Description of weak anisotropy and weak attenuation using the first-order perturbation theory

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ABSTRACT

Velocity anisotropy and attenuation in weakly anisotropic and weakly attenuating structures can be treated uniformly using the weak anisotropy-attenuation (WAA) parameters. The WAA parameters are constructed in a very analogous way to weak anisotropy (WA) parameters designed for weak elastic anisotropy. The WAA parameters generalize the WA parameters by incorporating the attenuation effects. The WAA parameters can be represented alternatively by one set of complex values or by two sets of real values. Assuming high-frequency waves and using the first-order perturbation theory, all basic wave quantities such as the slowness vector, polarization vector, propagation velocity, attenuation and quality factor are linear functions of the WAA parameters.

Numerical modeling shows that the perturbation formulas have different accuracy for different wave quantities. The propagation velocity is usually calculated with high accuracy. However, the attenuation and quality factor may be reproduced with appreciably lower accuracy. This happens mostly when strength of velocity anisotropy is higher than 10% and attenuation is moderate or weak (Q-factor > 20). In this case, the errors of the attenuation or quality factor can attain values comparable with strength of anisotropy or can be even higher. It is shown that a simple modification of the formulas by including some higher-order perturbations improves the accuracy three to four times.

Keywords: anisotropy, attenuation, perturbation theory, theory of wave propagation

INTRODUCTION

Anisotropic attenuating media are frequently met in exploration seismics and intensively studied in theory of seismic wave propagation (Carcione, 1994, 2000, 2007). Since a general approach valid for modeling of waves in anisotropic attenuating media with any strength of anisotropy and attenuation is complicated and computationally demanding (Carcione, 1990; Saenger and Bohlen, 2004) it is advantageous to adopt several assumptions simplifying the problem. Firstly, we often assume that the studied waves are of high frequency, and secondly that the medium is weakly anisotropic and/or weakly attenuating. Both conditions are reasonable and frequently met in seismic practice. Imposing these conditions is worthwhile because it allows us to apply the ray theory designed for the propagation of high-frequency waves (Červený, 2001) and the perturbation theory suitable for solving wave propagation problems related to weak anisotropy and weak attenuation.

So far the perturbation theory has mainly been applied to wave propagation problems in weakly anisotropic elastic media (Thomsen, 1986; Jech and Pšenčík, 1989; Vavryčuk, 1997, 2003; Farra, 2001, 2004; Song et al., 2001; Pšenčík and Vavryčuk, 2002). This medium is introduced as a perturbation of an isotropic elastic background, and anisotropic wave quantities are calculated as perturbations of isotropic wave quantities. The perturbation formulas depend linearly on the weak anisotropy (WA) parameters, which quantify elastic anisotropy of the medium (Thomsen, 1986; Mensch and Rasolofosaon, 1997; Rasolofosaon, 2000; Pšenčík and Farra, 2005; Farra and Pšenčík, 2008). A similar approach can be applied to weakly attenuating media, where wave quantities in attenuating media are calculated as perturbations of those in non-attenuating media. Finally, both approaches can be combined and the effects of weak anisotropy and weak attenuation can be treated simultaneously and uniformly.

In this paper, I have developed the perturbation theory applicable to propagation of high-frequency waves in weakly anisotropic and weakly attenuating media. All basic wave quantities are expressed in terms of the weak anisotropy-attenuation (WAA) parameters, which quantify the velocity and attenuation anisotropy and play a key role in the perturbation formulas. They can be defined either as complex-valued or real-valued quantities. The complex WAA parameters were first introduced by Rasolofosaon (2008) and applied to propagation of homogeneous plane waves in weakly anisotropic and weakly attenuating media of arbitrary symmetry. Rasolofosaon (2008) used the correspondence principle in his derivation and considered an anisotropic viscoelastic reference medium. The complete set of real-valued WAA parameters has not been published yet. The real-valued WAA parameters have a form similar to a linearized version of Thomsen's parameters known from studies of elastic and viscoelastic transverse isotropy (Zhu and Tsvankin, 2006) and orthorhombic anisotropy (Zhu and Tsvankin, 2007). All the previous approaches are based on the assumption of propagation of homogeneous plane waves. Since I deal with high-frequency waves which are generally inhomogeneous, the paper is also a further step from homogeneous plane-wave approaches (Carcione, 2000; Chichinina et al., 2006; Zhu and Tsvankin, 2006, 2007; Červený and Pšenčík, 2005, 2008a,b; Rasolofosaon, 2008) towards more realistic wave modeling. This mainly relates to calculating stationary slowness vectors, polarization vectors, and other wave quantities, which inherently depend on wave inhomogeneity. The wave inhomogeneity can be uniquely calculated in the ray theory from an experimental setup (i.e., from the source and receiver positions, medium parameters and boundary conditions) but must be a priori assumed in plane-wave approaches. Finally, the paper is also an extension of previous works (Vavryčuk, 2008) as it assumes the reference background medium as attenuating instead of purely elastic.

PERTURBATION FORMULAS

A weakly anisotropic and weakly attenuating medium can be viewed as a medium obtained by a small perturbation of an isotropic elastic or viscoelastic reference medium,

$$a_{ijkl} = a^0_{ijkl} + \Delta a_{ijkl} \,, \tag{1}$$

where a^0_{ijkl} defines the reference medium and Δa_{ijkl} its perturbation. The density-normalized viscoelastic stiffness parameters a^0_{ijkl} can be expressed in terms of the P- and S-wave velocities c^P_0 and c^S_0

$$a^0_{ijkl} = \left(\left(c^P_0\right)^2 - 2\left(c^S_0\right)^2\right)\delta_{ij}\delta_{kl} + \left(c^S_0\right)^2\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) \,, \tag{2}$$

where δ_{ij} denotes the Kronecker delta. If the reference medium is elastic, the reference parameters are real but the perturbations are complex,

$$a^0_{ijkl} = a^R_{ijkl} \,,\ \Delta a_{ijkl} = \Delta a^R_{ijkl} + i\,\Delta a^I_{ijkl} \;. \tag{3}$$

where perturbations Δa^R_{ijkl}, Δa^I_{ijkl} describe weak anisotropy and weak attenuation, respectively. If the reference medium is viscoelastic, both the reference parameters and perturbations are complex. In order to keep the approach as general as possible, the reference medium will be considered as viscoelastic.

Using the first-order perturbation theory, we can simplify the formulas for the phase and ray wave quantities derived for homogeneous media of arbitrarily strong anisotropy and attenuation (see Vavryčuk, 2007). The approach is basically the same as presented in Vavryčuk (2008), the only difference is that we now consider a different reference medium. The reference medium is assumed to be anisotropic elastic in Vavryčuk (2008) but isotropic viscoelastic in this paper. The ray direction is fixed during perturbations. The perturbation of the eigenvalue of the Christoffel tensor $G(\mathbf{n})$,

$$G(\mathbf{n}) = a_{ijkl} n_i n_l g_j g_k = c^2 , \tag{4}$$

reads

$$G = G_0 + \Delta G , \tag{5}$$

$$G_0 = c_0^2 , \; \Delta G = \Delta a_{ijkl} n_i^0 n_l^0 g_j^0 g_k^0 . \tag{6}$$

The eigenvalue G_0 and the perturbation ΔG are complex valued:

$$G_0 = G_0^R + iG_0^I , \;\; \Delta G = \Delta G^R + i\Delta G^I , \tag{7}$$

$$\Delta G^R = \Delta a_{ijkl}^R n_i^0 n_l^0 g_j^0 g_k^0 , \tag{8}$$

$$\Delta G^I = \Delta a_{ijkl}^I n_i^0 n_l^0 g_j^0 g_k^0 , \tag{9}$$

where slowness and polarization vectors $\mathbf{n}^0$ and $\mathbf{g}^0$ are real valued and correspond to an isotropic viscoelastic reference medium. For the P-wave, the polarization vector $\mathbf{g}^0$ equals to the slowness direction vector $\mathbf{n}^0$. For the S-waves, the polarization vectors $\mathbf{g}^0$ lie in the plane perpendicular to $\mathbf{n}^0$. Their orientation in this plane must be calculated according perturbation formulas designed for degenerate eigenvectors (see Vavryčuk, 2003, his Appendix A).

Formulas 8 and 9 are valid if both perturbations Δa_{ijkl}^R and Δa_{ijkl}^I are mutually comparable and small with respect to values of the reference medium. Since Δa_{ijkl}^I is often significantly smaller than Δa_{ijkl}^R, formula 9 can appear to be of low accuracy (see Section 5 Numerical examples). The inaccuracy is incorporated into formula 9 by identifying slowness direction **n** and polarization vector **g** in an anisotropic medium with those in the isotropic reference medium. Thus the effects of the velocity anisotropy are fully neglected in formula 9. The accuracy is improved if we adopt a modified formula for ΔG^I expressed as

$$\Delta G^I = \Delta a^I_{ijkl} n^R_i n^R_l g^0_j g^0_k \; . \qquad 10)$$

Even higher accuracy is achieved for ΔG^I expressed as

$$\Delta G^I = \Delta a^I_{ijkl} n^R_i n^R_l g^R_j g^R_k \; , \qquad (11)$$

where $\mathbf{n}^R$ and $\mathbf{g}^R$ are the real parts of the slowness and polarization vectors in a weakly anisotropic medium, respectively. Similarly, formula 8 for ΔG^R should be modified in an analogous way as formula 10 or 11 if we study details of very weakly anisotropic but strongly attenuating medium.

For the perturbation of a slowness vector, see Appendix A. The Appendix shows that the slowness vector is homogeneous in the reference medium, but generally inhomogeneous in a perturbed medium. However, the inhomogeneity is small being of the order of the first perturbation. For the perturbation of a polarization vector, see Appendix B. If we calculate complex energy velocity *v*, as the magnitude of complex energy velocity vector **v**,

$$v = \sqrt{v_i v_i} \; , \; v_i = a_{ijkl} p_l g_j g_k \; , \qquad (12)$$

and complex phase velocity *c* from formula 4, we obtain that velocities *v* and *c* are equal in the first-order perturbation theory and read

$$v = c = \sqrt{G} \; . \qquad (13)$$

Also other ray and phase quantities (for their definitions, see Vavryčuk, 2007) are equal in the first-order perturbation theory:

$$V^{\text{phase}} = V^{\text{ray}} , \; \left[Q^{\text{phase}}\right]^{-1} = \left[Q^{\text{ray}}\right]^{-1} , \; A^{\text{phase}} = A^{\text{ray}} , \qquad (14)$$

hence, hereafter I will not distinguish between the ray and phase quantities simply speaking of velocity V, quality factor Q and attenuation A. The velocity is expressed as

$$V = \sqrt{G^R} = V_0 \left(1 + \frac{1}{2} \frac{\Delta G^R}{G_0^R} \right). \quad (15)$$

This equation follows from the following expression, $V^2 = V_0^2 + \Delta G^R$, where V_0 corresponds to the isotropic elastic part of the reference medium, $V_0^2 = G_0^R$. The attenuation and quality factor read

$$Q^{-1} = Q_V^{-1} - \frac{\Delta G^I}{V^2}, \; A = A_V - \frac{\Delta G^I}{2V^3}, \quad (16)$$

or alternatively

$$Q^{-1} = Q_V^{-1} \left(1 + \frac{\Delta G^I}{G_0^I} \right), \; A = A_V \left(1 + \frac{\Delta G^I}{G_0^I} \right), \quad (17)$$

where

$$G_0 = c_0^2, \; G_0^R = \left(c_0^2\right)^R, \; G_0^I = \left(c_0^2\right)^I, \; Q_V^{-1} = -\frac{G_0^I}{V^2}, \; A_V = -\frac{G_0^I}{2V^3}. \quad (18)$$

Equations 16 and 17 follow from equations $Q^{-1} = -G^I / G^R$ and $A = Q^{-1} / 2V$ (see Vavryčuk, 2008, his formulas 51 and 59). Emphasize that Q_V^{-1} and A_V are not quantities describing an isotropic viscoelastic reference medium. They reflect the effects of weak velocity anisotropy being thus directionally dependent. The dependence on velocity V is acknowledged by using subscript V. Equations 16 hold for viscoelastic as well as for elastic reference media, equations 17 are restricted to the viscoelastic reference medium only (G_0^I in the denominator must be non-zero).

Note that although the phase and ray quantities are equal, the ray and slowness directions differ (see Pšenčík & Vavryčuk, 2002). Ray direction **N** is real and fixed and, therefore, does not change during perturbations: $\mathbf{N} = \mathbf{N}^0$. Ray direction **N** is equal to slowness direction $\mathbf{n}^0$ in the isotropic reference medium. However, slowness direction **n** in a perturbed medium deviates from $\mathbf{n}^0$ and **N** and is generally complex. The difference between both directions **n** and **N** is small being of the order of the first perturbation.

A similar observation about the approximate equality of the phase and ray attenuations (equation 14) is reported by Behura and Tsvankin (2009), who show that the so-called normalized group attenuation coefficient estimated along seismic rays practically coincides with the phase attenuation coefficient computed for a zero inhomogeneity angle. Under strong anisotropy and attenuation, however, the equality of the ray and phase attenuations is not fully valid and can be broken under some conditions (see Vavryčuk, 2007b; Behura and Tsvankin, 2009).

WEAK ANISOTROPY-ATTENUATION PARAMETERS

Instead of using perturbations Δa_{ijkl} in formulas for wave quantities it is often convenient to rearrange the formulas by introducing dimensionless constants called the "weak anisotropy-attenuation (WAA) parameters". The WAA parameters are constructed very similarly to "weak anisotropy (WA) parameters", which are used in weak elastic anisotropy. The WAA parameters generalize the WA parameters by incorporating also the attenuation effects. The WAA parameters can be defined alternatively as complex quantities or real quantities. The complex WAA parameters were first introduced by Rasolofosaon (2008). The WAA parameters describe a directional variation of the complex energy velocity or equivalently of the complex phase velocity. Since they are complex valued they reflect jointly both the velocity anisotropy and attenuation. One set of complex WAA parameters can be split into two sets of real WAA parameters, which describe the directional variations of real velocity and real attenuation separately.

Procedure

In order to construct the complex and real WAA parameters, we define dimensionless perturbations $\Delta\varepsilon_{ijkl}$, $\Delta\varepsilon^{V}_{ijkl}$ and $\Delta\varepsilon^{Q}_{ijkl}$:

$$\Delta\varepsilon_{ijkl} = \frac{\Delta a_{ijkl}}{G_0},\ \Delta\varepsilon^{V}_{ijkl} = \frac{\Delta a^{R}_{ijkl}}{G^{R}_0},\ \Delta\varepsilon^{Q}_{ijkl} = \frac{\Delta a^{I}_{ijkl}}{G^{I}_0}. \qquad (19)$$

Hence

$$G = G_0\left(1 + \Delta\varepsilon_{ijkl} n^0_i n^0_l g^0_j g^0_k\right),$$

$$G^R = G_0^R\left(1 + \Delta\varepsilon_{ijkl}^V n_i^0 n_l^0 g_j^0 g_k^0\right) , \tag{20}$$

$$G^I = G_0^I\left(1 + \Delta\varepsilon_{ijkl}^Q n_i^0 n_l^0 g_j^0 g_k^0\right) .$$

Using the following notation

$$\Delta\varepsilon = \Delta\varepsilon_{ijkl} n_i^0 n_l^0 g_j^0 g_k^0$$

$$\Delta\varepsilon^V = \Delta\varepsilon_{ijkl}^V n_i^0 n_l^0 g_j^0 g_k^0 , \tag{21}$$

$$\Delta\varepsilon^Q = \Delta\varepsilon_{ijkl}^Q n_i^0 n_l^0 g_j^0 g_k^0 ,$$

the formulas for the eigenvalue of the Christoffel tensor, phase velocity, Q-factor and attenuation are modified as:

$$G = G_0(1 + \Delta\varepsilon), \; V = V_0\left(1 + \frac{1}{2}\Delta\varepsilon^V\right), \; Q^{-1} = Q_V^{-1}\left(1 + \Delta\varepsilon^Q\right) , \; A = A_V\left(1 + \Delta\varepsilon^Q\right) , \tag{22}$$

where

$$G_0 = c_0^2 , \; V_0 = \sqrt{\left(c_0^2\right)^R} , \; Q_V^{-1} = -\frac{G_0^I}{V^2} , \; A_V = -\frac{G_0^I}{2V^3} . \tag{23}$$

Quantities G_0 and V_0 describe the isotropic reference medium and are directionally independent. Quality factor Q_V and attenuation A_V are directionally dependent.

Definition of complex WAA parameters

To keep the notation consistent with the WA parameters defined previously by Farra and Pšenčík (2008, their formula A1), the dimensionless perturbations $\Delta\varepsilon_{ijkl}$, $\Delta\varepsilon_{ijkl}^V$ and $\Delta\varepsilon_{ijkl}^Q$ will be expressed in the Voigt notation and slightly rearranged. Hence, the complex WAA parameters are finally defined as:

$$\varepsilon_x = \frac{a_{11} - G_0^P}{2G_0^P} , \; \varepsilon_y = \frac{a_{22} - G_0^P}{2G_0^P} , \; \varepsilon_z = \frac{a_{33} - G_0^P}{2G_0^P} ,$$

$$\delta_x = \frac{a_{23} + 2a_{44} - G_0^P}{G_0^P} , \; \delta_y = \frac{a_{13} + 2a_{55} - G_0^P}{G_0^P} , \; \delta_z = \frac{a_{12} + 2a_{66} - G_0^P}{G_0^P} ,$$

$$\gamma_x = \frac{a_{44} - G_0^S}{2G_0^S}, \ \gamma_y = \frac{a_{55} - G_0^S}{2G_0^S}, \ \gamma_z = \frac{a_{66} - G_0^S}{2G_0^S},$$

$$\chi_x = \frac{a_{14} + 2a_{56}}{G_0^P}, \ \chi_y = \frac{a_{25} + 2a_{46}}{G_0^P}, \ \chi_z = \frac{a_{36} + 2a_{45}}{G_0^P}, \tag{24}$$

$$\varepsilon_{15} = \frac{a_{15}}{G_0^P}, \ \varepsilon_{16} = \frac{a_{16}}{G_0^P}, \ \varepsilon_{24} = \frac{a_{24}}{G_0^P}, \ \varepsilon_{26} = \frac{a_{26}}{G_0^P}, \ \varepsilon_{34} = \frac{a_{34}}{G_0^P}, \ \varepsilon_{35} = \frac{a_{35}}{G_0^P},$$

$$\varepsilon_{46} = \frac{a_{46}}{G_0^S}, \ \varepsilon_{56} = \frac{a_{56}}{G_0^S}, \ \varepsilon_{45} = \frac{a_{45}}{G_0^S},$$

where a_{ij} are complex viscoelastic parameters in the Voigt notation, and G_0^P and G_0^S are complex eigenvalues of the Christoffel tensor in the isotropic viscoelastic reference medium. They can be calculated from real P- and S-wave velocities α and β, and quality factors Q_0^P and Q_0^S as follows

$$G_0^P = \alpha^2\left(1 - \frac{i}{Q_0^P}\right), \ G_0^S = \beta^2\left(1 - \frac{i}{Q_0^S}\right). \tag{25}$$

Definition of real WAA parameters

If we separate the effects of velocity anisotropy and attenuation, we obtain two sets of real WAA parameters: one set for the velocity anisotropy (with superscript *V*) and one set for the attenuation anisotropy (with superscript *Q*). Again, perturbations $\Delta\varepsilon_{ijkl}^V$ and $\Delta\varepsilon_{ijkl}^Q$ are rearranged in a similar way as in formula 24. For the velocity anisotropy parameters we obtain:

$$\varepsilon_x^V = \frac{a_{11}^R - \alpha^2}{2\alpha^2}, \ \varepsilon_y^V = \frac{a_{22}^R - \alpha^2}{2\alpha^2}, \ \varepsilon_z^V = \frac{a_{33}^R - \alpha^2}{2\alpha^2},$$

$$\delta_x^V = \frac{a_{23}^R + 2a_{44}^R - \alpha^2}{\alpha^2}, \ \delta_y^V = \frac{a_{13}^R + 2a_{55}^R - \alpha^2}{\alpha^2}, \ \delta_z^V = \frac{a_{12}^R + 2a_{66}^R - \alpha^2}{\alpha^2},$$

$$\gamma_x^V = \frac{a_{44}^R - \beta^2}{2\beta^2}, \ \gamma_y^V = \frac{a_{55}^R - \beta^2}{2\beta^2}, \ \gamma_z^V = \frac{a_{66}^R - \beta^2}{2\beta^2},$$

$$\chi_x^V = \frac{a_{14}^R + 2a_{56}^R}{\alpha^2}, \ \chi_y^V = \frac{a_{25}^R + 2a_{46}^R}{\alpha^2}, \ \chi_z^V = \frac{a_{36}^R + 2a_{45}^R}{\alpha^2}, \tag{26}$$

$$\varepsilon_{15}^{V}=\frac{a_{15}^{R}}{\alpha^2},\ \varepsilon_{16}^{V}=\frac{a_{16}^{R}}{\alpha^2},\ \varepsilon_{24}^{V}=\frac{a_{24}^{R}}{\alpha^2},\ \varepsilon_{26}^{V}=\frac{a_{26}^{V}}{\alpha^2},\ \varepsilon_{34}^{V}=\frac{a_{34}^{R}}{\alpha^2},\ \varepsilon_{35}^{V}=\frac{a_{35}^{R}}{\alpha^2},$$

$$\varepsilon_{46}^{V}=\frac{a_{46}^{R}}{\beta^2},\ \varepsilon_{56}^{V}=\frac{a_{56}^{R}}{\beta^2},\ \varepsilon_{45}^{V}=\frac{a_{45}^{R}}{\beta^2}.$$

The attenuation anisotropy parameters are defined analogously as the velocity anisotropy parameters but in terms of a_{ij}^{I}, Q_0^P and Q_0^S:

$$\varepsilon_x^Q=-\frac{a_{11}^I Q_0^P+\alpha^2}{2\alpha^2},\ \varepsilon_y^Q=-\frac{a_{22}^I Q_0^P+\alpha^2}{2\alpha^2},\ \varepsilon_z^Q=-\frac{a_{33}^I Q_0^P+\alpha^2}{2\alpha^2},$$

$$\delta_x^Q=-\frac{\left(a_{23}^I+2a_{44}^I\right)Q_0^P+\alpha^2}{\alpha^2},\ \delta_y^Q=-\frac{\left(a_{13}^I+2a_{55}^I\right)Q_0^P+\alpha^2}{\alpha^2},\ \delta_z^Q=-\frac{\left(a_{12}^I+2a_{66}^I\right)Q_0^P+\alpha^2}{\alpha^2},$$

$$\gamma_x^Q=-\frac{a_{44}^I Q_0^S+\beta^2}{2\beta^2},\ \gamma_y^Q=-\frac{a_{55}^I Q_0^S+\beta^2}{2\beta^2},\ \gamma_z^Q=-\frac{a_{66}^I Q_0^S+\beta^2}{2\beta^2},$$

$$\chi_x^Q=-\frac{a_{14}^I+2a_{56}^I}{\alpha^2}Q_0^P,\ \chi_y^Q=-\frac{a_{25}^I+2a_{46}^I}{\alpha^2}Q_0^P,\ \chi_z^Q=-\frac{a_{36}^I+2a_{45}^I}{\alpha^2}Q_0^P, \qquad (27)$$

$$\varepsilon_{15}^Q=-\frac{a_{15}^I}{\alpha^2}Q_0^P,\ \varepsilon_{16}^Q=-\frac{a_{16}^I}{\alpha^2}Q_0^P,\ \varepsilon_{24}^Q=-\frac{a_{24}^I}{\alpha^2}Q_0^P,\ \varepsilon_{26}^Q=-\frac{a_{26}^I}{\alpha^2}Q_0^P,\ \varepsilon_{34}^Q=-\frac{a_{34}^I}{\alpha^2}Q_0^P,$$

$$\varepsilon_{35}^Q=-\frac{a_{35}^I}{\alpha^2}Q_0^P,\ \varepsilon_{46}^Q=-\frac{a_{46}^I}{\beta^2}Q_0^S,\ \varepsilon_{56}^Q=-\frac{a_{56}^I}{\beta^2}Q_0^S,\ \varepsilon_{45}^Q=-\frac{a_{45}^I}{\beta^2}Q_0^S.$$

Note that the two sets of real WAA parameters do not coincide with the real and imaginary parts of the one set of complex WAA parameters. This is because the complex WAA parameters do not separate the effects of velocity and attenuation anisotropy (see equation 19). For example, the real parts of the complex WAA parameters are affected not only by the velocity anisotropy but also by the attenuation of the reference medium. On the other hand, the two sets of the real WAA parameters strictly separate the effects of the velocity and attenuation anisotropy. The velocity anisotropy parameters are not affected by attenuation and attenuation anisotropy parameters are independent of the elastic anisotropy or the elastic properties of the reference medium. Also the reader is reminded that the formulas for the attenuation anisotropy parameters fail for the elastic reference medium. In this case, only the approach with complex WAA parameters is applicable.

P-WAVE IN TRANSVERSELY ISOTROPIC MEDIA

In this section, the derived formulas are specified for the P-wave propagating in a transversely isotropic medium with a vertical axis of symmetry (VTI medium). The medium is described by the following parameters in the Voigt notation: a_{11}, $a_{22} = a_{11}$, a_{33}, a_{44}, $a_{55} = a_{44}$, a_{66}, a_{13}, $a_{23} = a_{13}$ and $a_{12} = a_{11} - 2a_{66}$. All other parameters are zero. The parameters a_{ij} are complex valued. The velocity anisotropy and attenuation are assumed to be weak. The wave quantities are studied in the x_1–x_3 plane. Perturbations for the SV-wave can be found analogously to the P-wave and the SH-wave quantities can easily be calculated exactly in the VTI medium.

Formulas using perturbations of viscoelastic parameters

The complex and real velocities, quality factors and attenuations for the P-waves are expressed by the following formulas:

$$c^2 = c_0^2 + \Delta G \, , \;\; V = V_0 \left(1 + \frac{1}{2} \frac{\Delta G^R}{V_0^2} \right), \;\; Q^{-1} = Q_V^{-1} - \frac{\Delta G^I}{V^2} \, , \;\; A = A_V - \frac{\Delta G^I}{2V^3} \, . \tag{28}$$

where

$$\Delta G = \Delta a_{11} N_1^4 + \Delta a_{33} N_3^4 + 2\left(\Delta a_{13} + 2\Delta a_{44}\right) N_1^2 N_3^2 \, ,$$

$$\Delta G^R = \Delta a_{11}^R N_1^4 + \Delta a_{33}^R N_3^4 + 2\left(\Delta a_{13}^R + 2\Delta a_{44}^R\right) N_1^2 N_3^2 \, , \tag{29}$$

$$\Delta G^I = \Delta a_{11}^I N_1^4 + \Delta a_{33}^I N_3^4 + 2\left(\Delta a_{13}^I + 2\Delta a_{44}^I\right) N_1^2 N_3^2 \, ,$$

The reference quantities in equation 28 read

$$c_0 = \alpha \sqrt{1 - \frac{i}{Q_0^P}} \, , \;\; V_0 = \alpha \, , \;\; Q_V^{-1} = \frac{\alpha^2}{V^2} \frac{1}{Q_0^P} \, , \;\; A_V = \frac{\alpha^2}{2V^3} \frac{1}{Q_0^P} \, . \tag{30}$$

Vector **N** is the real ray direction, $\mathbf{N} = (\sin\theta, 0, \cos\theta)^T$, quantities α and Q_0^P are the real P-wave velocity and quality factor in the isotropic viscoelastic reference medium, and angle θ defines the deviation of a ray from the symmetry axis..

Formulas using WAA parameters

The complex and real velocities, quality factors and attenuations for the P-waves are expressed in terms of the WAA parameters by the following formulas:

$$c^2 = c_0^2(1+\Delta\varepsilon), \quad V = V_0\left(1+\frac{1}{2}\Delta\varepsilon^V\right), \; Q^{-1} = Q_V^{-1}\left(1+\Delta\varepsilon^Q\right) \;, \; A = A_V\left(1+\Delta\varepsilon^Q\right) \;. \tag{31}$$

Perturbations $\Delta\varepsilon$, $\Delta\varepsilon^V$ and $\Delta\varepsilon^Q$ in 31 read

$$\Delta\varepsilon = 2\left(\varepsilon_x N_1^4 + \varepsilon_z N_3^4 + \delta_x N_1^2 N_3^2\right),$$

$$\Delta\varepsilon^V = 2\left(\varepsilon_x^V N_1^4 + \varepsilon_z^V N_3^4 + \delta_x^V N_1^2 N_3^2\right), \tag{32}$$

$$\Delta\varepsilon^Q = 2\left(\varepsilon_x^Q N_1^4 + \varepsilon_z^Q N_3^4 + \delta_x^Q N_1^2 N_3^2\right),$$

where $N_1 = \sin\theta$ and $N_3 = \cos\theta$. The reference quantities are defined in equation 30, and the WAA parameters in equations 24, 26 and 27.

Formulas with improved accuracy

As mentioned in the previous section, the accuracy of the first-order perturbations for attenuation *A* and quality factor *Q* can be improved by incorporating some higher-order perturbations. This can be done when treating the slowness vector in a more accurate way than in standard formulas. So far, slowness direction **n** was simply identified with ray direction **N** in formulas 29 and 32. This approximation works well for very weak anisotropy. The stronger the anisotropy, the lower the accuracy of this approximation. Hence, instead of using the slowness direction $\mathbf{n}^0 = \mathbf{N}$ in formula 9, we can utilize the linearized $\mathbf{n}^R$,

$$\mathbf{n}^R = \mathbf{n}^0 + \Delta\mathbf{n}^R = \mathbf{N} + \Delta\mathbf{n}^R . \tag{33}$$

The perturbation formula for $\Delta\mathbf{n}^R$ is derived in Appendix A for anisotropy of arbitrary symmetry, and in Appendix C for transverse isotropy. Hence in TI media, we obtain for the P-wave

$$\Delta n_1^{PR} = -2\frac{N_1}{\alpha^2}\left[A_1^R N_3^4 + A_2^R N_3^2\right], \; \Delta n_3^{PR} = -2\frac{N_3}{\alpha^2}\left[A_1^R N_3^4 + \left(A_2^R - A_1^R\right)N_3^2 - A_2^R\right], \tag{34}$$

Constants A_1^R and A_2^R are expressed in terms of perturbations Δa_{ijkl}^R as

$$A_1^R = -\Delta a_{11}^R + 2\Delta a_{13}^R - \Delta a_{33}^R + 4\Delta a_{44}^R \ , \ A_2^R = \Delta a_{11}^R - \Delta a_{13}^R - 2\Delta a_{44}^R \ , \tag{35}$$

and in terms of WAA parameters as

$$A_1^R = 2\alpha^2\left(\delta_x^V - \varepsilon_x^V - \varepsilon_z^V\right) , \ A_2^R = \alpha^2\left(-\delta_x^V + 2\varepsilon_x^V\right) . \tag{36}$$

Since we correct just the slowness direction but not the polarization vectors in formula 10, the substitution of ray direction **N** by the corrected slowness direction **n** in formulas 29 and 32 will read as follows:

$$\mathbf{n} = \mathbf{n}^0 + \frac{1}{2}\Delta\mathbf{n}^R = \mathbf{N} + \frac{1}{2}\Delta\mathbf{n}^R \ . \tag{37}$$

Hence, the corrected formula 32 reads

$$\Delta\varepsilon = 2\left(\varepsilon_x n_1^4 + \varepsilon_z n_3^4 + \delta_x n_1^2 n_3^2\right),$$

$$\Delta\varepsilon^V = 2\left(\varepsilon_x^V n_1^4 + \varepsilon_z^V n_3^4 + \delta_x^V n_1^2 n_3^2\right), \tag{38}$$

$$\Delta\varepsilon^Q = 2\left(\varepsilon_x^Q n_1^4 + \varepsilon_z^Q n_3^4 + \delta_x^Q n_1^2 n_3^2\right),$$

where

$$n_1 = N_1\left\{1 - \left(\delta_x^V - 2\varepsilon_z^V\right)N_3^2 - 2\left(\varepsilon_x^V + \varepsilon_z^V - \delta_x^V\right)N_1^2 N_3^2\right\},$$

$$n_3 = N_3\left\{1 + \left(\delta_x^V - 2\varepsilon_z^V\right)N_1^2 + 2\left(\varepsilon_x^V + \varepsilon_z^V - \delta_x^V\right)N_1^4\right\}, \tag{39}$$

and vector **n** is further normalized to be of unit length before inserting into formula 38..

NUMERICAL EXAMPLES

In this section, I demonstrate the accuracy of the perturbation formulas using numerical examples for the P-wave in homogeneous VTI media. I adopted four viscoelastic models with two strengths of anisotropy and two levels of attenuation. The models are denoted as models A2, A4, B2 and B4, being taken from Vavryčuk (2008). The anisotropy strength (i.e., the magnitude of the directional velocity variation) is 23% for models A2 and A4, and 10% for models B2 and B4. The average *Q*-factors are about 10 for models A2 and B2, and 40 for models A4 and B4. The *Q*-factor anisotropy is 45.5% for all four models (see Vavryčuk, 2008, his Table 3). The models with anisotropy strength of 23% cannot be considered as weakly anisotropic, but here they are used to illustrate how the accuracy of the perturbation formulas deteriorates in this case. The viscoelastic parameters of the models are summarized in Table 1. For detailed information on the models, see Vavryčuk (2008).

Table 1. Viscoelastic parameters. The two-index Voigt notation is used for the density-normalized elastic parameters and for quality parameters. Parameters a_{66}^R and Q_{66} are not listed because the P-wave is not sensitive to them.

Model	Elastic parameters				Attenuation parameters			
	a_{11}^R (km²/s²)	a_{13}^R (km²/s²)	a_{33}^R (km²/s²)	a_{44}^R (km²/s²)	Q_{11}	Q_{13}	Q_{33}	Q_{44}
A2	14.4	4.50	9.00	2.25	15	8	10	8
A4	14.4	4.50	9.00	2.25	60	32	40	32
B2	10.8	3.53	9.00	2.25	15	8	10	8
B4	10.8	3.53	9.00	2.25	60	32	40	32

Table 2. The values of the isotropic viscoelastic reference medium.

Model	α (km/s)	β (km/s)	Q_0^P	Q_0^S
A2	3.40	1.50	10.5	8.0
A4	3.40	1.50	42.0	32.0
B2	3.15	1.50	10.5	8.0
B4	3.15	1.50	42.0	32.0

Figure 1 shows the directional variations of the exact and approximate velocities, attenuations *A* and the *Q*-factors for models A2 (left-hand plots) and B2 (right-hand plots), respectively. Figure 2 shows the same quantities but for models A4 and B4. The angles range from 0° to 90°. The exact ray quantities (black solid line) are calculated according to formulas 21, 22 and 24 of Vavryčuk (2007b). The exact stationary slowness vector is calculated by a procedure described in Vavryčuk (2007b). The approximate velocities, attenuations *A* and the *Q*-factors are calculated using equations 31 and 32 (blue dashed line). The reference quantities needed in the approximate formulas are listed in Table 2. In the approximations, I do not distinguish between the ray and phase quantities because they are identical in the first-order perturbation theory. The figures show that the highest accuracy is achieved for the velocity having errors less than 3% for models A and less than 1% for models B. This result is satisfactory regarding that strength of the velocity anisotropy is 23% for models A, and 10% for models B. However, the accuracies of attenuation *A* and quality factor *Q* are considerably lower. Their accuracies are about 15% for models A2 and A4, and 10% for models B2 and B4 (see Tables 3 and 4).

In order to assess the effectiveness and accuracy of perturbation formulas 31 and 32, Figures 1 and 2 also show the approximate velocities, attenuations and *Q*-factors calculated using the alternative formulas derived for the P-wave propagating in TI media (red dashed lines) and exploiting the Thomsen-style parameters (Thomsen, 1986; Tsvankin, 2005; Zhu and Tsvankin, 2006):

$$V^{\mathrm{Th}} = V_0^{\mathrm{Th}}\left(1 + \delta N_1^2 N_3^2 + \varepsilon N_1^4\right), \tag{40}$$

$$A^{\mathrm{Th}} = \frac{A_0^{\mathrm{Th}}}{V^{\mathrm{Th}}}\left(1 + \delta_Q N_1^2 N_3^2 + \varepsilon_Q N_1^4\right), \tag{41}$$

$$Q^{\mathrm{Th}} = \frac{1}{2A^{\mathrm{Th}}V^{\mathrm{Th}}}, \tag{42}$$

where V_0^{Th} is the vertical velocity in the elastic reference VTI medium, ε and δ are Thomsen's parameters (Thomsen, 1986, his formulas 8a and 17), A_0^{Th} is the reference attenuation (Zhu and Tsvankin, 2006, their formula 22), and ε_Q and δ_Q are attenuation parameters (Zhu and Tsvankin, 2006, their formulas 28 and 31). Since the definition of attenuation *A* in this paper is slightly different from that in Zhu and Tsvankin (2006), formula 41 is not identical to the original formula 36 of Zhu and Tsvankin (2006). The values of the Thomsen-style parameters used in numerical modeling are summarized in Vavryčuk (2008, his Table 2).

Figures 1 and 2 show that the accuracy of formulas 41 ad 42 for attenuation and the *Q*-factor in models A2 and A4 is almost twice higher than that of the first-order perturbations 31 and 32. For models B2 and B4, the accuracy is roughly the same for both approaches. This demonstrates that formulas 41 and 42 are preferable in models with stronger velocity anisotropy. This is due to the fact that parameter δ_Q in formulas 41 and

42 depends not only on attenuation of the medium, but also on its velocity anisotropy. This property is lost in real-valued WAA parameters (formulas 26 and 27), where the effects of the velocity anisotropy and attenuation anisotropy are fully separated. Therefore, formulas 41 and 42 can be viewed as perturbation formulas which incorporate some of higher-order terms.

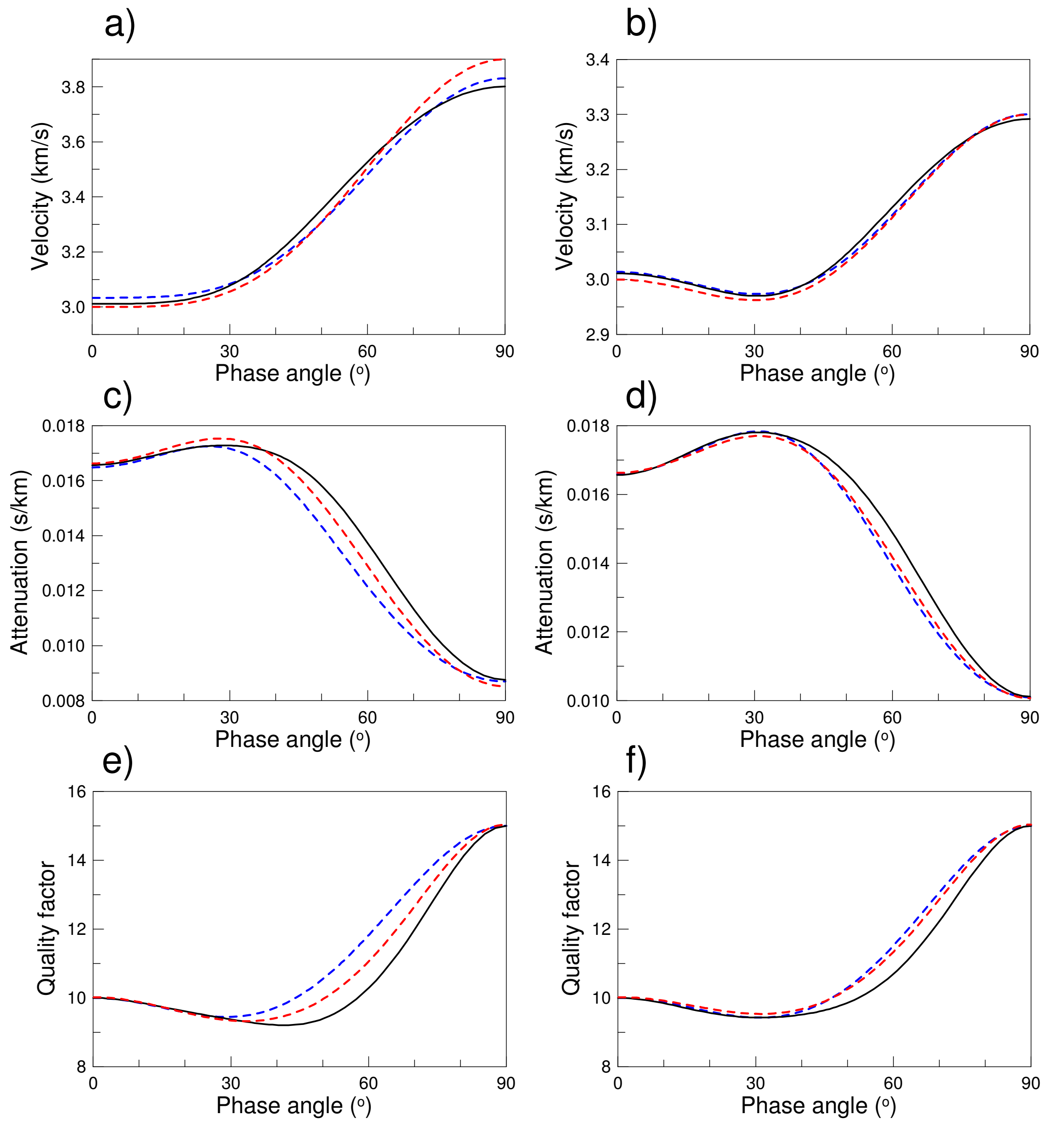

Figure 1. Exact and approximate velocities, attenuations and quality factors in models A2 (left-hand plots) and B2 (right-hand plots). Black solid lines show the exact phase quantities. Blue dashed lines show the approximate quantities calculated using formulas 31 and 32. Red dashed lines show the approximate solution 41 and 42 of Zhu and Tsvankin (2006). The phase angle denotes the deviation of the real part of the complex slowness vector from the symmetry axis.

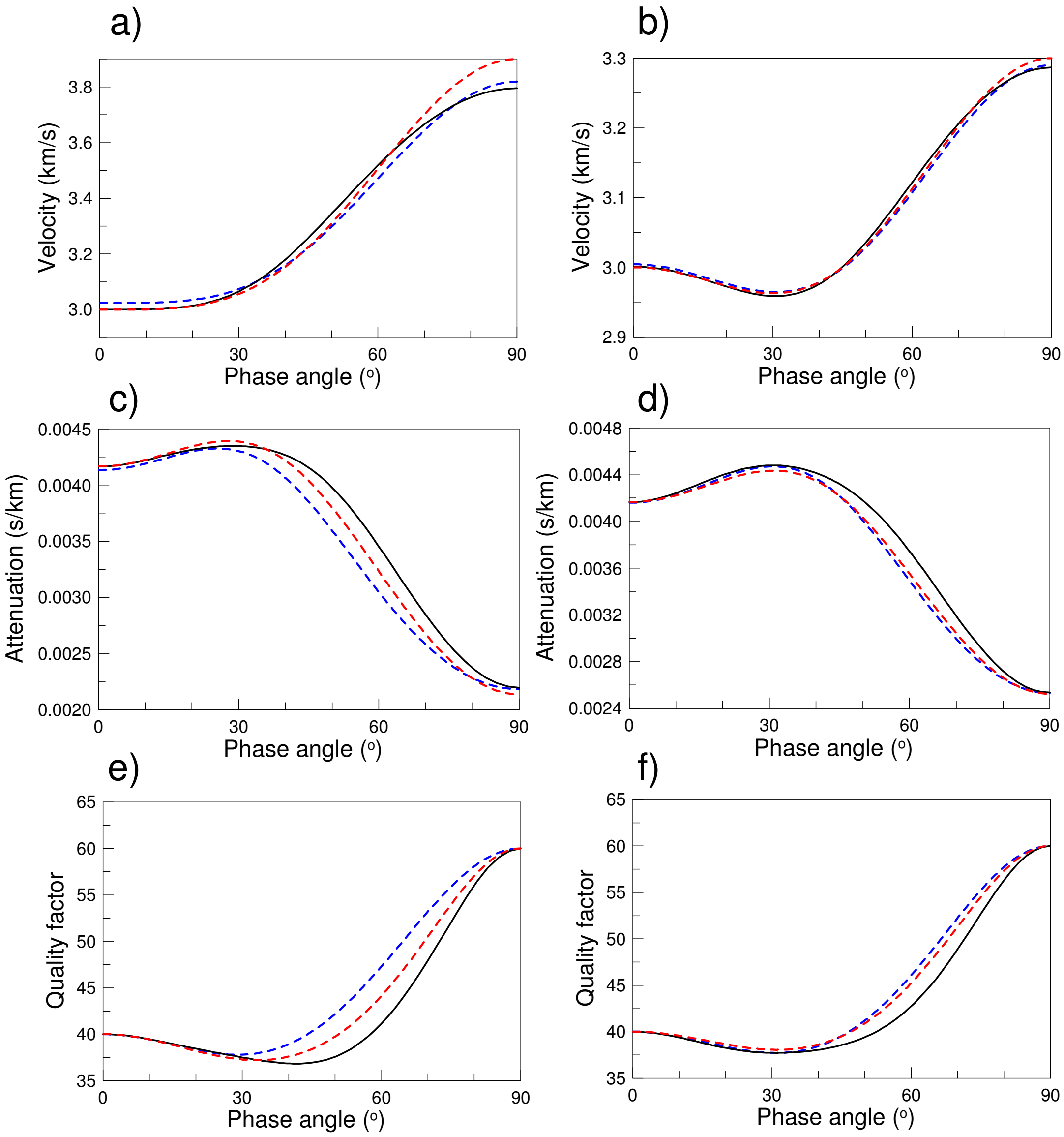

Figure 2. Exact and approximate velocities, attenuations and quality factors in models A4 (left-hand plots) and B4 (right-hand plots). For details see the caption of Figure 1.

Interestingly, the accuracy of approximate A and Q in Figures 1 and 2 does not depend on strength of attenuation, even though one would expect the perturbations to work better for less attenuating media (models A4 and B4). This observation is reported also by Zhu nad Tsvankin (2006) and it is explained by the fact that the accuracy of attenuation is not affected just by strength of attenuation, but also by strength of the velocity anisotropy. The velocity anisotropy and attenuation are both described by perturbations and their effects cannot be separated easily. Hence the accuracy of A and Q in the models studied is not primarily affected by strength of attenuation, but by strength of anisotropy. If we use the modified perturbation formula 10 for ΔG^1, the accuracy of A and Q improves. This is indicated in Figure 3 for models A2 and B2, and summarized in Tables 3 and 4. The figure and the tables also show errors of equations 41 and 42 derived by Zhu and Tsvankin (2006). Both approaches incorporate some of the higher-order perturbations and yield higher accuracy than the first-order perturbations. The accuracy of formulas 41 and 42 is almost twice higher than that of the standard first-order perturbations. The accuracy of formulas 31, 38 and 39 is almost three to four times higher than that of the standard first-order perturbations. Obviously, more complicated approximations (e.g., Zhu and Tsvankin, 2006, their formula 19) can yield even higher accuracy.

Table 3. Maximum errors of the perturbations of the attenuation. The error for a particular ray is calculated as $E = 100\left|U^{\text{exact}} - U^{\text{aprox}}\right| / U^{\text{exact}}$, where U^{exact} and U^{aprox} are the exact and approximate values of the respective quantity. The presented values are the maxima over all rays. ZT – perturbations of Zhu & Tsvankin (2006), V1 – formulas 31 and 32, V2 – formulas 31, 38 and 39.

Model	Error – ZT		Error – V1		Error – V2	
	A^{phase} (%)	A^{ray} (%)	A^{phase} (%)	A^{ray} (%)	A^{phase} (%)	A^{ray} (%)
A2	6.3	10.7	11.6	14.7	2.9	3.1
A4	6.6	11.0	11.8	14.9	2.7	3.3
B2	5.0	6.3	6.8	8.0	0.4	1.6
B4	5.3	6.6	6.9	8.3	0.5	1.8

Table 4. Maximum errors of the perturbations of the quality factor. The error for a particular ray is calculated as $E = 100\left|U^{\text{exact}} - U^{\text{aprox}}\right| / U^{\text{exact}}$, where U^{exact} and U^{aprox} are the exact and approximate values of the respective quantity. The presented values are the maxima over all rays. ZT – perturbations of Zhu & Tsvankin (2006), V1 – formulas 31 and 32, V2 – formulas 31, 38 and 39.

Model	Error – ZT		Error – V1		Error – V2	
	Q^{phase} (%)	Q^{ray} (%)	Q^{phase} (%)	Q^{ray} (%)	Q^{phase} (%)	Q^{ray} (%)
A2	7.6	8.0	15.0	14.2	5.1	4.0
A4	7.3	7.8	14.9	14.3	5.1	4.1
B2	6.1	6.1	8.0	8.0	2.0	2.1
B4	5.9	6.0	8.0	8.1	2.0	2.2

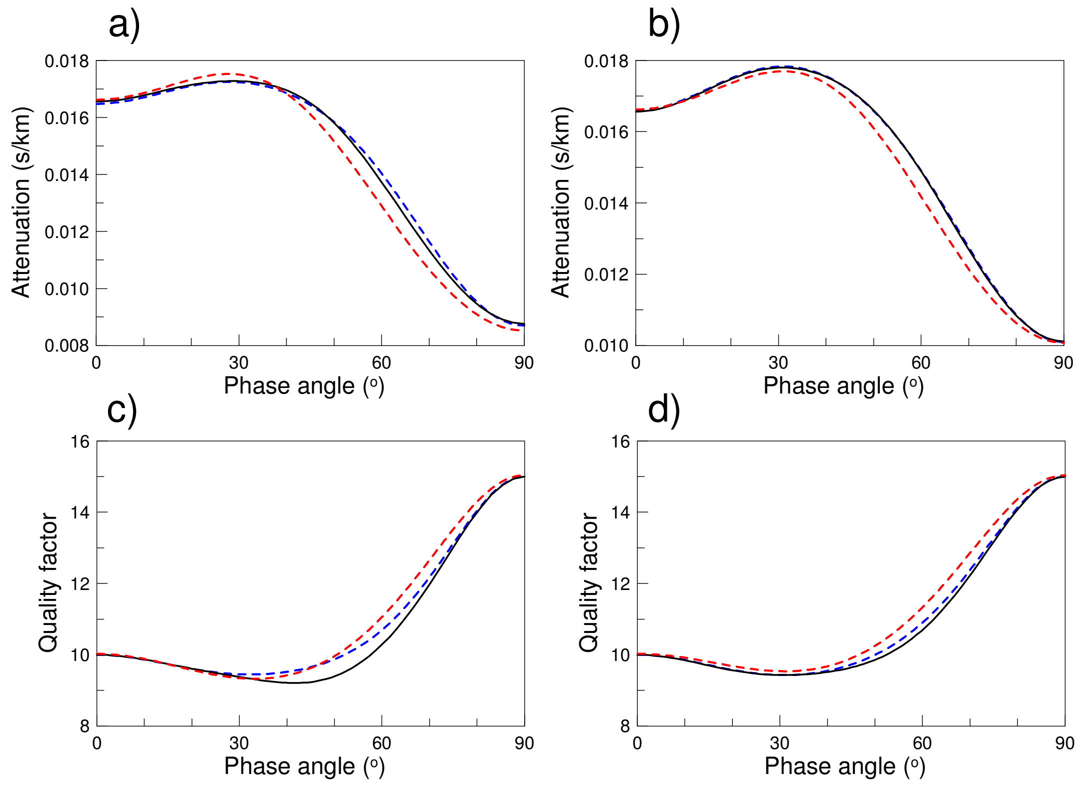

Figure 3. Exact and approximate attenuations and quality factors in models A2 (left-hand plots) and B2 (right-hand plots). Black solid lines show the exact phase quantities. Blue dashed lines show the approximate quantities of the improved accuracy calculated using formulas 31, 38 and 39. Red dashed lines show the approximate solution 41 and 42 of Zhu and Tsvankin (2006). The phase angle denotes the deviation of the real part of the complex slowness vector from the symmetry axis.

DISCUSSION

Numerical modeling shows that perturbation formulas differ in accuracy for different wave quantities. The propagation velocity is usually calculated with high accuracy. However, the attenuation and quality factor may be reproduced with appreciably lower accuracy. This happens mostly when the anisotropy strength is higher than 10% and the attenuation is moderate or weak ($Q > 20$). In this case, the first-order perturbations may appear to be a too rough approximation and a modified approach would be required. To overcome this difficulty, it is possible to introduce the real weak attenuation parameters in a slightly more complicated form than defined in this paper. This was done by Zhu and Tsvankin (2006, 2007) for TI and orthorhombic anisotropy. These definitions automatically include some effects of the velocity anisotropy (i.e., weak attenuation parameters depend on weak velocity parameters). Alternatively, we can incorporate some higher-order perturbations into formulas for attenuation and the *Q*-factor by considering the slowness direction calculated in an actual anisotropic medium, but not in an isotropic reference medium (see formula 10). The numerical examples prove that this approach is more accurate than the linearized approach by Zhu and Tsvankin (2006). Finally, it is also possible to use perturbations just for evaluating the slowness vector (formulas A8–A10), and possibly the polarization vector (formulas B7–B9). All other calculations can be performed exactly. Obviously, this approach yields the most accurate results (see Vavryčuk, 2008). A different highly accurate nonlinear approximation for the attenuation coefficient in TI media is given by Zhu and Tsvankin (2006, their equation 19).

CONCLUSIONS

The weak anisotropy-attenuation (WAA) parameters proved to be an effective tool for calculating wave quantities in weakly anisotropic attenuating media of arbitrary symmetry. The WAA parameters can be introduced alternatively as complex-valued or real-valued quantities. The use of complex-valued WAA parameters seems to be mathematically more elegant and less laborious when writing computer codes, but the real-valued WAA parameters are probably more comprehensible and their physical meaning more understandable. For example, the velocity anisotropy parameters are very similar to linearized versions of Thomsen's parameters widely used in seismic processing and inversion in transversely isotropic media. The only difference is that Thomsen's parameters use a fixed reference medium while the velocity anisotropy parameters use a reference medium which can be adjusted. Since the first-order perturbation formulas of the wave quantities depend linearly on the WAA parameters, the WAA parameters can easily be calculated in inverse problems.

The perturbation approach also has its limitations. Firstly, it is limited by strength of anisotropy and attenuation. Perturbations work well in anisotropic media where the phase and ray quantities are not very different. This is because the first-order perturbations do not distinguish between the phase and ray quantities. Obviously, the perturbations are not applicable to media with strong anisotropy or anisotropy displaying triplications. The standard perturbation formulas also do not work near singularities (acoustic axes), where the Christoffel tensor becomes nearly degenerate. In this case, the perturbation formulas must be modified.

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APPENDIX A: PERTURBATION OF THE SLOWNESS VECTOR

The slowness vector is calculated using the first-order perturbations and taken at a stationary point on the slowness surface. The stationary point is a point for which energy velocity vector **v** is homogeneous and its direction is parallel to the ray. The approach is basically the same as presented in Vavryčuk (2008). The only difference is that, instead of an anisotropic elastic medium assumed in Vavryčuk (2008), an isotropic viscoelastic medium is now considered. The perturbation of the P-wave stationary slowness vector for the anisotropic viscoelastic reference medium reads (see Vavryčuk, 2008, his equation 38)

$$\Delta p_l^{(1)} = \left[H_{il}^{0(1)}\right]^{-1}\left(\frac{v_i^{0(1)}}{2}\Delta a_{mjkl}\, p_m^{0(1)} p_l^{0(1)} g_j^{0(1)} g_k^{0(1)} - \Delta_a v_i^{(1)}\right), \tag{A1}$$

where

$$H_{il}^{0(1)} = a_{ijkl}^0 g_j^{0(1)} g_k^{0(1)} + \frac{v_i^{0(12)} v_l^{0(12)}}{G^{0(1)} - G^{0(2)}} + \frac{v_i^{0(13)} v_l^{0(13)}}{G^{0(1)} - G^{0(3)}}, \tag{A2}$$

$$\Delta_a v_i^{(1)} = \Delta a_{mjkl}\, p_l^{0(1)} g_j^{0(1)}\left[\delta_{im} g_k^{0(1)} + \frac{v_i^{0(12)} p_m^{0(1)} g_k^{0(2)}}{G^{0(1)} - G^{0(2)}} + \frac{v_i^{0(13)} p_m^{0(1)} g_k^{0(3)}}{G^{0(1)} - G^{0(3)}}\right], \tag{A3}$$

$$v_i^{0(12)} = a_{ijkl}^0 p_l^{0(1)}\left(g_j^{0(1)} g_k^{0(2)} + g_j^{0(2)} g_k^{0(1)}\right), \tag{A4}$$

$$v_i^{0(13)} = a_{ijkl}^0 p_l^{0(1)}\left(g_j^{0(1)} g_k^{0(3)} + g_j^{0(3)} g_k^{0(1)}\right), \tag{A5}$$

and δ_{ij} is the Kronecker delta. The superscript (1, 2 and 3) in brackets means the type of wave (P, S1 and S2). Quantity $H_{il}^{0(1)}$ is the P-wave metric tensor of the reference medium (see Vavryčuk, 2003). The formulas for the S1- and S2-wave stationary slowness vectors are analogous. Taking into account that in isotropic media

$$H_{il}^{0} = c_0^2\,\delta_{il}\,,\ \left[H_{il}^{0}\right]^{-1} = c_0^{-2}\delta_{il}\,,$$

$$v_i^{0(12)} g_i^{0(1)} = 0\,,\ v_i^{0(12)} g_i^{0(2)} = \frac{\left(c_0^P\right)^2 - \left(c_0^S\right)^2}{c_0}\,,\ v_i^{0(12)} g_k^{0(3)} = 0\,,$$

$$v_i^{0(13)} g_i^{0(1)} = 0\,,\ v_i^{0(13)} g_i^{0(2)} = 0\,,\ v_i^{0(13)} g_i^{0(3)} = \frac{\left(c_0^P\right)^2 - \left(c_0^S\right)^2}{c_0}\,, \tag{A6}$$

$$v_i^{0(23)} g_i^{0(1)} = 0\,,\ v_i^{0(23)} g_i^{0(2)} = 0\,,\ v_i^{0(23)} g_i^{0(3)} = 0\,,$$

we obtain

$$\Delta_a v_m^{(1)} g_m^{0(1)} = \frac{1}{c_0^P}\Delta a_{ijkl} n_i^0 n_l^0 n_j^0 n_k^0\,,\ \Delta_a v_m^{(1)} g_m^{0(2)} = \frac{2}{c_0^P}\Delta a_{ijkl} n_i^0 n_l^0 n_j^0 g_k^{0(2)}\,,$$

$$\Delta_a v_m^{(1)} g_m^{0(3)} = \frac{2}{c_0^P}\Delta a_{ijkl} n_i^0 n_l^0 n_j^0 g_k^{0(3)}\,,$$

$$\Delta_a v_m^{(2)} g_m^{0(1)} = \frac{1}{c_0^S}\Delta a_{ijkl} n_i^0 n_l^0 g_j^{0(2)} g_k^{0(2)}\,,\ \Delta_a v_m^{(2)} g_m^{0(2)} = \frac{1}{c_0^S}\Delta a_{ijkl} n_l^0 g_j^{0(2)}\left(g_i^{0(2)} g_k^{0(2)} - n_i^0 n_k^0\right)\,, \tag{A7}$$

$$\Delta_a v_m^{(2)} g_m^{0(3)} = \frac{1}{c_0^S}\Delta a_{ijkl} n_i^0 g_j^{0(2)} g_k^{0(2)} g_l^{0(3)}\,,$$

$$\Delta_a v_m^{(3)} g_m^{0(1)} = \frac{1}{c_0^S}\Delta a_{ijkl} n_i^0 n_l^0 g_j^{0(3)} g_k^{0(3)}\,,\ \Delta_a v_m^{(3)} g_m^{0(3)} = \frac{1}{c_0^S}\Delta a_{ijkl} n_l^0 g_j^{0(3)}\left(g_i^{0(3)} g_k^{0(3)} - n_i^0 n_k^0\right)\,,$$

$$\Delta_a v_m^{(3)} g_m^{0(2)} = \frac{1}{c_0^S}\Delta a_{ijkl} n_i^0 g_j^{0(3)} g_k^{0(3)} g_l^{0(2)}$$

and finally

$$\Delta p_m^{(1)} = -\frac{1}{2\left(c_0^P\right)^3}\Delta a_{ijkl} n_l^0 n_j^0 n_k^0\left(4\delta_{im} - 3n_i^0 n_m^0\right)\,, \tag{A8}$$

$$\Delta p_m^{(2)} = -\frac{1}{2\left(c_0^S\right)^3} \Delta a_{ijkl} n_l^0 g_j^{0(2)} \left[g_k^{0(2)} \left(2\delta_{im} - n_i^0 n_m^0 \right) - 2 n_i^0 n_k^0 g_m^{0(2)} \right] , \tag{A9}$$

$$\Delta p_m^{(3)} = -\frac{1}{2\left(c_0^S\right)^3} \Delta a_{ijkl} n_l^0 g_j^{0(3)} \left[g_k^{0(3)} \left(2\delta_{im} - n_i^0 n_m^0 \right) - 2 n_i^0 n_k^0 g_m^{0(3)} \right] . \tag{A10}$$

It follows from formulas A8–A10 that if perturbations Δa_{ijkl} are real valued, the perturbations of the slowness vector $\Delta \mathbf{p}^{(1)}$, $\Delta \mathbf{p}^{(2)}$ and $\Delta \mathbf{p}^{(3)}$ are also real valued. This means that a weakly anisotropic medium with isotropic attenuation or a weakly anisotropic elastic medium generate a homogeneous stationary slowness vector.

For the perturbation of the slowness direction we readily obtain

$$\Delta n_m^{(1)} = -\frac{2}{\left(c_0^P\right)^2} \Delta a_{ijkl} n_l^0 n_j^0 n_k^0 \left(\delta_{im} - n_i^0 n_m^0 \right) , \tag{A11}$$

$$\Delta n_m^{(2)} = -\frac{1}{\left(c_0^S\right)^2} \Delta a_{ijkl} n_l^0 g_j^{0(2)} \left[g_k^{0(2)} \left(\delta_{im} - n_i^0 n_m^0 \right) - n_i^0 n_k^0 g_m^{0(2)} \right] , \tag{A12}$$

$$\Delta n_m^{(3)} = -\frac{1}{\left(c_0^S\right)^2} \Delta a_{ijkl} n_l^0 g_j^{0(3)} \left[g_k^{0(3)} \left(\delta_{im} - n_i^0 n_m^0 \right) - n_i^0 n_k^0 g_m^{0(3)} \right] . \tag{A13}$$

APPENDIX B: PERTURBATION OF THE POLARIZATION VECTOR

The perturbation of the P-wave eigenvector $\mathbf{g}^{(1)}$ of the Christoffel tensor Γ_{jk} is expressed as a sum of perturbations projected into the directions of the S-wave polarization vectors $\mathbf{g}^{(2)}$ and $\mathbf{g}^{(3)}$,

$$\Delta g_i^{(1)} = \frac{\Delta G^{(12)}}{G^{0(1)} - G^{0(2)}} g_i^{0(2)} + \frac{\Delta G^{(13)}}{G^{0(1)} - G^{0(3)}} g_i^{0(3)} , \tag{B1}$$

where

$$\Delta G^{(rs)} = \Delta\Gamma_{jk} g_j^{0(r)} g_k^{0(s)}, \tag{B2}$$

$$\Delta\Gamma_{jk} = \Delta a_{ijkl} p_i^0 p_l^0 + a_{ijkl}^0 \left(p_i^0 \Delta p_l + p_l^0 \Delta p_i\right) . \tag{B3}$$

Taking into account formula A8 we can write

$$\Delta p_m^{(1)} g_m^{0(1)} = -\frac{1}{2\left(c_0^P\right)^3} \Delta a_{ijkl} n_i^0 n_l^0 n_j^0 n_k^0 ,$$

$$\Delta p_m^{(1)} g_m^{0(2)} = -\frac{2}{\left(c_0^P\right)^3} \Delta a_{ijkl} n_i^0 n_l^0 n_j^0 g_k^{0(2)} , \tag{B4}$$

$$\Delta p_m^{(1)} g_m^{0(3)} = -\frac{2}{\left(c_0^P\right)^3} \Delta a_{ijkl} n_i^0 n_l^0 n_j^0 g_k^{0(3)} .$$

Consequently, from formulas B2 and B3 we obtain,

$$\Delta\Gamma_{jk} g_j^{0(1)} g_k^{0(2)} = \frac{2\left(c_0^S\right)^2 - \left(c_0^P\right)^2}{\left(c_0^P\right)^4} \Delta a_{ijkl} n_i^0 n_l^0 n_j^0 g_k^{0(2)} , \tag{B5}$$

$$\Delta\Gamma_{jk} g_j^{0(1)} g_k^{0(3)} = \frac{2\left(c_0^S\right)^2 - \left(c_0^P\right)^2}{\left(c_0^P\right)^4} \Delta a_{ijkl} n_i^0 n_l^0 n_j^0 g_k^{0(3)}, \tag{B6}$$

and finally

$$\Delta g_m^{(1)} = \frac{1}{\left(c_0^P\right)^2} \frac{2\left(c_0^S\right)^2 - \left(c_0^P\right)^2}{\left(c_0^P\right)^2 - \left(c_0^S\right)^2} \Delta a_{ijkl} n_l^0 n_j^0 n_k^0 \left(\delta_{im} - n_i^0 n_m^0\right) . \tag{B7}$$

The perturbation of the S-wave polarization vectors $\mathbf{g}^{(2)}$ and $\mathbf{g}^{(3)}$ projected into the direction of the P-wave polarization vectors $\mathbf{g}^{0(1)}$ can be found in an analogous way. We obtain,

$$\Delta g_m^{(2)} = \left\{ \frac{1}{\left(c_0^S\right)^2} \Delta a_{ijkl} n_l^0 g_j^{0(2)} \left(g_i^{0(2)} g_k^{0(2)} - n_i^0 n_k^0\right) - \frac{1}{\left(c_0^P\right)^2 - \left(c_0^S\right)^2} \Delta a_{ijkl} n_i^0 n_l^0 n_j^0 g_k^{0(2)} \right\} n_m^0 , \tag{B8}$$

$$\Delta g_m^{(3)} = \left\{ \frac{1}{\left(c_0^S\right)^2} \Delta a_{ijkl} n_l^0 g_j^{0(3)} \left(g_i^{0(3)} g_k^{0(3)} - n_i^0 n_k^0 \right) - \frac{1}{\left(c_0^P\right)^2 - \left(c_0^S\right)^2} \Delta a_{ijkl} n_i^0 n_l^0 n_j^0 g_k^{0(3)} \right\} n_m^0 \ . \qquad \text{(B9)}$$

Since the isotropic reference medium is degenerate for the S-waves, the perturbation of polarization vectors $\mathbf{g}^{(2)}$ and $\mathbf{g}^{(3)}$ projected into the $\mathbf{g}^{0(2)} - \mathbf{g}^{0(3)}$ plane is calculated in a more complicated way (see Farra, 2001; Vavryčuk, 2003, his Appendix A) and is not presented here. For transversely isotropic medium, these projections of the SH- and SV-waves are identically zero.

APPENDIX C: PERTURBATION OF THE POLARIZATION VECTOR, SLOWNESS VECTOR AND SLOWNESS DIRECTION IN TI MEDIA

The perturbation formulas for stationary slowness vector $\Delta\mathbf{p}$ (Appendix A), its direction $\Delta\mathbf{n}$ (Appendix A), and polarization vector $\Delta\mathbf{g}$ (Appendix B) simplify in TI media. Substituting Δa_{ijkl} for TI and taking into account that the S1- and S2-waves become the SH- and SV-waves in TI, we obtain for the P-wave

$$\Delta p_1^P = N_1 C_p^P \left[3A_1 N_3^4 + 2A_2 N_3^2 + \Delta a_{11} \right], \Delta p_3^P = N_3 C_p^P \left[3A_1 N_3^4 + 2(A_2 - 2A_1) N_3^2 - 4A_2 + \Delta a_{11} \right], \ \text{(C1)}$$

$$\Delta n_1^P = N_1 C_n^P \left[A_1 N_3^4 + A_2 N_3^2 \right] , \ \Delta n_3^P = N_3 C_n^P \left[A_1 N_3^4 + (A_2 - A_1) N_3^2 - A_2 \right] , \qquad \text{(C2)}$$

$$\Delta g_1^P = N_1 C_g^P \left[A_1 N_3^4 + A_2 N_3^2 \right] , \ \ \Delta g_3^P = N_3 C_g^P \left[A_1 N_3^4 + (A_2 - A_1) N_3^2 - A_2 \right] , \qquad \text{(C3)}$$

where

$$C_p^P = -\frac{1}{2\left(c_0^P\right)^3} , \ \ C_n^P = -\frac{2}{\left(c_0^P\right)^2} , \ \ C_g^P = \frac{1}{\left(c_0^P\right)^2} \frac{2\left(c_0^S\right)^2 - \left(c_0^P\right)^2}{\left(c_0^P\right)^2 - \left(c_0^S\right)^2} \ , \qquad \text{(C4)}$$

and for the SV-wave

$$\Delta p_1^{SV} = N_1 C_p^{SV} \left[A_1 N_3^2 \left(1 - 3N_3^2\right) + \Delta a_{44} \right] , \ \Delta p_3^{SV} = N_3 C_p^{SV} \left[A_1 N_3^2 \left(5 - 3N_3^2\right) - 2A_1 + \Delta a_{44} \right] , \qquad \text{(C5)}$$

$$\Delta n_1^{SV} = N_1 C_n^{SV} A_1 N_3^2 \left(N_1^2 - N_3^2 \right) , \ \Delta n_3^{SV} = -N_3 C_n^{SV} A_1 N_1^2 \left(N_1^2 - N_3^2 \right) , \qquad \text{(C6)}$$

$$\Delta g_1^{SV} = \Delta n_3^{SV} + N_3 C_g^{SV} \left[A_1 N_3^4 + (A_2 - A_1) N_3^2 - A_2 \right], \Delta g_3^{SV} = -\Delta n_1^{SV} - N_1 C_g^{SV} \left[A_1 N_3^4 + A_2 N_3^2 \right], \ \text{(C7)}$$

where

$$C_p^{SV} = -\frac{1}{2\left(c_0^S\right)^3}\ ,\ \ C_n^{SV} = -\frac{1}{\left(c_0^S\right)^2}\ ,\ C_g^{SV} = \frac{1}{\left(c_0^P\right)^2 - \left(c_0^S\right)^2}\ . \tag{C8}$$

Using WAA parameters, the formulas for constants A_1 and A_2 are expressed in terms of perturbations Δa_{ijkl} as

$$A_1 = -\Delta a_{11} + 2\Delta a_{13} - \Delta a_{33} + 4\Delta a_{44}\ ,\ \ A_2 = \Delta a_{11} - \Delta a_{13} - 2\Delta a_{44}\ , \tag{C9}$$

and in terms of WAA parameters as

$$A_1 = 2\left(c_0^P\right)^2\left(\delta_x - \varepsilon_x - \varepsilon_z\right)\ ,\ A_2 = \left(c_0^P\right)^2\left(-\delta_x + 2\varepsilon_x\right)\ . \tag{C10}$$

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