Determination of the reference symmetry axis of a generally anisotropic medium which is approximately transversely isotropic

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Summary

For a given stiffness tensor (tensor of elastic moduli) of a generally anisotropic medium, we estimate to which extent is the medium transversely isotropic and determine the direction of its reference symmetry axis in terms of the reference symmetry vector. If the medium is exactly transversely isotropic, we obtain its symmetry axis. We can also calculate the first–order and second–order spatial derivatives of the reference symmetry vector. The proposed method is illustrated in various transversely isotropic and approximately transversely isotropic velocity models.

Keywords

Elastic anisotropy, stiffness tensor, elastic moduli, transverse isotropy, approximate transverse isotropy, reference symmetry axis.

1. Introduction

The coupling ray theory (Chapman & Shearer, 1989; Bulant & Klimeš, 2002; Klimeš & Bulant, 2012) is usually applied to common anisotropic rays (Bakker, 2002; Klimeš & Bulant, 2004; 2006; Klimeš, 2006; Bulant & Klimeš, 2008). On the other hand, the coupling ray theory is more accurate if it is applied to reference rays which are closer to the actual S–wave paths (Klimeš & Bulant, 2014a).

If we a priori know that a given medium is transversely isotropic, we can separate the slowness surface into the P–wave slowness sheet, the SH–wave slowness sheet and the SV–wave slowness sheet. We may then trace the SH rays and SV rays (Klimeš & Bulant, 2014a), and use them as the reference rays for the prevailing–frequency approximation of the coupling ray theory. In this case, the SH rays and SV rays are better reference rays than the common anisotropic reference rays.

Even if a given medium is not transversely isotropic but is approximately transversely isotropic, the SH and SV reference rays (Klimeš & Bulant, 2015) may represent better reference rays than the common anisotropic reference rays. Note that, in this case, the anisotropic–ray–theory rays often cannot be used as the reference rays (Bulant & Klimeš, 2014; Klimeš & Bulant, 2014b).

For a given stiffness tensor (tensor of elastic moduli) of a generally anisotropic medium, it is thus very useful to be able to estimate to which extent is the medium transversely isotropic and to determine the direction of its reference symmetry axis.

The stiffness tensor of a transversely isotropic medium is independent of the rotation around the symmetry axis.

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For a given stiffness tensor of a generally anisotropic medium and a given rotation axis, we calculate the derivative of the stiffness tensor with respect to the angle of rotation in Section 2.1. The norm of the derivative of the stiffness tensor with respect to the angle of rotation divided by the norm of the stiffness tensor characterizes the extent of the dependence of the stiffness tensor on the rotation.

In Section 2.2, we determine the rotation axis corresponding to the smallest norm of the angular derivative of the stiffness tensor and refer to it as the reference symmetry axis. The direction of the reference symmetry axis is specified in terms of the reference symmetry vector.

In Sections 2.3 and 2.4, we also calculate the first–order and second–order spatial derivatives of the reference symmetry vector, which may be useful for tracing the SH and SV reference rays (Klimeš & Bulant, 2015) in heterogeneous velocity models with spatially varying reference symmetry vector, and for solving the corresponding equations of geodesic deviation.

The proposed method is illustrated in various transversely isotropic and approximately transversely isotropic velocity models in Section 3.

The lower–case Roman indices take values 1, 2 and 3. The upper–case Roman indices take values 1 and 2. Indices in parentheses are used to index the eigenvalues and corresponding eigenvectors. The Einstein summation over repetitive lower–case Roman indices (without parentheses) corresponding to the 3 spatial coordinates, is used throughout the paper.

2. Reference symmetry axis

2.1. Derivative of the stiffness tensor with respect to the angle of rotation

Transformation matrix $R_{in}(\varphi)$ corresponding to the rotation of vectors about given unit vector t_i by angle φ is an orthogonal matrix, with $R_{in}(0) = \delta_{in}$, where Kronecker delta δ_{in} represents the elements of the identity matrix. The derivative $\omega_{in} = R'_{in} = \mathrm{d}R_{in}/\mathrm{d}\varphi$ of the transformation matrix at $\varphi = 0$ reads

$$\omega_{in} = -t_m \varepsilon_{min} \quad , \tag{1}$$

where ε_{ijk} is the Levi–Civita symbol.

For unit rotation vector t_i, the derivative of stiffness tensor a_{ijkl} with respect to the angle φ of rotation is

$$a'_{ijkl} = \omega_{in} a_{njkl} + \omega_{jn} a_{inkl} + \omega_{kn} a_{ijnl} + \omega_{ln} a_{ijnr} \quad . \tag{2}$$

We insert matrix (1) into angular derivative (2) and obtain

$$a'_{ijkl} = -t_m \left(\varepsilon_{min} a_{njkl} + \varepsilon_{mjn} a_{inkl} + \varepsilon_{mkn} a_{ijnl} + \varepsilon_{mln} a_{ijkn} \right) \quad . \tag{3}$$

We define tensor

$$d_{ijklm} = \varepsilon_{min} a_{njkl} + \varepsilon_{mjn} a_{inkl} + \varepsilon_{mkn} a_{ijnl} + \varepsilon_{mln} a_{ijkn} \tag{4}$$

and express angular derivative (3) as

$$a'_{ijkl} = -d_{ijklm} \, t_m \quad . \tag{5}$$

2.2. Reference symmetry vector

We choose the square

$$y = a'_{ijkl}\, a'_{ijkl} \tag{6}$$

of the norm of the derivative of the stiffness tensor with respect to the angle of rotation as the objective function. We insert relation (5) into objective function (6) and obtain

$$y = t_m B_{mn} t_n \quad , \tag{7}$$

where

$$B_{mn} = d_{ijklm}\, d_{ijkln} \quad . \tag{8}$$

For unit rotation vector t_i, objective function (7) has its minimum $y = B_{(3)}$ for rotation vector t_i given by the eigenvector $t_{i(3)}$ of matrix B_{mn} corresponding to the smallest eigenvalue $B_{(3)}$. We shall refer to this eigenvector $t_{i(3)}$ as the *reference symmetry vector* and to the corresponding direction as the *reference symmetry axis.*

The ratio

$$\rho = \sqrt{\frac{a'_{ijkl}\, a'_{ijkl}}{a_{ijkl}\, a_{ijkl}}} \tag{9}$$

of the norm $\sqrt{a'_{ijkl} a'_{ijkl}}$ of the derivative of the stiffness tensor with respect to the angle of rotation and the norm $\sqrt{a_{ijkl} a_{ijkl}}$ of the stiffness tensor characterizes the extent of the dependence of the stiffness tensor on the rotation. For the reference symmetry vector $t_i = t_{i(3)}$, ratio (9) reads

$$\rho_{(3)} = \sqrt{\frac{B_{(3)}}{a_{ijkl}\, a_{ijkl}}} \quad . \tag{10}$$

This ratio characterizes the extent to which the medium is not transversely isotropic. We shall thus refer to it as the *non–TI ratio.*

Note that the reference symmetry vector $t_{i(3)}$ is stable and has a good physical meaning only if the minimum eigenvalue $B_{(3)}$ of matrix B_{mn} is considerably smaller than other two eigenvalues $B_{(1)}$ and $B_{(2)}$, i.e., if $\rho_{(3)}$ is considerably smaller than ratios

$$\rho_{(A)} = \sqrt{\frac{B_{(A)}}{a_{ijkl}\, a_{ijkl}}} \quad . \tag{11}$$

If non–TI ratio (10) is zero within rounding errors, the medium is exactly transversely isotropic and reference symmetry vector $t_{i(3)}$ specifies its symmetry axis.

2.3. First–order spatial derivatives of the symmetry vector

Since the symmetry vector is a unit eigenvector of matrix (8), we can calculate its first–order and second–order partial derivatives with respect to spatial coordinates analogously as the derivatives of the eigenvectors of the Christoffel matrix.

In addition to reference symmetry vector $t_{i(3)}$ corresponding to the minimum eigenvalue $B_{(3)}$ of matrix (8), we introduce also other two unit eigenvectors $t_{i(A)}$ corresponding to eigenvalues $B_{(A)}$.

The first–order partial derivatives of tensor (4) with respect to spatial coordinates read

$$d_{ijklm,p} = \varepsilon_{min} a_{njkl,p} + \varepsilon_{mjn} a_{inkl,p} + \varepsilon_{mkn} a_{ijnl,p} + \varepsilon_{mln} a_{ijkn,p} \quad . \tag{12}$$

The first–order partial derivatives of matrix (8) with respect to spatial coordinates then read

$$B_{mn,p} = d_{ijklm,p}\, d_{ijkln} + d_{ijklm}\, d_{ijkln,p} \quad . \tag{13}$$

We transform the first–order partial derivatives of matrix (8) into eigenvectors $t_{i(a)}$,

$$B_{(ab),p} = t_{m(a)} B_{mn,p} t_{n(b)} \quad . \tag{14}$$

The first–order partial derivatives of eigenvector $t_i = t_{i(3)}$ with respect to spatial coordinates then read (Klimeš, 2006, eq. 17)

$$t_{i,p} = \sum_A t_{i(A)} \frac{B_{(A3),p}}{B_{(3)} - B_{(A)}} \quad . \tag{15}$$

2.4. Second–order spatial derivatives of the symmetry vector

The second–order partial derivatives of tensor (4) with respect to spatial coordinates read

$$d_{ijklm,pq} = \varepsilon_{min} a_{njkl,pq} + \varepsilon_{mjn} a_{inkl,pq} + \varepsilon_{mkn} a_{ijnl,pq} + \varepsilon_{mln} a_{ijkn,pq} \quad . \tag{16}$$

The second–order partial derivatives of matrix (8) with respect to spatial coordinates then read

$$B_{mn,pq} = d_{ijklm,pq}\, d_{ijkln} + d_{ijklm}\, d_{ijkln,pq} + d_{ijklm,p}\, d_{ijkln,q} + d_{ijklm,q}\, d_{ijkln,p} \quad . \tag{17}$$

We transform the second–order partial derivatives of matrix (8) into eigenvectors $t_{i(A)}$,

$$B_{(ab),pq} = t_{m(a)}\, B_{mn,pq}\, t_{n(b)} \quad . \tag{18}$$

The second–order partial derivatives of eigenvector $t_i = t_{i(3)}$ with respect to spatial coordinates then read (Klimeš & Bulant, 2015, eqs. 39, 40)

$$\begin{aligned} t_{i,pq} = \sum_A t_{i(A)} \Bigg(& \frac{B_{(A3),pq}}{B_{(3)} - B_{(A)}} + \sum_B \frac{B_{(AB),p} B_{(B3),q} + B_{(AB),q} B_{(B3),p}}{(B_{(3)} - B_{(A)})\,(B_{(3)} - B_{(B)})} \\ & - \frac{B_{(A3),p} B_{(33),q} + B_{(A3),q} B_{(33),p}}{(B_{(3)} - B_{(A)})^2} \Bigg) \\ - t_{i(3)} \sum_B & \frac{B_{(B3),p} B_{(B3),q}}{(B_{(3)} - B_{(B)})^2} \quad . \end{aligned} \tag{19}$$

3. Numerical examples

Unit reference symmetry vector $t_{i(3)}$ and non–TI ratio (10) for a given stiffness tensor a_{ijkl} is determined according to Sections 2.1 and 2.2 of this paper by new program `tiaxis.for` of software package FORMS (Bucha & Bulant, 2015).

3.1. Velocity model WA

The density reduced stiffness tensor in vertically heterogeneous 1–D anisotropic velocity model WA by Pšenčík & Dellinger (2001) at the surface (zero depth) reads

$$\begin{array}{c} \\ {\scriptstyle 11} \\ {\scriptstyle 22} \\ {\scriptstyle 33} \\ {\scriptstyle 23} \\ {\scriptstyle 13} \\ {\scriptstyle 12} \end{array}
\begin{array}{c} \begin{array}{cccccc} {\scriptstyle 11} & {\scriptstyle 22} & {\scriptstyle 33} & {\scriptstyle 23} & {\scriptstyle 13} & {\scriptstyle 12} \end{array} \\
\begin{pmatrix} 13.39 & 4.46 & 4.46 & 0.00 & 0.00 & 0.00 \\ & 15.71 & 5.04 & 0.00 & 0.00 & 0.00 \\ & & 15.71 & 0.00 & 0.00 & 0.00 \\ & & & 5.33 & 0.00 & 0.00 \\ & & & & 4.98 & 0.00 \\ & & & & & 4.98 \end{pmatrix} \end{array} \quad . \tag{20}$$

Non–TI ratio (10) determined for this medium is

$$\rho_{(3)} = 0.000847 \quad , \tag{21}$$

and the corresponding unit reference symmetry vector is

$$t_{i(3)} = (1.000000 \;\; 0.000000 \;\; 0.000000) \quad . \tag{22}$$

We see that the medium is not exactly transversely isotropic but is approximately transversely isotropic.

We now slightly change density reduced stiffness tensor (20) to density reduced stiffness tensor

$$\begin{array}{c} \\ {\scriptstyle 11} \\ {\scriptstyle 22} \\ {\scriptstyle 33} \\ {\scriptstyle 23} \\ {\scriptstyle 13} \\ {\scriptstyle 12} \end{array}
\begin{array}{c} \begin{array}{cccccc} {\scriptstyle 11} & {\scriptstyle 22} & {\scriptstyle 33} & {\scriptstyle 23} & {\scriptstyle 13} & {\scriptstyle 12} \end{array} \\
\begin{pmatrix} 13.39 & 4.46 & 4.46 & 0.00 & 0.00 & 0.00 \\ & 15.70 & 5.04 & 0.00 & 0.00 & 0.00 \\ & & 15.70 & 0.00 & 0.00 & 0.00 \\ & & & 5.33 & 0.00 & 0.00 \\ & & & & 4.98 & 0.00 \\ & & & & & 4.98 \end{pmatrix} \end{array} \tag{23}$$

of a transversely isotropic medium. Non–TI ratio (10) determined for this medium is

$$\rho_{(3)} = 0.000000 \quad , \tag{24}$$

and the corresponding unit reference symmetry vector is

$$t_{i(3)} = (1.000000 \;\; 0.000000 \;\; 0.000000) \quad . \tag{25}$$

3.2. Velocity model QI

Velocity model WA was rotated by 45° about the positive x_3 half–axis in order to create vertically heterogeneous 1–D anisotropic velocity model QI. The density reduced stiffness tensor in km^{2}s^{-2} in velocity model QI at the surface (zero depth) reads (Bulant & Klimeš, 2002, eq. 38; Klimeš & Bulant, 2004, eq. 57; Pšenčík, Farra & Tessmer, 2012, eq. 16)

$$\begin{array}{c} \\ 11 \\ 22 \\ 33 \\ 23 \\ 13 \\ 12 \end{array} \begin{array}{c} \begin{array}{cccccc} 11 & 22 & 33 & 23 & 13 & 12 \end{array} \\ \begin{pmatrix} 14.485 & 4.525 & 4.750 & 0.000 & 0.000 & -0.580 \\ & 14.485 & 4.750 & 0.000 & 0.000 & -0.580 \\ & & 15.710 & 0.000 & 0.000 & -0.290 \\ & & & 5.155 & -0.175 & 0.000 \\ & & & & 5.155 & 0.000 \\ & & & & & 5.045 \end{pmatrix} \end{array} . \tag{26}$$

Non–TI ratio (10) determined for this medium is

$$\rho_{(3)} = 0.000847 \quad , \tag{27}$$

and the corresponding unit reference symmetry vector is

$$t_{i(3)} = (0.707107 \ \ 0.707107 \ \ 0.000000) \quad . \tag{28}$$

We see that the medium is not exactly transversely isotropic but is approximately transversely isotropic, analogously to velocity model WA.

3.3. Velocity model KISS

Velocity model WA was rotated by 1° about the positive x_3 half–axis in order to create vertically heterogeneous 1–D anisotropic velocity model KISS. The density reduced stiffness tensor in km^{2}s^{-2} in velocity model KISS at the surface (zero depth) reads (Pšenčík, Farra & Tessmer, 2012, eq. 20)

$$\begin{array}{c} \\ 11 \\ 22 \\ 33 \\ 23 \\ 13 \\ 12 \end{array} \begin{array}{c} \begin{array}{cccccc} 11 & 22 & 33 & 23 & 13 & 12 \end{array} \\ \begin{pmatrix} 13.39063 & 4.46008 & 4.46018 & 0.00000 & 0.00000 & -.01797 \\ & 15.70921 & 5.03982 & 0.00000 & 0.00000 & -.02251 \\ & & 15.71000 & 0.00000 & 0.00000 & -.01012 \\ & & & 5.32989 & -.00611 & 0.00000 \\ & & & & 4.98011 & 0.00000 \\ & & & & & 4.98008 \end{pmatrix} \end{array} . \tag{29}$$

Non–TI ratio (10) determined for this medium is

$$\rho_{(3)} = 0.000848 \quad , \tag{30}$$

and the corresponding unit reference symmetry vector is

$$t_{i(3)} = (0.999848 \ \ 0.017452 \ \ 0.000000) \quad . \tag{31}$$

We see that the medium is not exactly transversely isotropic but is approximately transversely isotropic, analogously to velocity model WA.

3.4. Velocity model SC1_II

The density reduced stiffness tensor in homogeneous anisotropic velocity model 1 by Shearer & Chapman (1989) reads

$$
\begin{array}{c|c}
 & \begin{array}{cccccc} 11 & 22 & 33 & 23 & 13 & 12 \end{array} \\
\begin{array}{c} 11 \\ 22 \\ 33 \\ 23 \\ 13 \\ 12 \end{array} &
\begin{pmatrix}
20.04 & 7.41 & 7.41 & 0.00 & 0.00 & 0.00 \\
 & 20.22 & 7.46 & 0.00 & 0.00 & 0.00 \\
 & & 20.22 & 0.00 & 0.00 & 0.00 \\
 & & & 6.38 & 0.00 & 0.00 \\
 & & & & 5.10 & 0.00 \\
 & & & & & 5.10
\end{pmatrix}
\end{array} \quad . \tag{32}
$$

Non–TI ratio (10) determined for this medium is

$$\rho_{(3)} = 0.000000 \quad , \tag{33}$$

and the corresponding unit reference symmetry vector is

$$t_{i(3)} = (1.000000 \;\; 0.000000 \;\; 0.000000) \quad . \tag{34}$$

We see that the medium is transversely isotropic within the rounding errors. If we inspect manually stiffness tensor (32), we see that the medium is exactly transversely isotropic.

Velocity model 1 by Shearer & Chapman (1989) was first rotated by 45° about the positive x_2 half–axis and then rotated by 30° about the positive x_3 half–axis in order to create the stiffness tensor of vertically heterogeneous 1–D anisotropic velocity model SC1_II at the surface (zero depth). After these rotations, the symmetry vector should read

$$t_{i(3)} = \left(\sqrt{3/8} \;\; \sqrt{1/8} \;\; \sqrt{1/2} \right) \quad . \tag{35}$$

The density reduced stiffness tensor in $\mathrm{km}^2\mathrm{s}^{-2}$ in velocity model SC1_II at the surface (zero depth) reads (Pšenčík, Farra & Tessmer, 2012, eq. 19)

$$
\begin{array}{c|c}
 & \begin{array}{cccccc} 11 & 22 & 33 & 23 & 13 & 12 \end{array} \\
\begin{array}{c} 11 \\ 22 \\ 33 \\ 23 \\ 13 \\ 12 \end{array} &
\begin{pmatrix}
18.97125 & 7.67125 & 8.36125 & 0.46000 & -0.31177 & -0.15589 \\
 & 19.64625 & 7.74375 & -0.49500 & 0.25115 & -0.42868 \\
 & & 18.87000 & -0.02250 & -0.03897 & 0.53477 \\
 & & & 5.89500 & 0.26847 & -0.28146 \\
 & & & & 6.20500 & 0.15250 \\
 & & & & & 5.97625
\end{pmatrix}
\end{array} \quad . \tag{36}
$$

Non–TI ratio (10) determined for this medium is

$$\rho_{(3)} = 0.000054 \quad , \tag{37}$$

and the corresponding unit reference symmetry vector is

$$t_{i(3)} = (0.612372 \;\; 0.353554 \;\; 0.707107) \quad . \tag{38}$$

We see that the medium is not exactly transversely isotropic but is close to transversely isotropic at the surface (zero depth). Numerically determined unit reference symmetry vector (38) is equal to its theoretical estimate (35).

At the depth of 1.5 km, velocity model SC1_II is very close to isotropic, but is slightly cubic and its symmetry axes coincide with the coordinate axes. The density reduced stiffness tensor in $\mathrm{km}^2\mathrm{s}^{-2}$ in velocity model SC1_II at the at the depth of 1.5 km reads (Pšenčík, Farra & Tessmer, 2012, eq. 19)

$$\begin{array}{c c} & \begin{array}{cccccc} \scriptstyle 11 & \scriptstyle 22 & \scriptstyle 33 & \scriptstyle 23 & \scriptstyle 13 & \scriptstyle 12 \end{array} \\ \begin{array}{c} \scriptstyle 11 \\ \scriptstyle 22 \\ \scriptstyle 33 \\ \scriptstyle 23 \\ \scriptstyle 13 \\ \scriptstyle 12 \end{array} & \begin{pmatrix} 30.25 & 10.08 & 10.08 & 0.00 & 0.00 & 0.00 \\ & 30.25 & 10.08 & 0.00 & 0.00 & 0.00 \\ & & 30.25 & 0.00 & 0.00 & 0.00 \\ & & & 10.08 & 0.00 & 0.00 \\ & & & & 10.08 & 0.00 \\ & & & & & 10.08 \end{pmatrix} \end{array} \quad . \tag{39}$$

The elements of the stiffness tensor (elastic moduli) are linear functions of depth. This means that, at depths between 0 km and 1.5 km, velocity model SC1_II is close to transversely isotropic, but is slightly tetragonal. For example, at the depth of 1.4 km, non–TI ratio (10) is

$$\rho_{(3)} = 0.000397 \quad , \tag{40}$$

and the corresponding unit reference symmetry vector is

$$t_{i(3)} = (0.611611 \;\; 0.348810 \;\; 0.710115) \quad . \tag{41}$$

We see that the medium is less transversely isotropic at the depth of 1.4 km than at the surface.

4. Applications

Possibility to determine whether a given stiffness tensor corresponds to a transversely isotropic medium may be very useful in selecting the method for calculating the wave field. If the medium is transversely isotropic or approximately transversely isotropic, we may use its symmetry vector or reference symmetry vector for tracing the SH and SV rays or the SH and SV reference rays (Klimeš & Bulant, 2015). The non–TI ratio, which identifies how much the given medium is transversely isotropic, and the unit reference symmetry vector can be determined according to Sections 2.1 and 2.2 of this paper.

If the reference symmetry vector is spatially varying, we need also its first–order spatial derivatives for ray tracing, and its second–order spatial derivatives for solving the corresponding equations of geodesic deviation. The first–order spatial derivatives of the reference symmetry vector can be determined according to Section 2.3 of this paper. The second–order spatial derivatives of the reference symmetry vector can be determined according to Section 2.4 of this paper.

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