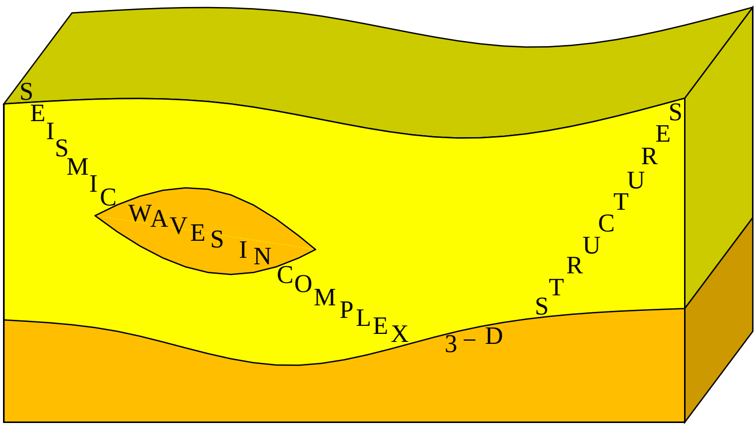
Superpositions of Gaussian beams and column Gaussian packets in heterogeneous anisotropic media

Luděk Klimeš

Department of Geophysics
Faculty of Mathematics and Physics
Charles University in Prague



http://sw3d.cz

The three-parametric integral superposition of Gaussian packets was proposed for isotropic media by Klimeš (1984), and was specified for anisotropic media by Klimeš (2014).

The two-parametric integral superposition of Gaussian beams is calculated along the reference surface. The paraxial Gaussian beams are expanded at the points of intersection of the reference rays with the reference surface.

The two-parametric integral superposition of Gaussian beams can be obtained from the three-parametric integral superposition of Gaussian packets by analytic asymptotic quadrature along the reference rays (Klimeš, 1984; 2015).

We consider a general system of reference lines instead of the reference rays for the one-parametric analytic asymptotic quadrature of the three-parametric integral superposition of Gaussian packets (Klimeš, 2015; 2016). We then obtain the two-parametric integral superposition of column Gaussian packets instead of the two-parametric integral superposition of Gaussian beams.

The two-parametric integral superposition of column Gaussian packets derived by Klimeš (2015) is expressed in terms of the partial derivatives of the reference travel time along the reference surface with respect to ray parameters.

Klimeš (2015) then demonstrated that the resulting column Gaussian packets are regular at caustics only if they coincide with Gaussian beams which represent a special case of column Gaussian packets.

The paper by Klimeš (2016) represents an extended version of the paper by Klimeš (2015). Since weighting factors of Gaussian beams in the superposition are usually expressed in ray-centred coordinates, the extended version of the paper is supplemented with new Section 5 which transforms the two-parametric integral superposition of Gaussian beams from the quantities defined with respect to ray parameters along the reference surface to the quantities defined with respect to ray-centred coordinates.

The two-parametric **integral superposition of column Gaussian packets** is performed along the reference surface, and represents a generalization of the two-parametric integral superposition of paraxial Gaussian beams.

Two-parametric integral superposition of column Gaussian packets along reference surface $\tilde{x}^m = \tilde{x}^m(\xi^K)$ parametrized by ray parameters ξ^1 and ξ^2 (Klimeš, 2015, eq. 33):

$$u_{i[j]}(x^m, \omega) = \frac{\omega}{2\pi} \iint \mathrm{d}\xi^1 \mathrm{d}\xi^2 \, A_{i[j]} \sqrt{\det}[\mathrm{i}(M_{AB} - \tilde{F}_{AB})] \, \exp\big(\mathrm{i}\omega\tilde{\theta}\big) \qquad (33)$$

$A_{i[j]}$... complex-valued vectorial or tensorial amplitude corresponding to the ray theory under consideration at point $\tilde{x}^m$,
ω... circular frequency,
M_{AB}... 2×2 matrix of the second-order partial derivatives of the reference travel time along the reference surface with respect to ξ^1 and ξ^2,
$\tilde{F}_{AB}$... 2×2 complex-valued matrix with positive imaginary part describing the shape of the column Gaussian packet centred at point $\tilde{x}^i$.
Function $\sqrt{\det}\big(B_{AB}\big)$ is the product of the square roots of the eigenvalues of matrix B_{AB}. The individual square roots are taken with positive real parts.

Paraxial complex-valued travel time:

$$\tilde{\theta} = \tau + (x^k - \tilde{x}^k)\, p_k + \tfrac{1}{2}(x^k - \tilde{x}^k)\, \tilde{f}_{kl}\, (x^l - \tilde{x}^l) \tag{17}$$

τ... travel time corresponding to the ray theory under consideration at point $\tilde{x}^i$ (need not be equal to the reference travel time),
p_k... reference slowness vector at point $\tilde{x}^i$.

Second-order partial derivatives of the complex-valued travel time at point $\tilde{x}^i$:

$$\tilde{f}_{ij} = Z_{Mi}\tilde{F}_{MN}Z_{Nj} + Z_{3i}Z_3^k N_{kj} + N_{il}Z_3^l Z_{3j} - Z_{3i}Z_3^k N_{kl} Z_3^l Z_{3j} \tag{27}$$

$\tilde{F}_{AB}$... 2×2 complex-valued matrix with positive imaginary part describing the shape of the column Gaussian packet centred at point $\tilde{x}^i$,
N_{kl}... 3×3 matrix of the second-order partial derivatives of the reference travel time at point $\tilde{x}^i$,
Z_3^i... contravariant vector tangent to the reference line,
Z_{3i}... covariant vector perpendicular to the reference surface, $Z_{3i}Z_3^i = 1$,
Z_{1i}, Z_{2i}... covariant vectors perpendicular to the reference line, scaled as $Z_{Ki}\,\partial\tilde{x}^i/\partial\xi^L = \delta_{KL}$, $K, L = 1, 2$.

The column Gaussian packets infinitely extend along the reference lines, whether the Gaussian beams infinitely extend along the reference rays.

Unfortunately, the paraxial expansion of the column Gaussian packets is generally singular at caustics. The column Gaussian packets are not singular at caustics only if the reference lines are tangent to the reference rays at the reference surface. If we choose the reference lines along the reference rays, the two-parametric integral superposition of column Gaussian packets along the reference surface reduces to the two-parametric integral superposition of paraxial Gaussian beams along the reference surface.

The two-parametric **integral superposition of Gaussian beams** over ray parameters ξ^1 and ξ^2, expressed in terms of the quantities defined with respect to ray-centred coordinates, reads (Klimeš, 2016, eq. 56):

$$u_{i[j]}(x^m) = \frac{\omega}{2\pi} \iint \mathrm{d}\xi^1 \mathrm{d}\xi^2 \, A_{i[j]} \left|\det(Q^K_L)\right| \sqrt{\det}[\mathrm{i}(M^{(q)}_{AB} - \tilde{f}^{(q)}_{AB})] \exp\!\left(\mathrm{i}\omega\tilde{\theta}\right) \tag{56}$$

$A_{i[j]}$... complex-valued vectorial or tensorial amplitude corresponding to the ray theory under consideration at point $\tilde{x}^m$,
ω... circular frequency,
Q^K_L... 2×2 matrix of geometrical spreading in ray-centred coordinates,
$M^{(q)}_{AB}$... 2×2 matrix of the second-order partial derivatives of the reference travel time in ray-centred coordinates,
$\tilde{f}^{(q)}_{AB}$... 2×2 complex-valued matrix with positive imaginary part describing the shape of the Gaussian beam centred at point $\tilde{x}^i$.
Function $\sqrt{\det}\left(B_{AB}\right)$ is the product of the square roots of the eigenvalues of matrix B_{AB}. The individual square roots are taken with positive real parts.

This integral superposition is identical to the integral superposition of Gaussian beams by Červený & Pšenčík (2015, eq. 59).

Paraxial complex-valued travel time:

$$\tilde{\theta} = \tau + (x^k - \tilde{x}^k)\, p_k + \tfrac{1}{2}(x^k - \tilde{x}^k)\, \tilde{f}_{kl}\, (x^l - \tilde{x}^l) \tag{17}$$

τ... travel time corresponding to the ray theory under consideration at point $\tilde{x}^i$ (need not be equal to the reference travel time),
p_k... reference slowness vector at point $\tilde{x}^i$.

Second-order partial derivatives of the complex-valued travel time at point $\tilde{x}^i$:

$$\tilde{f}_{ij} = h_{Ai} \tilde{f}^{(q)}_{AB} h_{Bj} + \left[-p_i H_{,j} - H_{,i} p_j + p_i p_j H_{,k} H^{,k} \frac{\mathrm{d}\gamma^3}{\mathrm{d}\tau} \right] \frac{\mathrm{d}\gamma^3}{\mathrm{d}\tau} \tag{44}$$

$\tilde{f}^{(q)}_{AB}$... 2×2 complex-valued matrix with positive imaginary part describing the shape of the Gaussian beam centred at point $\tilde{x}^i$.
h_{1i}, h_{1i}... covariant basis vectors of the ray-centred coordinate system,
$H_{,i}$, $H^{,j}$... phase-space derivatives of the Hamiltonian function,
γ^3... parameter along rays corresponding to the Hamiltonian function.

The integral superpositions of column Gaussian packets and paraxial Gaussian beams may correspond to the anisotropic ray theory, to the frequency-dependent coupling ray theory for S waves or to the prevailing-frequency approximation of the coupling ray theory (Klimeš & Bulant, 2012) in anisotropic media, or to the isotropic ray theory in isotropic media. The equations can be used in both Cartesian and curvilinear coordinates.

References:

Červený, V. & Pšenčík, I. (2015): Integral superposition of paraxial Gaussian beams in inhomogeneous anisotropic layered structures in Cartesian coordinates. *Seismic Waves in Complex 3–D Structures*, **25**, 109–155, online at “http://sw3d.cz”.

Klimeš, L. (1984): The relation between Gaussian beams and Maslov asymptotic theory. *Stud. geophys. geod.*, **28**, 237–247.

Klimeš, L. (2014): Superposition of Gaussian packets in heterogeneous anisotropic media. *Seismic Waves in Complex 3-D Structures*, **24**, 127–130, online at “http://sw3d.cz”.

Klimeš, L. (2015): Superpositions of Gaussian beams and column Gaussian packets in heterogeneous anisotropic media. *Seismic Waves in Complex 3-D Structures*, **25**, 103–108, online at “http://sw3d.cz”.

Klimeš, L. (2016): Superpositions of Gaussian beams and column Gaussian packets in heterogeneous anisotropic media. *Seismic Waves in Complex 3–D Structures*, **26**, 123–130, online at “http://sw3d.cz”.

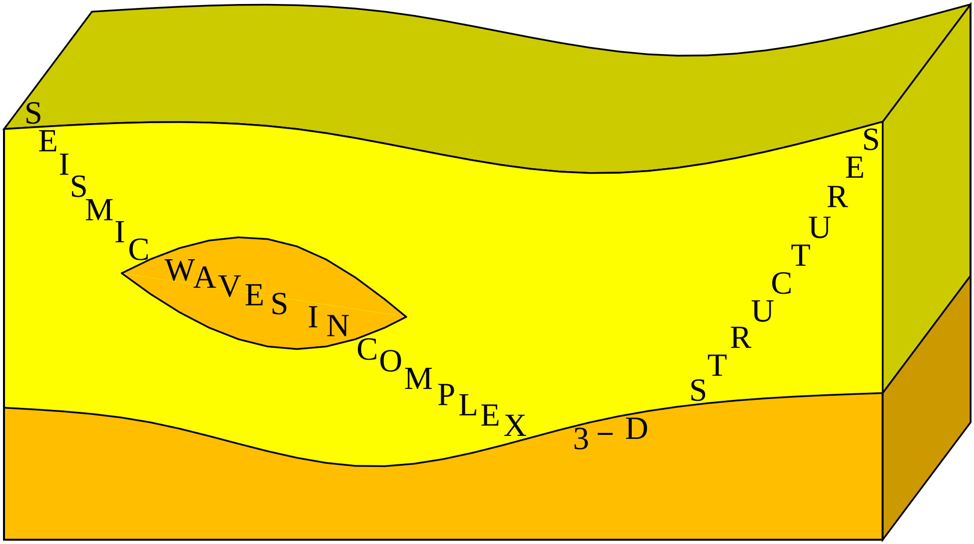
Klimeš, L. & Bulant, P. (2012): Single-frequency approximation of the coupling ray theory. In: *Seismic Waves in Complex 3-D Structures, Report 22*, pp. 143–167, Dep. Geophys., Charles Univ., Prague, online at “http://sw3d.cz”.

Acknowledgements

The research has been supported:

by the Grant Agency of the Czech Republic under contracts 16-01312S and 16-05237S,

and by the consortium "Seismic Waves in Complex 3-D Structures"



http://sw3d.cz