APPROXIMATE RELATION BETWEEN THE RAY VECTOR AND THE WAVE NORMAL IN WEAKLY ANISOTROPIC MEDIA

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ABSTRACT

Determination of the ray vector (the unit vector specifying the direction of the group velocity vector) corresponding to a given wave normal (the unit vector parallel to the phase velocity vector or slowness vector) in an arbitrary anisotropic medium can be performed using the exact formula following from the ray tracing equations. The determination of the wave normal from the ray vector is, generally, a more complicated task, which is usually solved iteratively. We present a first-order perturbation formula for the approximate determination of the ray vector from a given wave normal and vice versa. The formula is applicable to qP as well as qS waves in directions, in which the waves can be dealt with separately (i.e. outside singular directions of qS waves). Performance of the approximate formulae is illustrated on models of transversely isotropic and orthorhombic symmetry. We show that the formula for the determination of the ray vector from the wave normal yields rather accurate results even for strong anisotropy. The formula for the determination of the wave normal from the ray vector works reasonably well in directions, in which the considered waves have convex slowness surfaces. Otherwise, it can yield, especially for stronger anisotropy, rather distorted results.

Key words: wave normal, ray vector, weak anisotropy

1. INTRODUCTION

It is well known that the first-order perturbations of magnitudes of phase and group velocities due to perturbation of elastic parameters are equal, see *Backus (1965)*. However, the directions of the wave normal and of the ray vector generally differ. The determination of the ray vector corresponding to a given wave normal is an easy task (*Musgrave, 1970; Červený et al., 1977; Červený, 2001*). The determination of the wave normal from the ray vector is more complicated. It is because the wave normal and the ray vector are related by a complicated nonlinear formula (see the above references), which must be, in most cases, solved iteratively. Moreover, the formula can yield even multivalued solutions due to the triplication of *qS*-wave surfaces The problem simplifies for weakly anisotropic media. In this case, the first-order perturbation method can be applied (*Jech and Pšenčík, 1989*) and the formula for the ray vector can be linearized. This approach has been applied by *Vavryčuk (1997, Eq. 19)* for transversely isotropic media. In this paper, the approach is used for weak anisotropy of arbitrary symmetry (*Pšenčík, 1996*). We expect that the derived formulae can find applications in the

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estimation of elastic parameters of a homogeneous anisotropic medium from observed deviations of wave normal and ray vector. The formula can be also used in two-point ray tracing in homogeneous anisotropic media, to find the wave normal corresponding to the direction between two given points. Several applications are also mentioned by *Song and Every (2000)*.

In Sec. 2, we summarize basic results of the first-order perturbation method for weakly anisotropic media. We use them to derive an approximate formula relating the wave normal n_j and the ray vector N_j and vice versa in Sec. 3. We also give approximate formulae for the group velocity vectors of all three types of waves, expressed as a function of the ray vector. In Sec. 4. accuracy of the mentioned formulae is tested on three numerical examples. Two examples represent transversely isotropic media with vertical axes of symmetry (VTI media). The third example represents an orthorhombic medium with symmetry planes coinciding with coordinate planes.

2. APPROXIMATE FORMULAE FOR THE PHASE VELOCITY AND POLARIZATION VECTORS

In the following, the lowercase indices attain values 1, 2, 3, the uppercase indices only values 1 and 2. Einstein summation convention is used over repeating indices.

Let us consider a weakly anisotropic medium of an arbitrary symmetry specified by the tensor of the density-normalized elastic parameters a_{ijkl}, and an isotropic reference medium specified by the velocity c_0, where $c_0 = \alpha$ for the P wave and $c_0 = \beta$ for S waves. The reference medium is chosen so that its elastic properties do not differ much from the weakly anisotropic medium. Let us denote by n_j the wave normal and by $e_j^{(1)}$ and $e_j^{(2)}$ two unit vectors so that the three vectors form a mutually perpendicular triplet. In the plane perpendicular to n_j, the vectors $e_j^{(1)}$ and $e_j^{(2)}$ make an angle ϕ with vectors $i_j^{(1)}$ and $i_j^{(2)}$ spanning the same plane so that

$$e_j^{(1)} = i_j^{(1)} \cos\phi + i_j^{(2)} \sin\phi \ , \qquad e_j^{(2)} = -i_j^{(1)} \sin\phi + i_j^{(2)} \cos\phi \ . \tag{1}$$

The vectors $i_j^{(1)}$ and $i_j^{(2)}$ are mutually perpendicular unit vectors which can be defined, for example, as follows:

$$\vec{i}^{\,(1)} \equiv D^{-1}\left(n_1 n_3, n_2 n_3, n_3^2 - 1\right) \ , \qquad \vec{i}^{\,(2)} \equiv D^{-1}(-n_2, n_1, 0) \ , \tag{2}$$

where

$$D = \left(n_1^2 + n_2^2\right)^{1/2} \ , \qquad n_1^2 + n_2^2 + n_3^2 = 1 \ . \tag{3}$$

The vector $i_j^{(3)}$ is chosen so that $i_j^{(3)} \equiv e_j^{(3)} \equiv n_j$. For the above specification of a weakly anisotropic medium and of a wave normal, we can introduce a *weak anisotropy matrix* $B_{mn}^{(i)}$, see Eq. (8') in *Jech and Pšenčík (1989)* or Eq. (11) in *Pšenčík and Gajewski (1998)*,

$$B_{mn}^{(i)} = \Delta a_{ijkl} n_j n_l i_i^{(m)} i_k^{(n)} = a_{ijkl} n_j n_l i_i^{(m)} i_k^{(n)} - c_0^2 \delta_{mn} \ . \tag{4}$$

Here Δa_{ijkl} denotes perturbation from an isotropic reference medium, in which $c_0 = \alpha$ for $m = n = 3$ and $c_0 = \beta$ for $m = n = 1, 2$. The matrix $B_{mn}^{(i)}$ results from perturbation of the Christoffel matrix, see *Jech and Pšenčík (1989)* and *Červený (2001)*. It plays an important role in all formulae describing weak anisotropy. The superscript (i) indicates that the matrix $B_{mn}^{(i)}$ is calculated using the vectors $i_j^{(m)}$.

For a fixed wave normal, the perturbation theory yields the following formulae for the phase velocity and polarization vectors. The phase velocity in a weakly anisotropic medium is given by (see Eqs. (8) and (18) of *Jech and Pšenčík, 1989*)

$$c_3 = c_0 + \Delta c_3 \cong \alpha + \frac{1}{2\alpha} B_{33}^{(i)} \tag{5a}$$

for the *qP* wave and by

$$c_1 = c_0 + \Delta c_1 \cong \beta + \frac{1}{2\beta} B_{11}^{(e)} \ , \qquad c_2 = c_0 + \Delta c_2 \cong \beta + \frac{1}{2\beta} B_{22}^{(e)} \tag{5b}$$

for *qS* waves. The symbol Δc_3 denotes perturbation of the phase velocity for *qP* wave and Δc_1 and Δc_2 for *qS* waves. The symbol $B_{mn}^{(e)}$ denotes an element of the weak anisotropy matrix calculated using vectors $e_j^{(I)}$, see below.

The polarization vector of the *qP* wave is given by (see Eqs. (11) and (23) of *Jech and Pšenčík, 1989*)

$$g_j^{(3)} = n_j + \Delta g_j^{(3)} \cong n_j + \frac{B_{13}^{(i)} i_j^{(1)} + B_{23}^{(i)} i_j^{(2)}}{\alpha^2 - \beta^2} \ . \tag{6a}$$

The polarization vectors of *qS* waves are given by

$$g_j^{(1)} = e_j^{(1)} + \Delta g_j^{(1)} \cong e_j^{(1)} + \frac{B_{13}^{(e)}}{\beta^2 - \alpha^2} \left(n_j + \frac{B_{23}^{(e)} e_j^{(2)}}{B_{11}^{(e)} - B_{22}^{(e)}} \right) ,$$

$$g_j^{(2)} = e_j^{(2)} + \Delta g_j^{(2)} \cong e_j^{(2)} + \frac{B_{23}^{(e)}}{\beta^2 - \alpha^2} \left(n_j - \frac{B_{13}^{(e)} e_j^{(1)}}{B_{11}^{(e)} - B_{22}^{(e)}} \right) . \tag{6b}$$

The matrix $B_{mn}^{(e)}$ is a symmetric matrix whose elements are expressed in terms of elements of the matrix $B_{mn}^{(i)}$ and especially chosen angle ϕ for which $B_{12}^{(e)} = 0$. This choice of angle ϕ yields the vectors $e_j^{(I)}$, which can be used as zero-order approximation of the polarization vectors of qS waves in the isotropic reference medium. The elements of the matrices $B_{mn}^{(e)}$ and $B_{mn}^{(i)}$ are related in the following way:

$$B_{11}^{(e)} = B_{11}^{(i)} \cos^2\phi + 2B_{12}^{(i)} \cos\phi \sin\phi + B_{22}^{(i)} \sin^2\phi \ ,$$

$$B_{22}^{(e)} = B_{11}^{(i)} \sin^2\phi - 2B_{12}^{(i)} \cos\phi \sin\phi + B_{22}^{(i)} \cos^2\phi \ ,$$

$$B_{12}^{(e)} = \left(B_{22}^{(i)} - B_{11}^{(i)}\right)\cos\phi \sin\phi + B_{12}^{(i)} \left(\cos^2\phi - \sin^2\phi\right) \ ,$$

$$B_{33}^{(e)} = B_{33}^{(i)} \ , \quad B_{13}^{(e)} = B_{13}^{(i)} \cos\phi + B_{23}^{(i)} \sin\phi \ , \quad B_{23}^{(e)} = -B_{13}^{(i)} \sin\phi + B_{23}^{(i)} \cos\phi \ . \tag{7}$$

Let us note that equations (6b) fail when $B_{11}^{(e)}$ is close to $B_{22}^{(e)}$. This happens in the vicinity of the qS-wave singularities. From the equation $B_{12}^{(e)} = 0$, we can find the formula for the determination of the angle ϕ,

$$\tan 2\phi = \frac{2B_{12}^{(i)}}{B_{11}^{(i)} - B_{22}^{(i)}} \ . \tag{8}$$

Alternatively, the matrix $B_{mn}^{(e)}$, can be expressed in the following way analogous to (4)

$$B_{mn}^{(e)} = a_{ijkl} n_j n_l e_i^{(m)} e_k^{(n)} - c_0^2 \delta_{mn} \ . \tag{9}$$

The vectors $e_j^{(I)}$ in Eq. (9) are given by (1) with ϕ specified in Eq. (8). From (5) and (6), we can see that the matrix $B_{mn}^{(e)}$ appears only in the expressions for qS waves.

Let us mention that approximate values of phase velocities in the examples shown later were calculated by taking square roots of the following first-order formulae for the square of the qP-wave phase velocity,

$$c_3^2 \cong \alpha^2 + B_{33}^{(i)} \ , \tag{10a}$$

and for the squares of the qS-wave phase velocities

$$c_1^2 \cong \beta^2 + B_{11}^{(e)} \ , \qquad c_2^2 \cong \beta^2 + B_{22}^{(e)} \ . \tag{10b}$$

Eqs. (10) follow straightforwardly from (5) by neglecting second-order terms. Using Eq. (4) in Eq. (10a) we get for qP waves

$$c_3^2 \sim a_{ijkl} n_i n_j n_k n_l \ , \tag{11a}$$

see Eq. (14) in *Pšenčík and Gajewski (1998)*. Using Eq. (9) in Eqs. (10b), we get for qS waves

$$c_1^2 \cong a_{ijkl} n_j n_l e_i^{(1)} e_k^{(1)} \ , \qquad c_2^2 \cong a_{ijkl} n_j n_l e_i^{(2)} e_k^{(2)} \ . \tag{11b}$$

For similar expressions, see *Farra (2001)*. We can see that the expressions for the square of the phase velocity (11) are independent of the choice of the parameters of the reference medium.

3. RELATION BETWEEN WAVE NORMAL AND RAY VECTOR

The approximate formulae for the phase velocities and polarization vectors presented in the previous section follow from formulae derived by *Jech and Pšenčík (1989)*. In this section, we discuss a different topic, namely approximate relation of the ray vector and the wave normal, and approximate formulae for the group velocity. Let us start from the formula for the group velocity vector v_j in an anisotropic medium, see e.g. *Musgrave (1970)*, *Červený et al. (1977)*, *Červený (2001)*:

$$v_j = v N_j = c^{-1} a_{ijkl} n_l g_i g_k \ . \tag{12}$$

For simplicity, the superscripts denoting the type of the considered wave are omitted. In Eq. (12), v and c are the group and phase velocities and g_j denotes the polarization vector of a considered wave. Generally, the group- and phase-velocity vectors differ in anisotropic media by their magnitudes v and c as well as directions N_j and n_j. The determination of the ray vector N_j from the wave normal n_j can be performed with the use of exact relation (12). For approximate determination of n_j from N_j, we use the first-order perturbation method. We seek the ray vector N_j as a perturbation of the wave normal n_j,

$$N_j = n_j + \Delta N_j = n_j + A_1 e_j^{(1)} + A_2 e_j^{(2)} \ , \tag{13a}$$

where $A_I \,(I = 1, 2)$ are coefficients to be determined. The group velocity v can be expressed as

$$v = c_0 + \Delta v \ , \tag{13b}$$

where Δv is the perturbation of the group velocity

Using Eqs. (13) and neglecting second-order perturbations, the group velocity vector v_j can be expressed as a sum of the component parallel to the wave normal and a component perpendicular to it:

$$v_j = (c_0 + \Delta v) n_j + c_0 \left(A_1 e_j^{(1)} + A_2 e_j^{(2)} \right) \ . \tag{14}$$

Similarly, we can expand the RHS of Eq. (12) for wave normal kept fixed. In such a way, we get an alternative expression for the group velocity vector v_j:

$$v_j = c_0 n_j - \Delta c n_j + c_0^{-1} \big[a_{ijkl} n_l e_i e_k$$

$$+ \left(\alpha^2 - \beta^2\right) \left((n_k e_k) \Delta g_j - (n_k e_k) e_j + (n_k \Delta g_k) e_j \right) - \beta^2 n_j \big] \ . \tag{15}$$

Comparison of projections of group velocity vector v_j given by (14) and (15) onto the wave normal yields

$$\Delta v = \Delta c \quad . \tag{16}$$

This is a well-known identity (*Backus, 1965*), indicating that the phase and group velocities have equal magnitudes in the first-order approximation of the perturbation theory.

Comparing projections of the group velocity vector v_j given by (14) and (15) onto the vector $e_j^{(I)}$ yields sought coefficients A_I:

$$A_I = c_0^{-2} \Big[a_{ijkl} n_l e_i e_k e_j^{(I)} + \left(\alpha^2 - \beta^2\right)\Big((n_k e_k)\left(\Delta g_j e_j^{(I)}\right) - (n_k e_k)\left(e_j e_j^{(I)}\right) + (n_k \Delta g_k)\left(e_j e_j^{(I)}\right)\Big)\Big] \quad . \tag{17}$$

Using (13a) and (17) we can write an approximate expression relating the ray vector N_j and the wave normal n_j

$$N_j = n_j + \Delta N_j (n_k) \quad , \tag{18}$$

where the term ΔN_j reads for the *qP* wave:

$$\Delta N_j^{(3)} = 2\alpha^{-2} \left(B_{13}^{(i)} i_j^{(1)} + B_{23}^{(i)} i_j^{(2)} \right) \tag{19a}$$

and for *qS*1 and *qS*2 waves:

$$\Delta N_j^{(1)} = \beta^{-2} \left(\Delta a_{ijkl} n_l e_i^{(1)} e_k^{(1)} - B_{11}^{(e)} n_j - B_{13}^{(e)} e_j^{(1)} \right) \quad ,$$

$$\Delta N_j^{(2)} = \beta^{-2} \left(\Delta a_{ijkl} n_l e_i^{(2)} e_k^{(2)} - B_{22}^{(e)} n_j - B_{23}^{(e)} e_j^{(2)} \right) \quad . \tag{19b}$$

Since A_I is a first-order quantity, Eq. (13a) implies that everywhere, where a first-order quantity is multiplied by N_j or n_j, these two vectors are interchangeable. This means that, within the first-order approximation, with respect to Δa_{ijkl}, there is no difference if the perturbation ΔN_j is expressed with respect to n_j or N_j. This makes possible to rewrite Eq. (18) within the same first-order approximation, into the form

$$n_j = N_j - \Delta N_j (N_k) \quad . \tag{20}$$

In (20), $\Delta N_j\,(N_k)$ is determined from (19) with n_j substituted by N_j. Eq. (20) should work in directions, in which the considered wave has a convex slowness surface. It cannot work, however, when the slowness surface is concave or hyperbolic (this indicates triplication of the group velocity surface and three values of the function $n_j = n_j\,(N_k)$ for one N_k). Since Eq. (20) can yield only a single value of n_j for a value of N_k, the equation cannot, in principle, describe properly the exact behaviour of the function $n_j = n_j(N_k)$. In addition, we should expect a worse performance of Eq. (20) than of Eq. (18), caused by the strong nonlinearity of the term ΔN_j with respect to the wave normal n_j. Eqs. (18) and

(20) with (19) represent sought approximate formulae for the determination of the ray vector from wave normal and vice versa.

From (14), (16), (18) and (19), we can also derive approximate formulae for the group velocity vector in a weakly anisotropic medium. The formulae can be expressed both in terms of the wave normal n_j and the ray vector N_j. For qP waves, the group velocity vector has the form

$$v_j^{(3)} = (\alpha + \Delta c_3 (n_k)) n_j + \alpha \Delta N_j^{(3)} (n_k) = (\alpha + \Delta c_3 (N_k)) N_j^{(3)} \ . \tag{21a}$$

For $qS1$ and $qS2$ waves, the group velocity vectors are

$$v_j^{(1)} = (\beta + \Delta c_1 (n_k)) n_j + \beta \Delta N_j^{(1)} (n_k) = (\beta + \Delta c_1 (N_k)) N_j^{(1)} \ ,$$

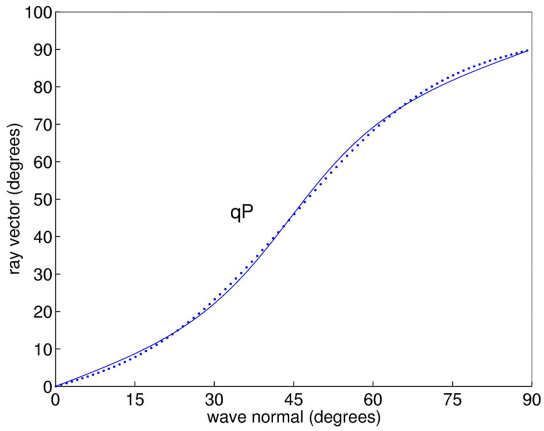
$$v_j^{(2)} = (\beta + \Delta c_2 (n_k)) n_j + \beta \Delta N_j^{(2)} (n_k) = (\beta + \Delta c_2 (N_k)) N_j^{(2)} \ . \tag{21b}$$

For Δc_k and $\Delta N_j^{(k)}$ see (5) and (19), respectively.

Let us mention an interesting phenomenon. If we compare Eq. (6a) and Eq. (21a) and take into account Eq. (19a), we can see that the qP-wave polarization vector, qP-wave group velocity vector and the wave normal are coplanar in weakly anisotropic media. This is an extension of the observation made by *Crampin (1981)* for general anisotropy in symmetry planes. As *Crampin (1981)*, we can also observe that the deviation of the group velocity and polarization vectors from the wave normal is in the same direction. All three vectors become parallel in the longitudinal directions (*Helbig, 1994*). In this case, the elements $B_{I3}^{(i)}$ of the weak anisotropy matrix vanish.

4. NUMERICAL EXAMPLES

We illustrate the accuracy of the derived formulae on three models of anisotropic media. Model A and Model B are transversely isotropic with vertical axes of symmetry (VTI). Model A is the *Shearer and Chapman (1989)* Model 1 (thin water-filled cracks), Model B is Model 4 of the same authors (thin water-filled cracks – extremely anisotropic). The Model C is orthorhombic, adopted from *Farra (2001)*. Anisotropy (calculated as $2\,(c_{max} - c_{min}) / (c_{max} + c_{min}) \times 100\%$) of Model A is about 3.5% for qP wave and about 11.2% for qS waves. Anisotropy of Model B is about 9% for qP wave, 29% and 30% for qS waves. We can see that anisotropy of qS waves in Model B is rather strong. Anisotropy of Model C is 14% for qP waves and 7% and 5% for qS waves. Reference velocities used are $\alpha = 4.41$ and $\beta = 2.42$ km/s in Model A, $\alpha = 4.30$ and $\beta = 2.28$ km/s in Model B, and $\alpha = 3.17$ and $\beta = 2.00$ km/s in Model C.



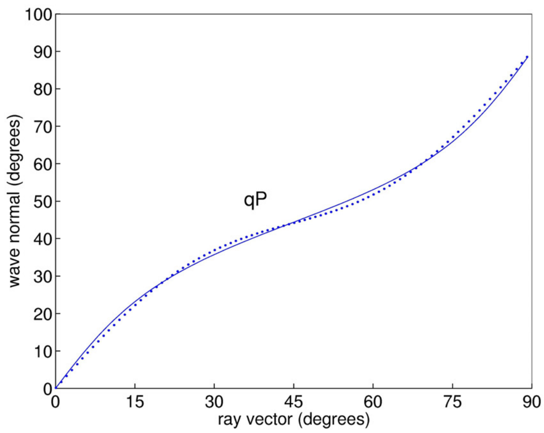


Fig. 1. Test of approximate formulae (18) (top) and (20) (bottom) for *qP* wave in Model B. Approximate curves (dotted lines) are compared with exact curves (solid lines). The wave normal and the ray vector are specified by their angles (in degrees) with the axis of symmetry.

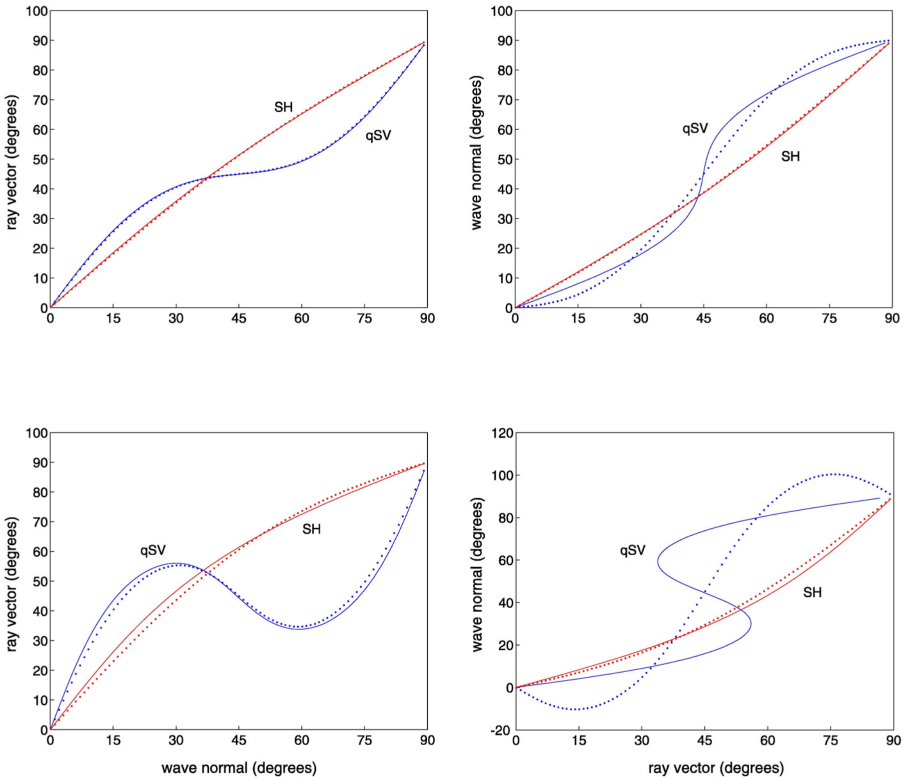


Fig. 2. Test of approximate formulae (18) (left column) and (20) (right column) for *qS* waves in Model A (top) and Model B (bottom). *qSV* wave – blue, *SH* wave – red. Approximate curves (dotted lines) are compared with exact curves (solid lines). The wave normal and the ray vector are specified by their angles (in degrees) with the axis of symmetry.

4.1 VTI Symmetry

Model A is characterized by the density-normalized elastic parameters A_{ij}, in $(\mathrm{km/s})^2$, with values: $A_{11} = A_{22} = 20.22$, $A_{33} = 20.04$, $A_{12} = 7.46$, $A_{13} = A_{23} = 7.41$, $A_{44} = A_{55} = 5.10$, $A_{66} = 6.38$. Model B is characterized by the density-normalized elastic parameters A_{ij}, in $(\mathrm{km/s})^2$, with values: $A_{11} = A_{22} = 20.16$, $A_{33} = 19.63$, $A_{12} = 7.40$, $A_{13} = A_{23} = 7.26$, $A_{44} = A_{55} = 3.48$, $A_{66} = 6.38$.

The accuracy of approximate formulae (18) and (20) for the determination of the ray vector N_j from a given wave normal n_j and vice versa is generally rather high for the *qP* wave as can be seen from Figure 1. The upper plot shows results obtained with approximate formula (18), the bottom plot with formula (20) for Model B. The wave normal and the ray vector are specified by their angles (in degrees) with the axis of symmetry. The comparison of approximate curves shown by dotted lines with exact curves shown by solid lines indicates quite a high accuracy of the approximate formulae.

The upper plots in Figure 2 show a comparison of the exact (solid line) and approximate (dotted line) curves for Model A. The bottom plots show the same for Model B. The left-hand plots show $N_j = N_j\,(n_k)$, see Eq. (18), the right-hand plots show $n_j = n_j\,(N_k)$, see Eq. (20). The angles of 0° correspond to the wave normal or ray vector along the symmetry axis, the angles of 90° correspond to the wave normal or ray vector perpendicular to the symmetry axis. For an isotropic medium the curves would be straight lines. Deviations from such lines indicate anisotropy. The exact and approximate curves $N_j = N_j\,(n_k)$ for Model A match each other very well for both *qS* waves The approximate formula $n_j = n_j\,(N_k)$ for the *SH* wave in Model A is of a similar accuracy. The accuracy is, however, rather low for the *qSV* wave. The approximate curve only indicates the basic trend of the exact curve. This effect is considerably more pronounced in the case of the *qSV* wave in Model B. The approximate formula fails because the single-valued approximate curve cannot fit the multi-valued exact curve. This is a consequence of the strong nonlinearity of the term ΔN_j with respect to the wave normal n_j. The approximate curves $N_j = N_j\,(n_k)$ and $n_j = n_j\,(N_k)$ for the *SH* wave for Model B show slightly greater deviations from the exact ones (anisotropy of Model B is rather strong) than in the Model A. The approximate curve $N_j = N_j\,(n_k)$ for the *qSV* wave has comparable accuracy. In this case, some ray vector directions correspond to three different phase normal directions. This is an indication of the triplication of the corresponding wavefront.

The accuracy of the formula (18) for Model A is illustrated in Figure 3. The upper plot shows exact angular deviation of the ray vector of *qS* waves from the wave normal as a function of the direction of the wave normal. The bottom plot shows the angular errors of the approximate formula (18): deviations of the approximate ray vector from the exact one in degrees. We can see in the upper plot that the deviation of the ray vector from the wave normal is always to one side and can reach 12° for the *qSV* wave and 6° for the *SH* wave at maximum. The maximum errors of the approximate formula (18) are about 0.45° for the *qSV* wave and nearly 0.5° for the *SH* wave. The above curves have similar forms also for Model B, only values are different. The maximum deviation of the ray vector from the wave normal can reach 30° for the *qSV* wave and 20° for the *SH* wave. The maximum errors of the approximate formula (18) are about 3.5° for both *qS* waves in Model B.

Figure 4 shows sections of the group velocity surfaces of *qSV* and *SH* waves as a function of the direction of the ray vector, see (21b). The ray vectors N_j in (21b) are determined from (18) and (19b) for regularly specified wave normals n_j. The upper plot corresponds to Model A, the bottom plot to Model B. We can see that (21b) yields, generally, less accurate results than previous formulae. It is due to the combination of two approximations contained in (21b): the approximation of the direction and the approximation of the value of the group velocity. Due to Eq. (16), the group velocity is equal to the phase velocity in the first-order perturbation approximation. An interesting phenomenon can be observed on the curve corresponding to the section of the *qSV* wave group-velocity surface in Model B. It shows that even the approximate formula can very roughly describe triplication of the group velocity surface.

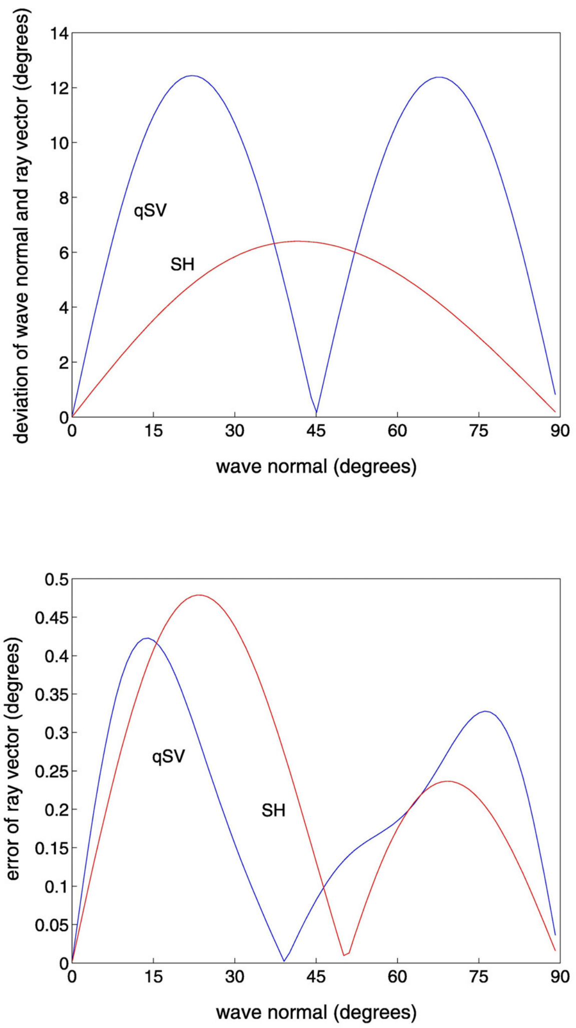


Fig. 3. Exact angular differences of the ray vector and the wave normal (top), and error plot of angular deviations of the approximate (Eq. (18)) and exact ray vectors (bottom) for *qS* waves in Model A. *qSV* wave – blue, *SH* wave – red. The wave normal is specified by its angle (in degrees) with the axis of symmetry.

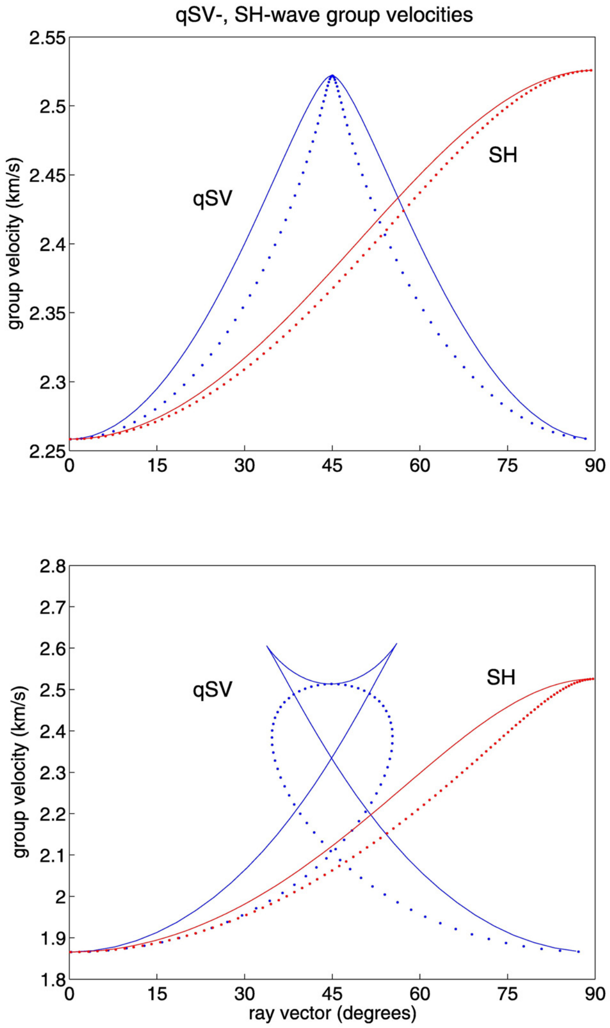


Fig. 4. Test of approximate formulae (21b) for the group velocity of *qS* waves in Model A (top) and Model B (bottom). *qSV* wave – blue, *SH* wave – red. The approximate curves (dotted lines) are compared with exact curves (solid lines). Ray vector is specified by its angle (in degrees) with the axis of symmetry.

4.2 Orthorhombic Symmetry

Model C is characterized by the density-normalized elastic parameters A_{ij}, in $(\mathrm{km/s})^2$, with values: $A_{11} = 10.8$, $A_{22} = 11.3$, $A_{33} = 8.5$, $A_{12} = 2.2$, $A_{13} = 1.9$, $A_{23} = 1.7$, $A_{44} = 3.6$, $A_{55} = 3.9$, $A_{66} = 4.3$.

Figure 5 illustrates accuracy of the approximate formula (18) for the *qS*1 wave (faster of the *qS* waves). The upper picture shows equal area plot of exact angular deviations (in degrees) of the ray vector and the wave normal. The bottom picture shows equal area plot of angular errors of the approximate formula (18). Both plots are parameterized by the wave normal. The maximum errors of the formula (18) are comparable with maximum exact differences between the wave normal and the ray vector. It is, however, necessary to emphasize that the maximum errors are concentrated to very narrow strips close to 45° inclination of the wave normal. In these directions the *qS* waves propagate with nearly the same phase velocities, i.e., the mentioned directions are singular directions, in which the studied formulae cannot, in principle, work properly. For the remaining directions, the errors are less than 1°. For the *qP* wave, the formula (18) yields errors of the same order everywhere. The errors are slightly larger for the *qS*2 wave. For all the waves, the approximate formulae work very well in the horizontal and vertical directions, which correspond to longitudinal directions.

5. CONCLUSIONS

Approximate formulae relating the ray vector and the wave normal were presented. The formulae are applicable to *qP* wave generally and to *qS* waves outside singular directions. The formulae can be also used for the approximate evaluation of the group velocity vector from the ray vector.

The accuracy of the formulae was tested on three examples of models of anisotropic media. The results show that the approximate formulae $N_j = N_j\,(n_k)$ and $n_j = n_j\,(N_k)$ work better for the *qP* wave than for the *qS* waves. For the *qS* waves, the above formulae fail in singular directions and their vicinities. Distorted results are also obtained from the formula $n_j = n_j\,(N_k)$ in directions of the triplication of the group velocity surface. The approximate formula $N_j = N_j\,(n_k)$ works rather well even in these directions. We have shown that the approximate formula for the group velocity as a function of the ray vector can even roughly describe a triplication.

The performed study shows that the approximate formulae relating the wave normal and the ray vector can be used quite safely for *qP* waves. For *qS* waves they must be used with a great care, especially when the wave normal is sought from a given ray vector.

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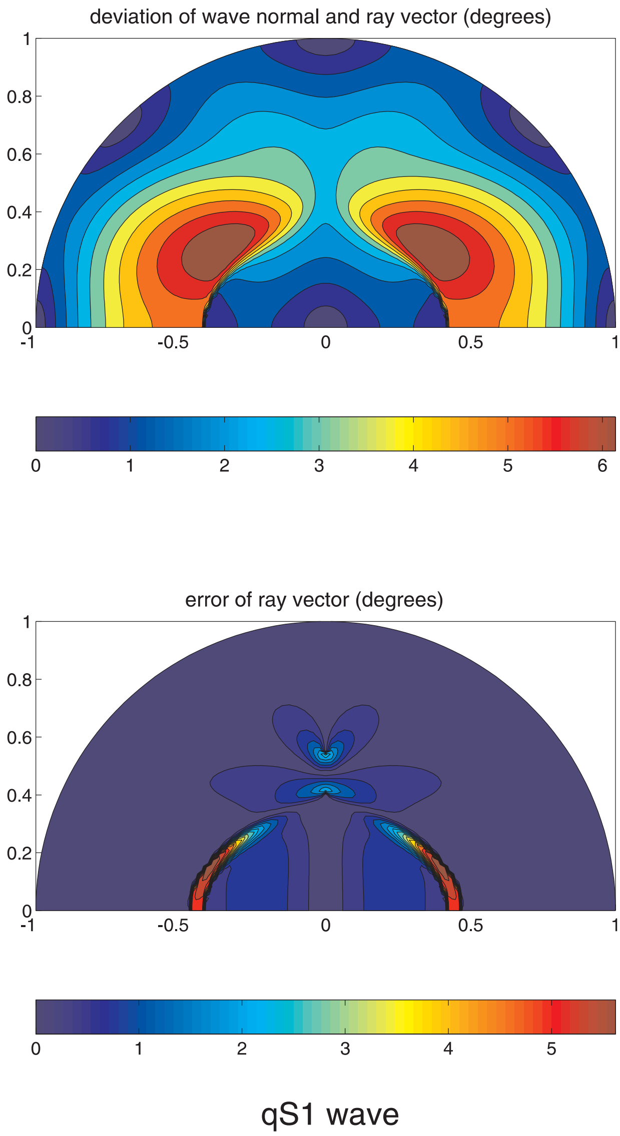

Fig. 5. Equal area plots of angular differences of the exact ray vector and the wave normal (top) and of angular deviations of the approximate (Eq. (18)) and exact ray vectors (bottom) for $qS1$ wave in Model C. Parameterization is in terms of wave normal, horizontal and vertical component of the wave normal along horizontal and vertical axis, respectively.

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