectors of the ray-centred coordinate system are

$$h^i_a = \frac{\partial x^i}{\partial q^a} \quad . \tag{8}$$

In the matrix notation, we shall denote the first two contravariant basis vectors as $\mathbf{h}_1$ and $\mathbf{h}_2$.

Amplitude of a general wavefield in terms of the paraxial vectors of geometrical spreading in Cartesian coordinates

$$\boxed{A = \frac{C}{\sqrt{\varrho \, |\varepsilon_{ijk} X_1^i X_2^j V^k|}} \, \exp(\mathrm{i}\varphi) \qquad (25)}$$

(Gajewski & Pšenčík, 1990, eq. 7; Kendall, Guest & Thomson, 1992, eqs. 3–4), where V^i is the ray velocity vector, C is the reduced amplitude, and φ is the phase shift due to caustics.

Paraxial vectors

$$X_A^i = \frac{\partial x^i}{\partial \gamma^A} \qquad (11)$$

of geometrical spreading represent the derivatives of Cartesian coordinates with respect to the ray parameters.

Amplitude of a general wavefield in terms of the matrix of geometrical spreading in ray-centred coordinates

$$\boxed{A(\mathbf{x},\tilde{x}) = \frac{C}{\sqrt{\varrho\, v\, |\mathbf{h}_1 \times \mathbf{h}_2|\, |\det(Q^I_{\ A})|}} \exp(\mathrm{i}\varphi) \qquad (37)}$$

(Klimeš, 2012, eqs. 7, 9), where C is the reduced amplitude and φ is the phase shift due to caustics.

Matrix

$$Q^I_{\ A} = \frac{\partial q^I}{\partial \gamma^A} \tag{13}$$

is the 2×2 paraxial matrix of geometrical spreading in ray-centred coordinates.

The contravariant basis vectors $\mathbf{h}_1$ and $\mathbf{h}_2$ of the ray-centred coordinate system are defined as

$$h^i_{\ A} = \frac{\partial x^i}{\partial q^A} \quad . \tag{8}$$

Amplitude of the elastic Green tensor

The scalar amplitude of the elastic Green tensor from point $\tilde{\mathbf{x}}$ to point $\mathbf{x}$ in the frequency domain:

$$\boxed{A^{\mathrm{G}}(\mathbf{x},\tilde{\mathbf{x}}) = \frac{1}{4\pi}\,\frac{1}{\sqrt{\varrho(\mathbf{x})\,v(\mathbf{x})\,\varrho(\tilde{\mathbf{x}})\,v(\tilde{\mathbf{x}})}\,L(\mathbf{x},\tilde{\mathbf{x}})}\,\exp[\mathrm{i}\,\varphi(\mathbf{x},\tilde{\mathbf{x}})] \qquad (38)}$$

(Klimeš, 2012, eq. 55), where ϱ is the density, v is the phase velocity, and $\varphi(\mathbf{x},\tilde{\mathbf{x}})$ is the phase shift of the Green tensor due to caustics.

Here $L(\mathbf{x},\tilde{\mathbf{x}})$ is the relative geometrical spreading.

Relative geometrical spreading in terms of the paraxial vectors in Cartesian coordinates

$$\boxed{L(\mathbf{x},\tilde{\mathbf{x}}) = \sqrt{\frac{|\varepsilon_{ijk} X_1^i(\mathbf{x}) X_2^j(\mathbf{x}) V^k(\mathbf{x})|\, V(\tilde{\mathbf{x}})}{v(\mathbf{x})\, |\mathbf{Y}_1(\tilde{\mathbf{x}}) \times \mathbf{Y}_2(\tilde{\mathbf{x}})|\, v(\tilde{\mathbf{x}})}}} \tag{61}$$

(Chapman, 2004, eq. 5.4.19), where V^i is the ray velocity vector, V is the ray velocity, and v is the phase velocity.

Paraxial vectors

$$X_A^i = \frac{\partial x^i}{\partial \gamma^A} \tag{11}$$

of geometrical spreading represent the derivatives of Cartesian coordinates with respect to the ray parameters.

Paraxial vectors $\mathbf{Y}_1$ and $\mathbf{Y}_2$, defined as

$$Y_{iA} = \frac{\partial p_i}{\partial \gamma^A} \quad , \tag{12}$$

represent the derivatives of the slowness vector with respect to the ray parameters.

Relative geometrical spreading in terms of the paraxial matrices in ray-centred coordinates

$$\boxed{L(\mathbf{x},\tilde{\mathbf{x}}) = \sqrt{|\mathbf{h}_1(\mathbf{x})\times\mathbf{h}_2(\mathbf{x})|\,|\det[\mathbf{Q}(\mathbf{x})]|\,|\det[\mathbf{P}(\tilde{\mathbf{x}})]|^{-1}\,|\mathbf{h}_1(\tilde{\mathbf{x}})\times\mathbf{h}_2(\tilde{\mathbf{x}})|} \quad . \qquad (51)}$$

Matrix $\mathbf{Q}$, defined as

$$Q^I_A = \frac{\partial q^I}{\partial \gamma^A} \quad , \tag{13}$$

is the 2×2 paraxial matrix of geometrical spreading in ray-centred coordinates q^a.

Matrix $\mathbf{P}$, defined as

$$P_{IA} = \frac{\partial p^{(q)}_I}{\partial \gamma^A} \quad , \tag{14}$$

represents the 2×2 paraxial matrix of derivatives of slowness vector $p^{(q)}_i$ in ray-centred coordinates with respect to the ray parameters.

The contravariant basis vectors $\mathbf{h}_1$ and $\mathbf{h}_2$ of the ray-centred coordinate system are defined as

$$h^i_A = \frac{\partial x^i}{\partial q^A} \quad . \tag{8}$$

Relative geometrical spreading in terms of the propagator matrix of geodesic deviation in Cartesian coordinates

$$\boxed{L(\mathbf{x},\tilde{\mathbf{x}}) = \sqrt{\frac{|V^k(\mathbf{x})\, C_{kl}(\mathbf{x},\tilde{\mathbf{x}}) V^l(\tilde{\mathbf{x}})|}{v(\mathbf{x})\, v(\tilde{\mathbf{x}})}} \qquad (70)}$$

(Kendall, Guest & Thomson, 1992, eq. 17b; Chapman, 2004, eq. 5.4.23), where V^i is the ray velocity vector and v is the phase velocity.

Here

$$C_{kl}(\mathbf{x},\tilde{\mathbf{x}}) = \tfrac{1}{2}\, \varepsilon_{kij}\, \varepsilon_{lmn}\, X_2^{im}(\mathbf{x},\tilde{\mathbf{x}})\, X_2^{jn}(\mathbf{x},\tilde{\mathbf{x}}) \tag{69}$$

is the matrix of the cofactors of the 3×3 upper right submatrix $X_2^{im}(\mathbf{x},\tilde{\mathbf{x}})$ of the 6×6 propagator matrix of geodesic deviation in Cartesian coordinates, defined as the derivative

$$X_2^{ij} = \frac{\partial x^i}{\partial \tilde{p}_j} \tag{19}$$

of Cartesian coordinates x^i with respect to initial slowness vector $\tilde{p}_j$.

Relative geometrical spreading in terms of the propagator matrix of geodesic deviation in ray-centred coordinates

$$\boxed{L(\mathbf{x},\tilde{\mathbf{x}}) = \sqrt{|\mathbf{h}_1(\mathbf{x})\times\mathbf{h}_2(\mathbf{x})|\,|\det[\mathbf{Q}_2(\mathbf{x},\tilde{\mathbf{x}})]|\,|\mathbf{h}_1(\tilde{\mathbf{x}})\times\mathbf{h}_2(\tilde{\mathbf{x}})|} \qquad (39)}$$

(Klimeš, 2012, eq. 13).

The 2×2 upper right submatrix $\mathbf{Q}_2(\mathbf{x},\tilde{\mathbf{x}})$ of the 4×4 propagator matrix of geodesic deviation in ray-centred coordinates is defined as the derivative

$$Q_2^{IJ} = \frac{\partial q^I}{\partial \tilde{p}_J^{(q)}} \tag{20}$$

of ray-centred coordinates q^I with respect to initial slowness vector $\tilde{p}_J^{(q)}$ in ray-centred coordinates.

The contravariant basis vectors $\mathbf{h}_1$ and $\mathbf{h}_2$ of the ray-centred coordinate system are defined as

$$h_A^i = \frac{\partial x^i}{\partial q^A} \quad . \tag{8}$$

Relative geometrical spreading in terms of the second-order derivatives of the characteristic function in Cartesian coordinates

$$\boxed{L(\mathbf{x}, \tilde{\mathbf{x}}) = 1 \Big/ \sqrt{|p_i(\mathbf{x})\, W^{il}(\mathbf{x}, \tilde{\mathbf{x}})\, p_l(\tilde{\mathbf{x}})|\, v(\mathbf{x})\, v(\tilde{\mathbf{x}})} \quad , \qquad (91)}$$

where p_i is the slowness vector and v is the phase velocity.

Here

$$W^{il}(\mathbf{x}, \tilde{\mathbf{x}}) = \tfrac{1}{2} \varepsilon^{ijk} \varepsilon^{lmn} \frac{\partial^2 \tau}{\partial x^j \partial \tilde{x}^m}(\mathbf{x}, \tilde{\mathbf{x}}) \frac{\partial^2 \tau}{\partial x^k \partial \tilde{x}^n}(\mathbf{x}, \tilde{\mathbf{x}}) \qquad (88)$$

is the matrix of the cofactors of matrix

$$\frac{\partial^2 \tau}{\partial x^k \partial \tilde{x}^l}(\mathbf{x}, \tilde{\mathbf{x}}) \qquad (86)$$

of the mixed second-order derivatives of the characteristic function with respect to source coordinates $\tilde{x}^l$ and receiver coordinates x^k.

Relative geometrical spreading in terms of the second-order derivatives of the characteristic function in ray-centred coordinates

$$\boxed{L(\mathbf{x},\tilde{\mathbf{x}}) = \sqrt{|\mathbf{h}_1(\mathbf{x})\times\mathbf{h}_2(\mathbf{x})|\,\left|\det\!\left(\frac{\partial^2\tau}{\partial q^A\partial\tilde{q}^B}(\mathbf{x},\tilde{\mathbf{x}})\right)\right|^{-1}|\mathbf{h}_1(\tilde{\mathbf{x}})\times\mathbf{h}_2(\tilde{\mathbf{x}})|} \quad . \qquad (84)}$$

Here

$$\frac{\partial^2\tau}{\partial q^A\partial\tilde{q}^B}(\mathbf{x},\tilde{\mathbf{x}}) = h^k_A(\mathbf{x})\,\frac{\partial^2\tau}{\partial x^k\partial\tilde{x}^l}(\mathbf{x},\tilde{\mathbf{x}})\,h^l_B(\tilde{\mathbf{x}}) \qquad (76)$$

is the 2×2 matrix of the mixed second-order derivatives of the characteristic function with respect to source ray-centred coordinates $\tilde{q}^B$ and receiver ray-centred coordinates q^A.

The contravariant basis vectors $\mathbf{h}_1$ and $\mathbf{h}_2$ of the ray-centred coordinate system are defined as

$$h^i_A = \frac{\partial x^i}{\partial q^A} \quad . \qquad (8)$$

References

Chapman, C.H. (2004): *Fundamentals of Seismic Wave Propagation.* Cambridge Univ. Press, Cambridge.

Gajewski, D. & Pšenčík, I. (1990): Vertical seismic profile synthetics by dynamic ray tracing in laterally varying layered anisotropic structures. *J. geophys. Res.*, **95B**, 11301–11315.

Kendall, J-M., Guest, W.S. & Thomson, C.J. (1992): Ray-theory Green's function reciprocity and ray-centred coordinates in anisotropic media. *Geophys. J. int.*, **108**, 364–371.

Klimeš, L. (2006): Ray-centred coordinate systems in anisotropic media. *Stud. geophys. geod.*, **50**, 431–447.

Klimeš, L. (2010): Phase shift of the Green tensor due to caustics in anisotropic media. *Stud. geophys. geod.*, **54**, 268–289.

Klimeš, L. (2012): Zero-order ray-theory Green tensor in a heterogeneous anisotropic elastic medium. *Stud. geophys. geod.*, **56**, 373–382.

Klimeš, L. (2014a): Phase shift of a general wavefield due to caustics in anisotropic media. *Seismic Waves in Complex 3-D Structures*, **24**, 95–109, online at "http://sw3d.cz".

Klimeš, L. (2014b): Calculation of the amplitudes of elastic waves in anisotropic media in Cartesian or ray-centred coordinates. *Seismic Waves in Complex 3-D Structures*, **24**, 111–126, online at "http://sw3d.cz".

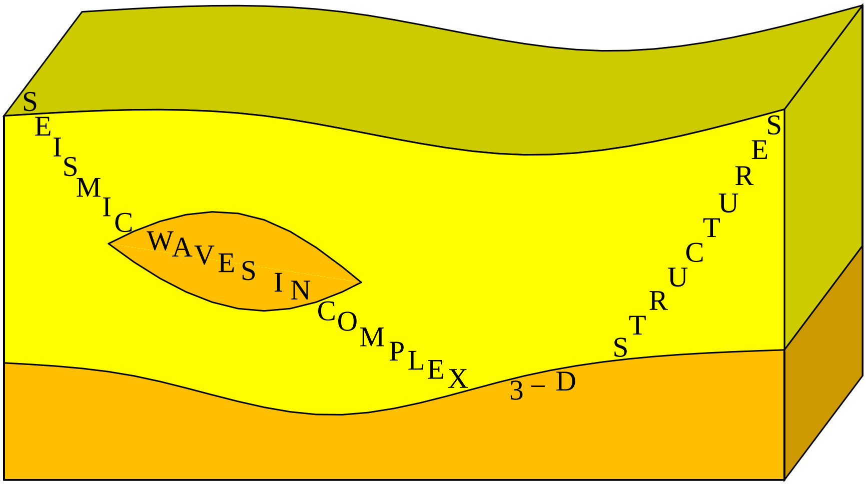
Acknowledgements

The research has been supported:

by the Grant Agency of the Czech Republic under contract P210/10/0736,

by the Ministry of Education of the Czech Republic within research project MSM0021620860,

and by the consortium "Seismic Waves in Complex 3-D Structures"



http://sw3d.cz