Superpositions of Gaussian beams and column Gaussian packets in heterogeneous anisotropic media

Luděk Klimeš

Department of Geophysics, Faculty of Mathematics and Physics, Charles University in Prague, Ke Karlovu 3, 121 16 Praha 2, Czech Republic, http://sw3d.cz/staff/klimes.htm

Summary

Based on the integral superposition of Gaussian packets, we derive the equations for the integral superposition of Gaussian beams and for the integral superposition of column Gaussian packets in smoothly heterogeneous media. Whereas Gaussian beams extend along their central rays, column Gaussian packets extend along an arbitrary system of lines.

The equations are applicable to both the anisotropic ray theory and the coupling ray theory in anisotropic media, or to the isotropic ray theory in isotropic media. The superpositions corresponding to the coupling ray theory can be applied to various kinds of reference rays. The equations can be used in both Cartesian and curvilinear coordinates.

Keywords

Summation of Gaussian packets, summation of Gaussian beams, summation of column Gaussian packets, anisotropy, heterogeneous media, wave propagation, travel time, amplitude, Green tensor, S–wave coupling.

1. Introduction

The equations for the integral superposition of Gaussian packets in heterogeneous isotropic media were derived using the Maslov asymptotic theory applied to a general 3–D subspace of the 6–D complex phase space by Klimeš (1984), and demonstrated in the computation of seismic wave fields by Klimeš (1989). These equations were rederived for heterogeneous anisotropic media by Klimeš (2014).

In this paper, we start with the three–parametric superposition of Gaussian packets in a smoothly heterogeneous generally anisotropic medium by Klimeš (2014). We consider the endpoints of rays along a reference surface, and a system of reference lines intersecting the reference surface. We then asymptotically calculate the one–parametric superpositions along the reference lines and in this way convert the three–parametric superposition of Gaussian packets into a two–parametric superposition of column Gaussian packets, which infinitely extend along the reference lines. The two–parametric superposition of column Gaussian packets is performed along the reference surface, and represents a generalization of the two–parametric superposition of paraxial Gaussian beams.

We then demonstrate that the column Gaussian packets are not singular at caustics only if the reference lines are tangent to the reference rays at the reference surface. If we choose the reference lines along the reference rays, the two–parametric superposition of column Gaussian packets reduces to the two–parametric superposition of paraxial Gaussian beams along the reference surface.

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The presented integral superpositions of column Gaussian packets and paraxial Gaussian beams may correspond to the anisotropic ray theory, to the frequency–dependent coupling ray theory for S waves or to the prevailing–frequency approximation of the coupling ray theory (Klimeš & Bulant, 2012) in anisotropic media, or to the isotropic ray theory in isotropic media. The equations can be used in both Cartesian and curvilinear coordinates.

The lower–case indices $i, j, ...$ take the values of $1, 2, 3$. The upper–case indices $I, J, ...$ take the values of $1, 2$. The Einstein summation convention over repeated indices is used.

2. Integral superposition of Gaussian packets

We consider Cartesian or curvilinear coordinates x^i in 3–D space. We consider an orthonomic system $x^i = \tilde{x}^i(\gamma^a)$ of rays parametrized by ray coordinates γ^a, where γ^1 and γ^2 are the ray parameters, and γ^3 is the parameter along rays determined by the form of the Hamiltonian function. In the case of the coupling ray theory, the orthonomic system of rays is represented by the system of reference rays along which the coupling equations are solved. The reference rays are calculated using the reference Hamiltonian function, and correspond to the reference travel time and the reference slowness vector. The time–harmonic integral superposition of Gaussian packets, Gaussian beams or column Gaussian packets will be considered with respect to the reference rays.

Along the reference rays, we calculate the ray–theory travel time and the corresponding ray–theory amplitude using an arbitrary ray method. The ray–theory travel time may differ from the reference travel time, but will be assumed to represent just a perturbation of the reference travel time. In this way, we shall substitute the first–order and second–order spatial derivatives of the reference travel time for the corresponding spatial derivatives of the ray–theory travel time.

The integral superposition of paraxial Gaussian packets in heterogeneous isotropic media was derived using the Maslov asymptotic theory applied to a general 3–D subspace of the 6–D complex phase space by Klimeš (1984, eq. 51). In a heterogeneous anisotropic medium, the time–harmonic integral superposition of paraxial Gaussian packets reads (Klimeš, 2014, eq. 11)

$$\begin{aligned} u_{i[j]}(x^m,\omega) = \left(\frac{\omega}{2\pi}\right)^{\frac{3}{2}} \iiint \mathrm{d}\tilde{x}^1 \mathrm{d}\tilde{x}^2 \mathrm{d}\tilde{x}^3 \, A_{i[j]}(\tilde{x}^m,\omega) \sqrt{\det}\big\{\mathrm{i}\big[N_{ab}(\tilde{x}^n) - f_{ab}(\tilde{x}^n)\big]\big\} \\ \times \exp\big\{\mathrm{i}\omega\big[\tau(\tilde{x}^n) + (x^k - \tilde{x}^k)\, p_k(\tilde{x}^n) + \tfrac{1}{2}(x^k - \tilde{x}^k)\, f_{kl}(\tilde{x}^n)\, (x^l - \tilde{x}^l)\big]\big\} \ , \qquad (1) \end{aligned}$$

where function $\sqrt{\det}(M_{ab})$ is the product of the square roots of the eigenvalues of matrix M_{ab}. The individual square roots are taken with positive real parts. Here ω is the circular frequency, $\tau(\tilde{x}^n)$ is the travel time at point $\tilde{x}^i$, $p_k(\tilde{x}^n)$ is the slowness vector corresponding to the orthonomic system of rays at point $\tilde{x}^i$, $f_{kl}(\tilde{x}^n)$ is the complex–valued matrix with positive imaginary part describing the shape of the paraxial Gaussian packet centred at point $\tilde{x}^i$, $N_{kl}(\tilde{x}^m)$ is the matrix of the second–order partial derivatives of the reference travel time at point $\tilde{x}^i$, and $A_{i[j]}(\tilde{x}^m,\omega)$ is the complex–valued vectorial or tensorial ray–theory amplitude.

The amplitudes of Gaussian packets corresponding to a particular source are vectorial without optional subscript $_{[j]}$. The amplitudes of Gaussian packets designed to compose the Green tensor (Klimeš, 2012) are tensorial with optional subscript $_{[j]}$.

In anisotropic media, vectorial or tensorial amplitude $A_{i[j]}(\tilde{x}^m,\omega)$ may represent either the anisotropic–ray–theory vectorial or tensorial amplitude, or the frequency–dependent coupling–ray–theory vectorial or tensorial amplitude calculated using the scalar reference amplitude corresponding to the orthonomic system of reference rays, or the vectorial or tensorial amplitude of the prevailing–frequency approximation of the coupling ray theory (Klimeš & Bulant, 2012). In isotropic media, vectorial or tensorial amplitude $A_{i[j]}(\tilde{x}^m,\omega)$ represents the isotropic–ray–theory vectorial or tensorial amplitude.

Matrix $f_{kl}(\tilde{x}^n)$ may be chosen arbitrarily, but must be smoothly varying with coordinates $\tilde{x}^n$. Matrix $f_{kl}(\tilde{x}^n)$ need not satisfy the equations for Gaussian packets propagating along the rays, because Gaussian packets arriving to different points of the same ray may correspond to different initial conditions for the shape of Gaussian packets. Matrix $f_{kl}(\tilde{x}^n)$ may also depend on frequency ω.

3. General integral superposition of column Gaussian packets

We consider the points of intersection of rays determined by ray parameters γ^1 and γ^2 with the reference surface. In the integral superposition, we introduce curvilinear coordinates ξ^i,

$$\tilde{x}^i = \tilde{x}^i(\xi^M,\xi^3) \quad , \tag{2}$$

with $\xi^3 = 0$ along the reference surface which becomes the coordinate surface. We choose $\xi^I = \gamma^I$ along the reference surface. The reference lines for one–parametric analytical asymptotic integration in superposition (1) are represented by the ξ^3 coordinate lines.

We change the coordinates in integral superposition (1),

$$u_{i[j]}(x^m) = \left(\frac{\omega}{2\pi}\right)^{\frac{3}{2}} \iiint \mathrm{d}\xi^1 \mathrm{d}\xi^2 \mathrm{d}\xi^3\, A_{i[j]} \left|\det\!\left(\frac{\partial x^m}{\partial \xi^n}\right)\right| \sqrt{\det}[\mathrm{i}(N_{ab}-f_{ab})]\, \exp(\mathrm{i}\omega\theta) \quad , \tag{3}$$

where

$$\theta = \tau + (x^k - \tilde{x}^k)\, p_k + \tfrac{1}{2}(x^k - \tilde{x}^k)\, f_{kl}\, (x^l - \tilde{x}^l) \quad . \tag{4}$$

For fixed ξ^I, we consider the dependence of quantities in the integral superposition on ξ^3. The quadratic Taylor expansion of coordinates with respect to ξ^3 reads

$$\tilde{x}^i(\xi^3) \simeq \tilde{x}^i(0) + Z^i_3(0)\,\xi^3 + \tfrac{1}{2}\, Z^i_{33}(0)\,(\xi^3)^2 \quad , \tag{5}$$

where

$$Z^i_3 = \frac{\partial x^i}{\partial \xi^3} \tag{6}$$

and

$$Z^i_{33} = \frac{\partial^2 x^i}{\partial \xi^3 \partial \xi^3} \quad . \tag{7}$$

Note that contravariant vector Z^i_3 is tangent to the reference lines.

The quadratic Taylor expansion of the reference travel time with respect to ξ^3 reads

$$\tau(\xi^3) \simeq \tau(0) + p_i(0)\, Z^i_3(0)\,\xi^3 + \tfrac{1}{2}\, p_i(0) Z^i_{33}(0)\,(\xi^3)^2 + \tfrac{1}{2}\, Z^i_3(0)\, N_{ij}(0)\, Z^j_3(0)\,(\xi^3)^2 \quad . \tag{8}$$

Since we assume that travel time $\tau(\tilde{x}^n)$ in superposition (1) is a perturbation of the reference travel time, we apply expansion (8) approximately also to the travel time in superposition (1). The perturbation is contained in term $\tau(0)$.

The linear Taylor expansion of the reference slowness vector with respect to ξ^3 reads

$$p_i(\xi^3) \simeq p_i(0) + N_{ij}(0)\, Z_3^j(0)\, \xi^3 \quad . \tag{9}$$

The mixed quadratic Taylor expansion of

$$\theta(\xi^3) \simeq \tau(\xi^3) + [x^k - \tilde{x}^k(\xi^3)]\, p_k(\xi^3) + \tfrac{1}{2}[x^k - \tilde{x}^k(\xi^3)]\, f_{kl}(\xi^3)\, [x^l - \tilde{x}^l(\xi^3)]]\} \tag{10}$$

with respect to ξ^3 and $x^i - \tilde{x}^i(0)$ then reads

$$\begin{aligned}\theta(\xi^3) \simeq\ & \theta(0) + p_i(0) Z_3^i(0)\, \xi^3 + \tfrac{1}{2}\, p_i(0) Z_{33}^i(0)\, (\xi^3)^2 + \tfrac{1}{2}\, Z_3^i(0)\, N_{ij}(0)\, Z_3^j(0)\, (\xi^3)^2 \\ & - p_i(0)\, Z_3^i(0)\, \xi^3 - \tfrac{1}{2}\, p_i(0)\, Z_{33}^i(0)\, (\xi^3)^2 + [x^i - \tilde{x}^i(0)]\, N_{ij}(0)\, Z_3^j(0)\, \xi^3 \\ & - Z_3^i(0)\, N_{ij}(0)\, Z_3^j(0)\, (\xi^3)^2 - [x^i - \tilde{x}^i(0)]\, f_{ij}(0)\, Z_3^j(0)\, \xi^3 \\ & + \tfrac{1}{2}\, Z_3^i(0)\, f_{ij}(0)\, Z_3^j(0)\, (\xi^3)^2 \quad ,\end{aligned} \tag{11}$$

where

$$\theta(0) \simeq \tau(0) + [x^k - \tilde{x}^k(0)]\, p_k(0) + \tfrac{1}{2}[x^k - \tilde{x}^k(0)]\, f_{kl}(0)\, [x^l - \tilde{x}^l(0)]]\} \quad . \tag{12}$$

We perform the summations in (11) and obtain expansion

$$\theta(\xi^3) \simeq \theta(0) + \tfrac{1}{2}\, Z_3^i(0)\, [f_{ij}(0) - N_{ij}(0)]\, Z_3^j(0)\, (\xi^3)^2 - [x^i - \tilde{x}^i(0)]\, [f_{ij}(0) - N_{ij}(0)]\, Z_3^j(0)\, \xi^3 \quad , \tag{13}$$

which does not contain $Z_{33}^i(0)$ and is thus independent of the curvature of the reference lines $x^i = \tilde{x}^i(\xi^3)$. Expansion (13) may be expressed as

$$\begin{aligned}\theta(\xi^3) \simeq\ & \theta(0) + \frac{1}{2}\, Z_3^k(0)\, [f_{kl}(0) - N_{kl}(0)]\, Z_3^l(0) \left(\xi^3 - \frac{[x^i - \tilde{x}^i(0)]\, [f_{ij}(0) - N_{ij}(0)]\, Z_3^j(0)}{Z_3^r(0)\, [f_{rs}(0) - N_{rs}(0)]\, Z_3^s(0)} \right)^2 \\ & - \frac{1}{2}\, \frac{\{(x^i - \tilde{x}^i(0)]\, [f_{ij}(0) - N_{ij}(0)]\, Z_3^j(0)\}^2}{Z_3^r(0)\, [f_{rs}(0) - N_{rs}(0)]\, Z_3^s(0)} \quad .\end{aligned} \tag{14}$$

We now integrate superposition (3) with respect to ξ^3 and obtain two–parametric integral superposition

$$u_{i[j]}(x^m) = \frac{\omega}{2\pi} \iint \mathrm{d}\xi^1 \mathrm{d}\xi^2\, A_{i[j]} \left| \det\!\left(\frac{\partial x^m}{\partial \xi^n} \right) \right| \frac{\sqrt{\det}[\mathrm{i}(N_{ab} - f_{ab})]}{\sqrt{\mathrm{i} Z_3^r (N_{rs} - f_{rs}) Z_3^s}}\, \exp\!\left(\mathrm{i}\omega\tilde{\theta}\right) \quad , \tag{15}$$

where $\xi^1 = \gamma^1$ and $\xi^2 = \gamma^2$ along the reference surface of integration. Complex–valued travel time

$$\tilde{\theta} = \theta(0) - \frac{1}{2}\, \frac{\{(x^i - \tilde{x}^i(0)]\, [f_{ij}(0) - N_{ij}(0)]\, Z_3^j(0)\}^2}{Z_3^r(0)\, [f_{rs}(0) - N_{rs}(0)]\, Z_3^s(0)} \tag{16}$$

can be expressed as

$$\boxed{\tilde{\theta} = \tau + (x^k - \tilde{x}^k)\, p_k + \tfrac{1}{2}(x^k - \tilde{x}^k)\, \tilde{f}_{kl}\, (x^l - \tilde{x}^l) \qquad (17)}$$

with

$$\tilde{f}_{ij} = f_{ij} - \frac{(f_{ik} - N_{ik}) Z_3^k Z_3^l (f_{lj} - N_{lj})}{Z_3^r (f_{rs} - N_{rs}) Z_3^s} \quad , \tag{18}$$

where all quantities are taken at the reference surface. Note that

$$(\tilde{f}_{ik} - N_{ik})\, Z_3^k = 0 \quad . \tag{19}$$

We define the derivatives

$$Z^i_M = \frac{\partial x^i}{\partial \xi^M} \tag{20}$$

of coordinates x^i along the reference surface with respect to ray parameters $\xi^M = \gamma^M$. Contravariant vectors Z^i_1 and Z^i_2 are tangent to the reference surface.

We introduce 2×2 matrix

$$\tilde{F}_{MN} = Z^i_M f_{ij} Z^j_N - \frac{Z^i_M(f_{ik}-N_{ik})Z^k_3 Z^l_3(f_{lj}-N_{lj})Z^j_N}{Z^r_3(f_{rs}-N_{rs})Z^s_3} \quad . \tag{21}$$

We define transformation matrix

$$Z_{mi} = \frac{\partial \xi^m}{\partial x^i} \tag{22}$$

which represents the inverse matrix to matrix Z^i_m given by definitions (5) and (20),

$$Z_{mi}\, Z^i_n = \delta_{mn} \quad . \tag{23}$$

Covariant vectors Z_{1i} and Z_{2i} are perpendicular to the reference ξ^3 lines. Covariant vector Z_{3i} is perpendicular to the reference surface.

We may consider decomposition $\tilde{f}_{ij} = (\tilde{f}_{ij} - N_{ij}) + N_{ij}$, apply identity (19) and definition (21) to term $(\tilde{f}_{ij} - N_{ij})$, and express matrix (18) as

$$\tilde{f}_{ij} = Z_{Mi}(\tilde{F}_{MN} - Z^k_M N_{kl} Z^l_N) Z_{Nj} + N_{ij} \quad . \tag{24}$$

Considering identity

$$Z_{mi}\, Z^j_m = \delta^j_i \tag{25}$$

following from definition (23), we express matrix (24) as

$$\tilde{f}_{ij} = Z_{Mi}\tilde{F}_{MN} Z_{Nj} - (\delta^k_i - Z_{3i}Z^k_3) N_{kl} (\delta^l_j - Z^l_3 Z_{3j}) + N_{ij} \quad , \tag{26}$$

which reads

$$\boxed{\tilde{f}_{ij} = Z_{Mi}\tilde{F}_{MN} Z_{Nj} + Z_{3i} Z^k_3 N_{kj} + N_{il} Z^l_3 Z_{3j} - Z_{3i} Z^k_3 N_{kl} Z^l_3 Z_{3j} \quad , \qquad (27)}$$

where all quantities are taken at the reference surface.

We covariantly transform matrices N_{ij} and f_{ij} from general coordinates x^i to reference coordinates ξ^m using the transformation matrix given by definitions (5) and (20),

$$M_{mn} = Z^i_m N_{ij} Z^j_n \quad , \tag{28}$$

$$F_{mn} = Z^i_m f_{ij} Z^j_n \quad . \tag{29}$$

Two–parametric integral superposition (15) then reads

$$u_{i[j]}(x^m) = \frac{\omega}{2\pi} \iint \mathrm{d}\xi^1 \mathrm{d}\xi^2\, A_{i[j]}\, \frac{\sqrt{\det}[\mathrm{i}(M_{ab} - F_{ab})]}{\sqrt{\mathrm{i}(M_{33} - F_{33})}} \exp\!\left(\mathrm{i}\omega\tilde{\theta}\right) \quad , \tag{30}$$

and matrix (21) reads

$$\tilde{F}_{MN} = F_{MN} - \frac{(F_{M3}-M_{M3})(F_{3N}-M_{3N})}{F_{33}-M_{33}} \quad . \tag{31}$$

We calculate the determinant of 2×2 matrix $\tilde{F}_{MN} - M_{MN}$ using expression (31) and obtain

$$\det(\tilde{F}_{MN} - M_{MN}) = \frac{\det(F_{ab} - M_{ab})}{F_{33} - M_{33}} \quad . \tag{32}$$

We insert relation (32) into integral superposition (30) and obtain two–parametric integral superposition

$$\boxed{u_{i[j]}(x^m) = \frac{\omega}{2\pi}\iint \mathrm{d}\xi^1 \mathrm{d}\xi^2\, A_{i[j]}\, \sqrt{\det}[\mathrm{i}(M_{AB} - \tilde{F}_{AB})]\, \exp\!\left(\mathrm{i}\omega\tilde{\theta}\right) \quad . \qquad (33)}$$

4. Integral superpositions of Gaussian beams

Since

$$\tilde{f}_{ik}\, Z_3^k = N_{ik}\, Z_3^k \quad , \tag{34}$$

matrix $\tilde{f}_{ik}$ is always finite only if $N_{ik}\, Z_3^k$ is always finite. To keep $N_{ik}\, Z_3^k$ finite at caustics, we must choose contravariant vector Z_3^k tangent to the reference ray. Curvilinear coordinate ξ^3 then represents a local parameter along the ray. Covariant vectors Z_{1i} and Z_{2i} are perpendicular to the reference rays, and covariant vector Z_{3i} is perpendicular to the reference surface.

Hamilton's equations of rays (ray tracing equations) yield

$$H^{,k} = \tfrac{\mathrm{d}x^k}{\mathrm{d}\gamma^3} \quad . \tag{35}$$

For the reference lines coinciding with rays, we may thus express

$$Z_3^k = H^{,k} \tfrac{\mathrm{d}\gamma^3}{\mathrm{d}\xi^3} \quad . \tag{36}$$

Considering identity

$$N_{ik} H^{,k} = -H_{,i} \quad , \tag{37}$$

we express matrix (27) for the superposition of Gaussian beams as

$$\tilde{f}_{ij} = Z_{Mi}\tilde{F}_{MN}Z_{Nj} - Z_{3i}H_{,j}\tfrac{\mathrm{d}\gamma^3}{\mathrm{d}\xi^3} - H_{,i}Z_{3j}\tfrac{\mathrm{d}\gamma^3}{\mathrm{d}\xi^3} + Z_{3i}Z_3^k H_{,k}Z_{3j}\tfrac{\mathrm{d}\gamma^3}{\mathrm{d}\xi^3} \quad . \tag{38}$$

For a special case of the reference surface coinciding with a wavefront, relation (38) is analogous to the transformation of the second–order derivatives of travel time from ray–centred coordinates to general coordinates (Červený & Klimeš, 2010, eq. 36).

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