Zero-order ray-theory Green tensor in a heterogeneous anisotropic medium

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Summary

The frequency–domain zero–order ray–theory Green tensor in a heterogeneous anisotropic elastic medium is derived from the zero–order ray–theory approximation using the representation theorem applied in ray–centred coordinates.

Keywords

Ray theory, Green tensor, travel time, amplitude, anisotropy, heterogeneous media, paraxial approximation, wave propagation.

1. Introduction

High–frequency asymptotic approximation of the Green tensor for a homogeneous anisotropic elastic medium has been derived by many authors, for details refer to Červený (2001, sec. 2.5.5). The zero–order ray–theory Green tensor in a heterogeneous anisotropic medium may be obtained by comparing a general zero–order ray–theory approximation for a heterogeneous anisotropic medium with the high–frequency asymptotic approximation of the Green tensor for a homogeneous anisotropic medium in a vicinity of a point source. However, this matching derivation is not very comfortable.

In this paper, we derive the zero–order ray–theory Green tensor right in a heterogeneous anisotropic medium. The derivation is performed in the frequency domain. We start with the zero–order ray–theory approximation of the solution of the elastodynamic equation, and use the representation theorem in ray–centred coordinates to derive the amplitude coefficients of the Green tensor and the phase shift of the Green tensor.

The Einstein summation over the pairs of identical Roman indices (both subscripts and superscripts) $i, j, k, ... = 1, 2, 3$ or $I, J, K, ... = 1, 2$ is used throughout this paper.

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2. Elastodynamic equation and the Green tensor

Seismic wavefield in an anisotropic elastic medium specified in terms of elastic moduli $c_{ijkl} = c_{ijkl}(\mathbf{x})$ and density $\varrho = \varrho(\mathbf{x})$ is subject to the elastodynamic equation (Červený, 2001, eq. 2.1.17)

$$[c_{ijkl}(\mathbf{x})\, u_{k,l}(\mathbf{x},t)]_{,j} - \varrho(\mathbf{x})\, \ddot{u}_i(\mathbf{x},t) + f_i(\mathbf{x},t) = 0 \qquad (1)$$

for displacement $u_i(\mathbf{x},t)$, where the dot $\dot{}$ stands for the derivative with respect to time t, and subscript $_{,j}$ following a comma stands for the partial derivative with respect to Cartesian spatial coordinate x^j. Force density $f_i(\mathbf{x},t)$ represents the source of the wavefield.

Elastodynamic Green tensor $G_{km}(\mathbf{x},\mathbf{x}',t)$, corresponding to elastodynamic equation (1), is defined by equation (Červený, 2001, eq. 2.5.37)

$$[c_{ijkl}(\mathbf{x})\, G_{km,l}(\mathbf{x},\mathbf{x}',t)]_{,j} - \varrho(\mathbf{x})\, \ddot{G}_{im}(\mathbf{x},\mathbf{x}',t) + \delta_{im}\, \delta(\mathbf{x}-\mathbf{x}')\, \delta(t) = 0 \qquad (2)$$

with the zero initial conditions for $t \le 0$. The spatial partial derivatives in elastodynamic equation (2) are related to coordinates x^i. Here $\delta(\mathbf{x})$ and $\delta(t)$ are the 3–D and 1–D Dirac distributions.

The elastic moduli obey symmetry relations

$$c_{ijkl}(\mathbf{x}) = c_{jikl}(\mathbf{x}) = c_{ijlk}(\mathbf{x}) \quad . \qquad (3)$$

We assume that the elastic moduli also obey symmetry relation

$$c_{ijkl}(\mathbf{x}) = c_{klij}(\mathbf{x}) \quad . \qquad (4)$$

In this paper, we shall mostly work in the frequency domain with 1–D Fourier transform

$$G_{im}(\mathbf{x},\mathbf{x}',\omega) = \widehat{\delta}(\omega) \int \mathrm{d}t\; G_{im}(\mathbf{x},\mathbf{x}',t)\; \exp(\mathrm{i}\omega t) \qquad (5)$$

of the elastodynamic Green tensor, and with the analogous Fourier transform of the displacement. Here $\widehat{\delta}(\omega)$ is a constant equal to the 1–D Fourier transform of the 1–D Dirac distribution $\delta(t)$.

We shall use equal symbols for a function of time and for its Fourier transform, and distinguish them by arguments t for time and ω for circular frequency. The only exception is constant $\widehat{\delta}(\omega)$, because $\delta(\omega)$ could be misleading.

Note that the phase shifts derived in this paper correspond to factor $\exp(\mathrm{i}\omega t)$ in (5), and would be opposite for factor $\exp(-\mathrm{i}\omega t)$.

Anisotropic elastodynamic equation (1) for the displacement in the frequency domain then reads (Červený, 2001, eq. 2.1.27)

$$\big[c_{ijkl}(\mathbf{x})\, u_{k,l}(\mathbf{x},\omega)\big]_{,j} + \omega^2 \varrho(\mathbf{x})\, u_i(\mathbf{x},\omega) + f_i(\mathbf{x},\omega) = 0 \quad , \qquad (6)$$

and the frequency–domain Green tensor for an elastic medium is the solution of equation (Červený, 2001, eq. 2.5.38)

$$\big[c_{ijkl}(\mathbf{x})\, G_{km,l}(\mathbf{x},\mathbf{x}',\omega)\big]_{,j} + \omega^2 \varrho(\mathbf{x})\, G_{im}(\mathbf{x},\mathbf{x}',\omega) + \delta_{im}\, \delta(\mathbf{x}-\mathbf{x}')\, \widehat{\delta}(\omega) = 0 \quad , \qquad (7)$$

analytical with respect to the inverse Fourier transform.

3. Representation theorem

Here we consider volume V which need not contain the support of force density $f_i(\mathbf{x},\omega)$. We multiply equation (7) by $u_i(\mathbf{x})$, subtract the product of equation (6) with Green tensor $G_{im}(\mathbf{x},\mathbf{x}',\omega)$, and integrate over V,

$$u_m(\mathbf{x}',\omega) = \frac{1}{\widehat{\delta}(\omega)} \int_V \mathrm{d}^3\mathbf{x} \Big\{ f_i(\mathbf{x},\omega)\, G_{im}(\mathbf{x},\mathbf{x}',\omega) - \big[c_{ijkl}(\mathbf{x},\omega)\, G_{km,l}(\mathbf{x},\mathbf{x}',\omega)\big]_{,j} u_i(\mathbf{x},\omega) + \big[c_{ijkl}(\mathbf{x})\, u_{k,l}(\mathbf{x},\omega)\big]_{,j} G_{im}(\mathbf{x},\mathbf{x}',\omega) \Big\} \ . \quad (8)$$

We apply symmetry relation (4) and obtain

$$u_m(\mathbf{x}',\omega) = \frac{1}{\widehat{\delta}(\omega)} \int_V \mathrm{d}^3\mathbf{x} \Big\{ G_{im}(\mathbf{x},\mathbf{x}',\omega)\, f_i(\mathbf{x},\omega) - \big[G_{im,j}(\mathbf{x},\mathbf{x}',\omega)\, c_{ijkl}(\mathbf{x})\big]_{,l} u_k(\mathbf{x},\omega) + G_{im}(\mathbf{x},\mathbf{x}',\omega) \big[c_{ijkl}(\mathbf{x})\, u_{k,l}(\mathbf{x},\omega)\big]_{,j} \Big\} \ , \quad (9)$$

which reads

$$u_m(\mathbf{x}',\omega) = \frac{1}{\widehat{\delta}(\omega)} \int_V \mathrm{d}^3\mathbf{x} \Big\{ G_{im}(\mathbf{x},\mathbf{x}',\omega)\, f_i(\mathbf{x},\omega) - \big[G_{im,j}(\mathbf{x},\mathbf{x}',\omega)\, c_{ijkl}(\mathbf{x})\, u_k(\mathbf{x},\omega)\big]_{,l} + \big[G_{im}(\mathbf{x},\mathbf{x}',\omega)\, c_{ijkl}(\mathbf{x})\, u_{k,l}(\mathbf{x},\omega)\big]_{,j} \Big\} \ . \quad (10)$$

We apply the divergence theorem to (10) and arrive at the representation theorem (Červený, 2001, eq. 2.6.4)

$$u_m(\mathbf{x}',\omega) = \frac{1}{\widehat{\delta}(\omega)} \int_V \mathrm{d}^3\mathbf{x}\; G_{im}(\mathbf{x},\mathbf{x}',\omega)\, f_i(\mathbf{x},\omega) + \frac{1}{\widehat{\delta}(\omega)} \oint_{\partial V} \mathrm{d}S(\mathbf{x})\; \big[G_{im}(\mathbf{x},\mathbf{x}',\omega)\, n_j(\mathbf{x})\, c_{ijkl}(\mathbf{x})\, u_{k,l}(\mathbf{x},\omega) - G_{im,j}(\mathbf{x},\mathbf{x}',\omega)\, c_{ijkl}(\mathbf{x})\, u_k(\mathbf{x},\omega)\, n_l(\mathbf{x})\big] \quad , \quad (11)$$

where $n_i(\mathbf{x})$ is the unit normal to the surface ∂V of volume V pointing outside V.

The integral over volume V represents the wavefield corresponding to the sources situated inside V. The integral over the surface ∂V of V represents the wavefield corresponding to the sources situated outside V, and is zero if all sources are situated inside V.

For $f_i(\mathbf{x},\omega) = \delta_{in}\,\delta(\mathbf{x}-\mathbf{x}'')\,\widehat{\delta}(\omega)$, the above representation theorem (11) yields $u_m(\mathbf{x}',\omega) = G_{mn}(\mathbf{x}',\mathbf{x}'',\omega)$. Integrating over the whole space, the surface integral in (11) vanishes and we obtain the reciprocity relation (Červený, 2001, eq. 2.6.5)

$$G_{mn}(\mathbf{x}',\mathbf{x}'',\omega) = G_{nm}(\mathbf{x}'',\mathbf{x}',\omega) \quad . \quad (12)$$

4. Zero–order ray–theory approximation

Zero–order ray–theory approximation of a solution of the elastodynamic equation (6) may be composed of individual arrivals. Hereinafter, we shall consider just one of these arrivals. The zero–order ray–theory approximation of one arrival in a smooth heterogeneous anisotropic medium without structural interfaces reads

$$u_i(\mathbf{x},\omega) \simeq \frac{g_i(\mathbf{x})}{\sqrt{\varrho(\mathbf{x})\, v(\mathbf{x})\, L(\mathbf{x})}}\, C \,\exp[\mathrm{i}\,\varphi(\mathbf{x})]\, \exp[\mathrm{i}\omega\,\tau(\mathbf{x})] \quad , \quad (13)$$

where C is a constant, $\varrho(\mathbf{x})$ is the density, $\tau(\mathbf{x})$ is the travel time, $v(\mathbf{x})$ is the corresponding phase velocity, $\varphi(\mathbf{x})$ is the phase shift due to caustics, and $L(\mathbf{x})$ is the geometrical spreading measured along wavefronts.

Hereinafter, notation $\simeq$ means that we have neglected the terms which can asymptotically be neglected for $\omega \to +\infty$.

Geometrical spreading $L(\mathbf{x})$ may be expressed in various forms. At each point $\mathbf{x}$ of the ray, we choose two linearly independent vectors $\mathbf{h}_1(\mathbf{x})$ and $\mathbf{h}_2(\mathbf{x})$ situated in the wavefront tangent plane. We may then parametrize points $\mathbf{x}'''$ of the wavefront tangent plane by parameters $q^1(\mathbf{x})$ and $q^2(\mathbf{x})$ called ray–centred coordinates:

$$\mathbf{x}''' = \mathbf{h}_1(\mathbf{x})\, q^1(\mathbf{x}) + \mathbf{h}_2(\mathbf{x})\, q^2(\mathbf{x}) \quad . \tag{14}$$

For more detailed description of ray–centred coordinates refer to Klimeš (2006).

For the geometrical spreading measured along wavefronts, we use expression (Červený, 2001, eq. 4.14.39.3)

$$L(\mathbf{x}) = \sqrt{|\mathbf{h}_1(\mathbf{x})\times\mathbf{h}_2(\mathbf{x})|\,|\det[\mathbf{Q}(\mathbf{x})]|} \quad , \tag{15}$$

where $|\mathbf{h}_1\times\mathbf{h}_2|$ is the norm of the cross product of the contravariant basis vectors $\mathbf{h}_1$ and $\mathbf{h}_2$ of the ray–centred coordinate system, and $\mathbf{Q}$ is the 2×2 matrix of geometrical spreading in ray–centred coordinates,

$$Q^{AB}(\mathbf{x}) = \frac{\partial q^A(\mathbf{x})}{\partial \gamma_B} \quad , \tag{16}$$

where γ_B are the ray parameters.

Note that constant C in (13) depends on the initial conditions and on the choice of ray parameters γ_B. Term $|\mathbf{h}_1\times\mathbf{h}_2|$ is required if the contravariant basis vectors $\mathbf{h}_1$ and $\mathbf{h}_2$ of the ray–centred coordinate system are not orthonormal. We may choose vectors $\mathbf{h}_1$ and $\mathbf{h}_2$ orthonormal and remove $|\mathbf{h}_1\times\mathbf{h}_2|$ from all equations without a loss of generality.

Green tensor $G_{im}(\mathbf{x},\mathbf{x}',\omega)$ is the solution of the elastodynamic equation corresponding to a point source at point $\mathbf{x}'$. The rays from a point source can be parametrized by two components $p^{(q)}_K = \partial\tau/\partial q^K$ of the slowness vector in ray–centred coordinates. The special case of matrix (16) of geometrical spreading for $\gamma_B = p^{(q)}_B(\mathbf{x}')$ reads

$$Q_2^{AB}(\mathbf{x},\mathbf{x}') = \frac{\partial q^A(\mathbf{x})}{\partial p^{(q)}_B(\mathbf{x}')} \quad . \tag{17}$$

The zero–order ray–theory approximation (13) specified to the Green tensor reads

$$G_{im}(\mathbf{x},\mathbf{x}',\omega) \simeq \frac{g_i(\mathbf{x})}{\sqrt{\varrho(\mathbf{x})\, v(\mathbf{x})}\, L(\mathbf{x},\mathbf{x}')}\, C_m(\mathbf{x},\mathbf{x}')\, \exp[\mathrm{i}\,\varphi(\mathbf{x},\mathbf{x}')]\, \exp[\mathrm{i}\omega\,\tau(\mathbf{x},\mathbf{x}')] \quad , \tag{18}$$

where

$$L(\mathbf{x},\mathbf{x}') = \sqrt{|\mathbf{h}_1(\mathbf{x})\times\mathbf{h}_2(\mathbf{x})|\,|\det[\mathbf{Q}_2(\mathbf{x},\mathbf{x}')]|\,|\mathbf{h}_1(\mathbf{x}')\times\mathbf{h}_2(\mathbf{x}')|} \tag{19}$$

is the relative geometrical spreading. We have included constant $\sqrt{|\mathbf{h}_1(\mathbf{x}')\times\mathbf{h}_2(\mathbf{x}')|}$ in definition (19) in order to make the relative geometrical spreading independent of the choice of ray–centred coordinates and equivalent to the definition of Červený (2001, eq. 4.14.45).

Quantities $\varrho(\mathbf{x})$, $v(\mathbf{x})$, $g_i(\mathbf{x})$, $\tau(\mathbf{x},\mathbf{x}')$ and $L(\mathbf{x},\mathbf{x}')$ correspond to the ray from point $\mathbf{x}'$ to point $\mathbf{x}$ and are already defined. We just need to determine constants $C_m(\mathbf{x},\mathbf{x}')$ and phase shift $\varphi(\mathbf{x},\mathbf{x}')$ corresponding to the ray from $\mathbf{x}'$ to $\mathbf{x}$.

5. Application of the reciprocity relation

We know that the two–point travel time is reciprocal:

$$\tau(\mathbf{x},\mathbf{x}') = \tau(\mathbf{x}',\mathbf{x}) \quad . \tag{20}$$

We insert the zero–order ray–theory approximation (18) of Green tensor into the reciprocity relation (13), and obtain equations

$$\frac{g_i(\mathbf{x})\, C_m(\mathbf{x},\mathbf{x}')}{\sqrt{\varrho(\mathbf{x})\, v(\mathbf{x})}\, L(\mathbf{x},\mathbf{x}')} = \frac{g_m(\mathbf{x}')\, C_i(\mathbf{x}',\mathbf{x})}{\sqrt{\varrho(\mathbf{x}')\, v(\mathbf{x}')}\, L(\mathbf{x}',\mathbf{x})} \quad , \tag{21}$$

and

$$\varphi(\mathbf{x},\mathbf{x}') = \varphi(\mathbf{x}',\mathbf{x}) \qquad (\mathrm{mod}\ 2\pi) \quad . \tag{22}$$

The decomposition of vector $C_m(\mathbf{x},\mathbf{x}')$ into its norm and a unit vector then must read

$$C_m(\mathbf{x},\mathbf{x}') = C(\mathbf{x},\mathbf{x}')\, g_m(\mathbf{x}') \quad . \tag{23}$$

We now need to determine constant $C(\mathbf{x},\mathbf{x}')$ and phase shift $\varphi(\mathbf{x},\mathbf{x}')$ corresponding to the ray from $\mathbf{x}'$ to $\mathbf{x}$. These quantities can be determined using the representation theorem.

6. Application of the representation theorem

We assume that point $\mathbf{x}$ is situated inside volume V and point $\mathbf{x}'$ outside volume V. Than the representation theorem (11) applied to the Green tensor reads

$$G_{mn}(\mathbf{x},\mathbf{x}',\omega) = \frac{1}{\widehat{\delta}(\omega)} \oint_{\partial V} \mathrm{d}S(\mathbf{x}''') \big[G_{im}(\mathbf{x}''',\mathbf{x},\omega)\, n_j(\mathbf{x}''')\, c_{ijkl}(\mathbf{x}''')\, G_{kn,l}(\mathbf{x}''',\mathbf{x}',\omega) - G_{im,j}(\mathbf{x}''',\mathbf{x},\omega)\, c_{ijkl}(\mathbf{x}''')\, G_{kn}(\mathbf{x}''',\mathbf{x}',\omega)\, n_l(\mathbf{x}''') \big] \ . \tag{24}$$

We apply the high–frequency approximations

$$G_{kn,l}(\mathbf{x}''',\mathbf{x}',\omega) \simeq \mathrm{i}\omega\, G_{kn}(\mathbf{x}''',\mathbf{x}',\omega)\, p_l(\mathbf{x}''',\mathbf{x}') \tag{25}$$

and

$$G_{kn,l}(\mathbf{x}''',\mathbf{x},\omega) \simeq \mathrm{i}\omega\, G_{kn}(\mathbf{x}''',\mathbf{x},\omega)\, p_l(\mathbf{x}''',\mathbf{x}) \tag{26}$$

of the spatial derivatives of the Green tensors to (24), and arrive at approximation

$$G_{mn}(\mathbf{x},\mathbf{x}',\omega) \simeq \frac{\mathrm{i}\omega}{\widehat{\delta}(\omega)} \oint_{\partial V} \mathrm{d}S(\mathbf{x}''')\, G_{im}(\mathbf{x}''',\mathbf{x},\omega)\, c_{ijkl}(\mathbf{x}''')\, G_{kn}(\mathbf{x}''',\mathbf{x}',\omega) \times \big[n_j(\mathbf{x}''')\, p_l(\mathbf{x}''',\mathbf{x}') - p_j(\mathbf{x}''',\mathbf{x})\, n_l(\mathbf{x}''') \big] \quad . \tag{27}$$

We separate points $\mathbf{x}$ and $\mathbf{x}'$ by a surface coinciding with the wavefront tangent plane in the vicinity of the ray in which the contributions to integral (27) are not negligible. We parametrize the wavefront tangent plane by ray–centred coordinates q^1 and q^2, see (14). Then

$$\mathrm{d}S(\mathbf{x}''') = |\mathbf{h}_1(\mathbf{x}'') \times \mathbf{h}_2(\mathbf{x}'')|\, \mathrm{d}q^1\, \mathrm{d}q^2 \tag{28}$$

and

$$n_i(\mathbf{x}''') = -v(\mathbf{x}'')\, p_i(\mathbf{x}'',\mathbf{x}') \quad . \tag{29}$$

We simultaneously apply approximations

$$p_i(\mathbf{x}''',\mathbf{x}') \simeq p_i(\mathbf{x}'',\mathbf{x}') \quad , \tag{30}$$

$$p_i(\mathbf{x}''',\mathbf{x}) \simeq p_i(\mathbf{x}'',\mathbf{x}) = -p_i(\mathbf{x}'',\mathbf{x}') \quad , \tag{31}$$

and

$$c_{ijkl}(\mathbf{x}''') \simeq c_{ijkl}(\mathbf{x}'') \tag{32}$$

to (27), which then reads

$$G_{mn}(\mathbf{x},\mathbf{x}',\omega) \simeq \frac{-2\mathrm{i}\omega}{\widehat{\delta}(\omega)} \iint \mathrm{d}q^1 \mathrm{d}q^2 \, G_{im}(\mathbf{x}''',\mathbf{x},\omega)\, c_{ijkl}(\mathbf{x}'')\, G_{kn}(\mathbf{x}''',\mathbf{x}',\omega) \\ \times p_j(\mathbf{x}'',\mathbf{x}')\, p_l(\mathbf{x}'',\mathbf{x}')\, v(\mathbf{x}'')\, |\mathbf{h}_1(\mathbf{x}'')\times\mathbf{h}_2(\mathbf{x}'')| \quad , \tag{33}$$

where $\mathbf{x}''' = \mathbf{x}'''(q^1,q^2)$. We insert the paraxial approximation

$$G_{im}(\mathbf{x}''',\mathbf{x}',\omega) \simeq \frac{g_i(\mathbf{x}'')\, g_m(\mathbf{x}')\, C(\mathbf{x}'',\mathbf{x}')}{\sqrt{\varrho(\mathbf{x}'')\, v(\mathbf{x}'')}\, L(\mathbf{x}'',\mathbf{x}')} \exp[\mathrm{i}\varphi(\mathbf{x}'',\mathbf{x}')] \exp[\mathrm{i}\omega\,\tau(\mathbf{x}''',\mathbf{x}')] \tag{34}$$

with

$$\tau(\mathbf{x}''',\mathbf{x}') \simeq \tau(\mathbf{x}'',\mathbf{x}') + \tfrac{1}{2} q^K M_{KL}(\mathbf{x}'',\mathbf{x}') q^L \tag{35}$$

of the Green tensor (18) with (23) into the integrand of (33). In paraxial expansion (35), we have denoted

$$M_{KL} = \frac{\partial \tau(\mathbf{x}'',\mathbf{x}')}{\partial q^A \partial q^B} \quad . \tag{36}$$

Considering identity

$$c_{ijkl}(\mathbf{x}'') g_i(\mathbf{x}'') p_j(\mathbf{x}'',\mathbf{x}') g_k(\mathbf{x}'') p_l(\mathbf{x}'',\mathbf{x}') = \varrho(\mathbf{x}'') \quad , \tag{37}$$

relation (33) reads

$$G_{mn}(\mathbf{x},\mathbf{x}',\omega) \simeq \frac{g_m(\mathbf{x})\, g_n(\mathbf{x}')\, C(\mathbf{x}'',\mathbf{x}')\, C(\mathbf{x}'',\mathbf{x})}{\widehat{\delta}(\omega)\, L(\mathbf{x}'',\mathbf{x})\, L(\mathbf{x}'',\mathbf{x}')} |\mathbf{h}_1(\mathbf{x}'')\times\mathbf{h}_2(\mathbf{x}'')|\, I(\mathbf{x},\mathbf{x}'',\mathbf{x}') \\ \times \exp\{\mathrm{i}\,[\varphi(\mathbf{x}'',\mathbf{x}) + \varphi(\mathbf{x}'',\mathbf{x}')]\} \exp\{\mathrm{i}\omega\,[\tau(\mathbf{x}'',\mathbf{x}) + \tau(\mathbf{x}'',\mathbf{x}')]\} \quad , \tag{38}$$

where

$$I(\mathbf{x},\mathbf{x}'',\mathbf{x}') = -2\mathrm{i}\omega \iint \mathrm{d}q^1 \mathrm{d}q^2 \exp\left\{\mathrm{i}\omega\,\tfrac{1}{2} q^K [M_{KL}(\mathbf{x}'',\mathbf{x}) + M_{KL}(\mathbf{x}'',\mathbf{x}')] q^L\right\} \quad . \tag{39}$$

We replace Green tensor $G_{mn}(\mathbf{x},\mathbf{x}',\omega)$ in (38) by (18) with (23) and obtain equation

$$\frac{C(\mathbf{x},\mathbf{x}')\; \exp[\mathrm{i}\varphi(\mathbf{x},\mathbf{x}')]}{\sqrt{\varrho(\mathbf{x})\, v(\mathbf{x})}\, L(\mathbf{x},\mathbf{x}')} = \frac{C(\mathbf{x}'',\mathbf{x}')\, C(\mathbf{x}'',\mathbf{x})}{\widehat{\delta}(\omega)\, L(\mathbf{x}'',\mathbf{x})\, L(\mathbf{x}'',\mathbf{x}')} |\mathbf{h}_1(\mathbf{x}'')\times\mathbf{h}_2(\mathbf{x}'')|\, I(\mathbf{x},\mathbf{x}'',\mathbf{x}') \\ \times \exp\{\mathrm{i}\,[\varphi(\mathbf{x}'',\mathbf{x}) + \varphi(\mathbf{x}'',\mathbf{x}')]\} \quad . \tag{40}$$

We calculate integral (39),

$$I(\mathbf{x},\mathbf{x}'',\mathbf{x}') = -2\mathrm{i}\, \frac{2\pi\; \exp\left\{\mathrm{i}\frac{\pi}{4}\mathrm{sgn}[M_{KL}(\mathbf{x}'',\mathbf{x}) + M_{KL}(\mathbf{x}'',\mathbf{x}')]\right\}}{\sqrt{|\det[M_{KL}(\mathbf{x}'',\mathbf{x}) + M_{KL}(\mathbf{x}'',\mathbf{x}')]|}} \quad , \tag{41}$$

and insert it into relation (40). The complex modulus of the resulting relation reads

$$\frac{C(\mathbf{x},\mathbf{x}')}{\sqrt{\varrho(\mathbf{x})\, v(\mathbf{x})}\, L(\mathbf{x},\mathbf{x}')} = \frac{4\pi}{\widehat{\delta}(\omega)} \frac{|\mathbf{h}_1(\mathbf{x}'')\times\mathbf{h}_2(\mathbf{x}'')|\, C(\mathbf{x}'',\mathbf{x}')\, C(\mathbf{x}'',\mathbf{x})}{L(\mathbf{x}'',\mathbf{x})\, L(\mathbf{x}'',\mathbf{x}') \sqrt{|\det[M_{KL}(\mathbf{x}'',\mathbf{x}) + M_{KL}(\mathbf{x}'',\mathbf{x}')]|}} \quad , \tag{42}$$

and the complex argument reads

$$\varphi(\mathbf{x},\mathbf{x}') = \varphi(\mathbf{x}'',\mathbf{x}) + \varphi(\mathbf{x}'',\mathbf{x}') + \tfrac{\pi}{4}\left\{\mathrm{sgn}[M_{KL}(\mathbf{x}'',\mathbf{x}) + M_{KL}(\mathbf{x}'',\mathbf{x}')] - 2\right\} \\ (\mathrm{mod}\ 2\pi) \quad . \tag{43}$$

7. Propagator matrix of geodesic deviation in ray–centred coordinates

We assume that the basis vectors $\mathbf{h}_K(\mathbf{x}'')$ of the ray–centred coordinate system along the ray from $\mathbf{x}$ to $\mathbf{x}'$ and along the ray from $\mathbf{x}'$ to $\mathbf{x}$ are equal.

We decompose the 4×4 propagator matrix of geodesic deviation in the ray–centred coordinates into four 2×2 submatrices:

$$\mathbf{\Pi}(\mathbf{x},\mathbf{x}') = \begin{pmatrix} \mathbf{Q}_1(\mathbf{x},\mathbf{x}') & \mathbf{Q}_2(\mathbf{x},\mathbf{x}') \\ \mathbf{P}_1(\mathbf{x},\mathbf{x}') & \mathbf{P}_2(\mathbf{x},\mathbf{x}') \end{pmatrix} \quad . \tag{44}$$

The definitions of individual 2×2 submatrices read

$$Q_1^{AB}(\mathbf{x},\mathbf{x}') = \frac{\partial q^A(\mathbf{x})}{\partial q^B(\mathbf{x}')} \quad , \qquad Q_2^{AB}(\mathbf{x},\mathbf{x}') = \frac{\partial q^A(\mathbf{x})}{\partial p_B^{(q)}(\mathbf{x}')} \quad ,$$

$$P_1^{AB}(\mathbf{x},\mathbf{x}') = \frac{\partial p_A^{(q)}(\mathbf{x})}{\partial q^B(\mathbf{x}')} \quad , \qquad P_2^{AB}(\mathbf{x},\mathbf{x}') = \frac{\partial p_A^{(q)}(\mathbf{x})}{\partial p_B^{(q)}(\mathbf{x}')} \quad . \tag{45}$$

Note that submatrix $\mathbf{Q}_2(\mathbf{x},\mathbf{x}')$ has already been defined by (17). We analogously decompose and define propagator matrices $\mathbf{\Pi}(\mathbf{x}'',\mathbf{x}')$ and $\mathbf{\Pi}(\mathbf{x},\mathbf{x}'')$.

Because the propagator matrices are symplectic, the inverse matrix to $\mathbf{\Pi}(\mathbf{x},\mathbf{x}'')$ reads

$$[\mathbf{\Pi}(\mathbf{x},\mathbf{x}'')]^{-1} = \begin{pmatrix} \mathbf{P}_2^{\mathrm{T}}(\mathbf{x},\mathbf{x}'') & -\mathbf{Q}_2^{\mathrm{T}}(\mathbf{x},\mathbf{x}'') \\ -\mathbf{P}_1^{\mathrm{T}}(\mathbf{x},\mathbf{x}'') & \mathbf{Q}_1^{\mathrm{T}}(\mathbf{x},\mathbf{x}'') \end{pmatrix} \quad . \tag{46}$$

Propagator matrix $\mathbf{\Pi}(\mathbf{x}'',\mathbf{x})$ differs from matrix (46) just by the direction of propagation along the same ray segment, and can be expressed in form

$$\mathbf{\Pi}(\mathbf{x}'',\mathbf{x}) = \begin{pmatrix} \mathbf{P}_2^{\mathrm{T}}(\mathbf{x},\mathbf{x}'') & \mathbf{Q}_2^{\mathrm{T}}(\mathbf{x},\mathbf{x}'') \\ \mathbf{P}_1^{\mathrm{T}}(\mathbf{x},\mathbf{x}'') & \mathbf{Q}_1^{\mathrm{T}}(\mathbf{x},\mathbf{x}'') \end{pmatrix} \quad . \tag{47}$$

The 2×2 matrices (36) of the second–order derivatives of travel time in the ray–centred coordinates read

$$\mathbf{M}(\mathbf{x}'',\mathbf{x}') = \mathbf{P}_2(\mathbf{x}'',\mathbf{x}')\,[\mathbf{Q}_2(\mathbf{x}'',\mathbf{x}')]^{-1} \tag{48}$$

and

$$\mathbf{M}(\mathbf{x}'',\mathbf{x}) = \mathbf{Q}_1^{\mathrm{T}}(\mathbf{x},\mathbf{x}'')\,[\mathbf{Q}_2^{\mathrm{T}}(\mathbf{x},\mathbf{x}'')]^{-1} = [\mathbf{Q}_2(\mathbf{x},\mathbf{x}'')]^{-1}\mathbf{Q}_1(\mathbf{x},\mathbf{x}'') \quad . \tag{49}$$

Then

$$\begin{aligned} &\mathbf{M}(\mathbf{x}'',\mathbf{x}) + \mathbf{M}(\mathbf{x}'',\mathbf{x}') \\ &\quad = [\mathbf{Q}_2(\mathbf{x},\mathbf{x}'')]^{-1}[\mathbf{Q}_1(\mathbf{x},\mathbf{x}'')\mathbf{Q}_2(\mathbf{x}'',\mathbf{x}') + \mathbf{Q}_2(\mathbf{x},\mathbf{x}'')\mathbf{P}_2(\mathbf{x}'',\mathbf{x}')]\,[\mathbf{Q}_2(\mathbf{x}'',\mathbf{x}')]^{-1} \quad . \end{aligned} \tag{50}$$

Chain rule

$$\mathbf{\Pi}(\mathbf{x},\mathbf{x}') = \mathbf{\Pi}(\mathbf{x},\mathbf{x}'')\,\mathbf{\Pi}(\mathbf{x}'',\mathbf{x}') \tag{51}$$

for propagator matrices (44) implies relation

$$\mathbf{Q}_2(\mathbf{x},\mathbf{x}') = \mathbf{Q}_1(\mathbf{x},\mathbf{x}'')\mathbf{Q}_2(\mathbf{x}'',\mathbf{x}') + \mathbf{Q}_2(\mathbf{x},\mathbf{x}'')\mathbf{P}_2(\mathbf{x}'',\mathbf{x}') \quad . \tag{52}$$

Relation (50) with (52) reads

$$\mathbf{M}(\mathbf{x}'',\mathbf{x}) + \mathbf{M}(\mathbf{x}'',\mathbf{x}') = [\mathbf{Q}_2(\mathbf{x},\mathbf{x}'')]^{-1}\mathbf{Q}_2(\mathbf{x},\mathbf{x}')\,[\mathbf{Q}_2(\mathbf{x}'',\mathbf{x}')]^{-1} \quad , \tag{53}$$

and we see that

$$\big|\det[\mathbf{M}(\mathbf{x}'',\mathbf{x}) + \mathbf{M}(\mathbf{x}'',\mathbf{x}')]\big| = \big|\det[\mathbf{Q}_2(\mathbf{x},\mathbf{x}'')]\big|^{-1}\big|\det[\mathbf{Q}_2(\mathbf{x},\mathbf{x}')]\big|\,\big|\det[\mathbf{Q}_2(\mathbf{x}'',\mathbf{x}')]\big|^{-1} \quad . \tag{54}$$

8. Completing the derivation

We now apply the results of previous Sections 5, 6 and 7 to determining constants $C_m(\mathbf{x},\mathbf{x}')$ and phase shift $\varphi(\mathbf{x},\mathbf{x}')$ in approximation (18).

8.1. Amplitude coefficient of the Green tensor

We insert relations (19) and (54) into equation (42) and arrive at relation

$$\frac{C(\mathbf{x},\mathbf{x}')}{\sqrt{\varrho(\mathbf{x})\,v(\mathbf{x})}} = \frac{4\pi}{\widehat{\delta}(\omega)}\,C(\mathbf{x}'',\mathbf{x}')\,C(\mathbf{x}'',\mathbf{x}) \quad . \tag{55}$$

We put $\mathbf{x}''=\mathbf{x}$ in (55) and obtain

$$C(\mathbf{x},\mathbf{x}) = \frac{\widehat{\delta}(\omega)}{4\pi}\,\frac{1}{\sqrt{\varrho(\mathbf{x})\,v(\mathbf{x})}} \quad . \tag{56}$$

Since $C(\mathbf{x}',\mathbf{x})=C(\mathbf{x},\mathbf{x})$ and $C(\mathbf{x},\mathbf{x}')=C(\mathbf{x}',\mathbf{x}')$, we analogously obtain

$$C(\mathbf{x},\mathbf{x}') = \frac{\widehat{\delta}(\omega)}{4\pi}\,\frac{1}{\sqrt{\varrho(\mathbf{x}')\,v(\mathbf{x}')}} \quad . \tag{57}$$

8.2. Phase shift due to caustics

It is obvious from relation (53) that, for fixed $\mathbf{x}$ and $\mathbf{x}'$, $\mathrm{sgn}[M_{KL}(\mathbf{x}'',\mathbf{x})+M_{KL}(\mathbf{x}'',\mathbf{x}')]$ is constant outside caustics indicated by matrix $\mathbf{Q}_2(\mathbf{x},\mathbf{x}'')=\mathbf{Q}_2^{\mathrm{T}}(\mathbf{x}'',\mathbf{x})$ or by matrix $\mathbf{Q}_2(\mathbf{x}'',\mathbf{x}')$. We know that phase shift $\varphi(\mathbf{x}'',\mathbf{x}')$ changes at caustic indicated by matrix $\mathbf{Q}_2(\mathbf{x}'',\mathbf{x}')$, where matrix $M_{KL}(\mathbf{x}'',\mathbf{x}')$ changes its signature through infinity, and that phase shift $\varphi(\mathbf{x}'',\mathbf{x})$ changes at caustic indicated by matrix $\mathbf{Q}_2^{\mathrm{T}}(\mathbf{x}'',\mathbf{x})$, where matrix $M_{KL}(\mathbf{x}'',\mathbf{x})$ changes its signature through infinity. If one eigenvalue of matrix $M_{KL}(\mathbf{x}'',\mathbf{x}')$ changes its sign from negative to positive through infinity, we define $\Delta\mathrm{sgn}[M_{KL}(\mathbf{x}'',\mathbf{x}')]=+2$, and analogously for other changes of its signature due to the caustics indicated by matrix $\mathbf{Q}_2(\mathbf{x}'',\mathbf{x}')$. Phase shift $\varphi(\mathbf{x}'',\mathbf{x}')$ changes at the caustics indicated by matrix $\mathbf{Q}_2(\mathbf{x}'',\mathbf{x}')$, where $\mathrm{sgn}[M_{KL}(\mathbf{x}'',\mathbf{x})+M_{KL}(\mathbf{x}'',\mathbf{x}')]$ changes by $\Delta\mathrm{sgn}[M_{KL}(\mathbf{x}'',\mathbf{x}')]$, and analogously for phase shift $\varphi(\mathbf{x}'',\mathbf{x})$. The increment of phase shift $\varphi(\mathbf{x}'',\mathbf{x}')$ at caustic $\mathbf{x}''$ is thus

$$\Delta\varphi(\mathbf{x}'',\mathbf{x}') = -\tfrac{\pi}{4}\,\Delta\mathrm{sgn}[M_{KL}(\mathbf{x}'',\mathbf{x}')] \quad . \tag{58}$$

This phase–shift rule is identical to the phase–shift rule of Lewis (1965, eq. F.19), and is equivalent to the phase–shift rules of Bakker (1998) and Klimeš (2010).

8.3. Initial phase shift

We assume that point $\mathbf{x}$ is not situated at a caustic corresponding to a point source at $\mathbf{x}'$. If point $\mathbf{x}''$ is approaching point $\mathbf{x}'$ against the direction of propagation, matrix $M_{KL}(\mathbf{x}'',\mathbf{x}')$ is increasing to infinity, and we may neglect finite matrix $M_{KL}(\mathbf{x}'',\mathbf{x})$. The limit of relation (43) for $\mathbf{x}''\to\mathbf{x}'$ thus yields

$$\varphi(\mathbf{x},\mathbf{x}') = \varphi(\mathbf{x}',\mathbf{x}) + \varphi(\mathbf{x}',\mathbf{x}') + \tfrac{\pi}{4}\big\{\lim_{\mathbf{x}''\to\mathbf{x}'}\mathrm{sgn}[M_{KL}(\mathbf{x}'',\mathbf{x}')]-2\big\} \qquad (\mathrm{mod}\ 2\pi) \ . \tag{59}$$

Because of the reciprocity (22) of the phase shift, equation (59) yields

$$\varphi(\mathbf{x}',\mathbf{x}') = \tfrac{\pi}{4}\big\{2-\lim_{\mathbf{x}''\to\mathbf{x}'}\mathrm{sgn}[M_{KL}(\mathbf{x}'',\mathbf{x}')]\big\} \qquad (\mathrm{mod}\ 2\pi) \quad . \tag{60}$$

Note that point $\mathbf{x}''$ is approaching initial point $\mathbf{x}'$ along the ray against the direction of propagation.

9. Conclusions

We insert relation (23) with (57) into approximation (18). The zero–order ray–theory approximation of the Green tensor in heterogeneous anisotropic elastic media then reads

$$G_{im}(\mathbf{x},\mathbf{x}',\omega) \simeq \frac{\widehat{\delta}(\omega)}{4\pi} \frac{g_i(\mathbf{x})\, g_m(\mathbf{x}')}{\sqrt{\varrho(\mathbf{x})\, v(\mathbf{x})\, \varrho(\mathbf{x}')\, v(\mathbf{x}')}\, L(\mathbf{x},\mathbf{x}')} \exp[\mathrm{i}\,\varphi(\mathbf{x},\mathbf{x}')] \exp[\mathrm{i}\omega\,\tau(\mathbf{x},\mathbf{x}')] \quad . \tag{61}$$

The initial phase shift $\varphi(\mathbf{x}',\mathbf{x}')$ is given by relation (60), and its increment due to caustics by relation (58). The sign of the phase shift as well as constant $\widehat{\delta}(\omega)$ correspond to Fourier transform (5).

The generalization of the Green tensor to velocity models with structural interfaces is straightforward (Červený, 2001, eq. 5.4.17).

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