Frequency-domain ray series for viscoelastic waves with a non-symmetric stiffness matrix

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Summary

In an elastic medium, it was proved that the stiffness tensor is symmetric with respect to the exchange of the first pair of indices and the second pair of indices, but the proof does not apply to a viscoelastic medium. In this paper, we thus propose the frequency–domain ray series for viscoelastic waves with a non–symmetric stiffness tensor.

Keywords

Viscoelastic media, stiffness tensor, wave propagation, ray theory, ray series, travel time, amplitude.

1. Introduction

The 3×3×3×3 frequency–domain stiffness tensor (elastic tensor, tensor of elastic moduli) $c^{ijkl} = c^{ijkl}(x^m, \omega)$ is symmetric with respect to the first pair of indices

$$c^{ijkl} = c^{jikl} \tag{1}$$

and with respect to the second pair of indices

$$c^{ijkl} = c^{ijlk} \quad . \tag{2}$$

It is thus frequently expressed in the form of the 6×6 stiffness matrix which lines correspond to the first pair of indices and columns to the second pair of indices.

In an elastic medium, it was proved that the stiffness tensor is symmetric with respect to the exchange of the first pair of indices and the second pair of indices,

$$c^{ijkl} = c^{klij} \quad . \tag{3}$$

The 6×6 stiffness matrix is thus symmetric in an elastic medium.

However, the above mentioned proof does not apply to a viscoelastic medium. In this paper, we thus propose the frequency–domain ray series for viscoelastic waves with a *non–symmetric stiffness matrix*,

$$c^{ijkl} \neq c^{klij} \quad . \tag{4}$$

The lower–case Roman indices take values 1, 2 and 3. The Einstein summation over repetitive lower–case Roman indices is used throughout the paper.

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2. Standard ray series

In the frequency domain, the viscoelastodynamic equation for complex–valued displacement $u_i = u_i(x^m, \omega)$ reads

$$(c^{ijkl} u_{l,k})_{,j} - (\mathrm{i}\omega)^2 \varrho \, u_i = 0 \quad , \tag{5}$$

where lower–case Roman subscript $_{,k}$ following a comma denotes the partial derivative with respect to corresponding spatial coordinate x^k. Here $c^{ijkl} = c^{ijkl}(x^m, \omega)$ is the frequency–domain stiffness tensor, $\varrho = \varrho(x^m)$ is the density and ω is the circular frequency.

We express the displacement in terms of its amplitude $U_i = u_i(x^m, \omega)$ and travel time $\tau = \tau(x^m)$ as

$$u_i = U_i \exp(\mathrm{i}\omega\tau) \quad . \tag{6}$$

We expand the frequency–dependent amplitude into high–frequency asymptotic series

$$U_i = \sum_{n=0}^{\infty} (\mathrm{i}\omega)^{-n} \, U_i^{[n]} \quad . \tag{7}$$

We insert displacement (6) into viscoelastodynamic equation (5) and obtain equation

$$(\mathrm{i}\omega)^2 N^i(U_m, \tau_{,n}) + \mathrm{i}\omega M^i(U_m, \tau_{,n}) + L^i(U_m) = 0 \quad , \tag{8}$$

where the differential operators are defined as

$$N^i(U_m, \tau_{,n}) = \varrho \, [\Gamma^{il}(x^m, \tau_{,n}) \, U_l - U_i] \quad , \tag{9}$$

$$M^i(U_m, \tau_{,n}) = (c^{ijkl} \tau_{,k} U_l)_{,j} + c^{ijkl} \tau_{,j} U_{l,k} \quad , \tag{10}$$

and

$$L^i(U_m) = (c^{ijkl} U_{l,k})_{,j} \quad . \tag{11}$$

Here the Christoffel matrix, defined as

$$\Gamma^{il}(x^m, p_n) = a^{ijkl}(x^m) \; p_j \, p_k \quad , \tag{12}$$

is a function of six phase–space coordinates x^m, p_n formed by three spatial coordinates x^m and three slowness–vector components p_n. In definition (12),

$$a^{ijkl}(x^m) = c^{ijkl}(x^m) \left[\varrho(x^n)\right]^{-1} \tag{13}$$

is the density–reduced stiffness tensor.

Inserting series (7) into viscoelastodynamic equation (8) and sorting the terms according to the order of iω, we obtain the system of equations

$$N^i\big(U_k^{[n]}, \tau_{,l}\big) + M^i\big(U_k^{[n-1]}, \tau_{,l}\big) + L^i\big(U_k^{[n-2]}\big) = 0 \tag{14}$$

for each order $n = 0, 1, 2, ...$ Here $U_k^{[-1]} = 0$ and $U_k^{[-2]} = 0$, i.e., operator M^i is missing in this equation for $n = 0$ and operator L^i is missing in this equation for $n = 0, 1$.

3. Christoffel equation and eikonal equation

Equation (14) for $n = 0$ constitutes the matrix Christoffel equation

$$\Gamma^{il}(x^m, \tau_{,n})\, U_l^{[0]} - U_i^{[0]} = 0 \quad . \tag{15}$$

The three eigenvalues of Christoffel matrix (12) correspond to three waves: the P wave and two S waves. Unlike as in the elastic case, the Christoffel matrix is not symmetric.

We select one of three eigenvalues of Christoffel matrix (12) and denote it as $G = G(x^m, \tau_{,n})$. We denote the corresponding right–hand eigenvector of the Christoffel matrix as $g_i = g_i(x^m, \tau_{,n})$,

$$\Gamma^{il}\, g_l = G\, g_i \quad , \tag{16}$$

and the corresponding left–hand eigenvector of the Christoffel matrix as $\vec{g}_i = \vec{g}_i(x^m, \tau_{,n})$,

$$\vec{g}_i\, \Gamma^{il} = \vec{g}_l\, G \quad . \tag{17}$$

The three right–hand eigenvectors of the Christoffel matrix and the three left–hand eigenvectors of the Christoffel matrix are mutually biorthogonal.

The lengths of the three right–hand eigenvectors of the Christoffel matrix are not determined, but we choose the lengths of the corresponding left–hand eigenvectors so that the three right–hand eigenvectors and the three left–hand eigenvectors are mutually biorthonormal,

$$\vec{g}_k g_k = 1 \quad . \tag{18}$$

The zero–order vectorial amplitude then reads

$$U_i^{[0]} = U^{[0]}\, g_i \quad , \tag{19}$$

where the zero–order ray–theory amplitude $U^{[0]}$ will be determined by the *transport equation* in Section 5.

In order to satisfy Christoffel equation (15), selected eigenvalue G must be unit,

$$\boxed{G(x^m, \tau_{,n}) = 1 \quad .} \tag{20}$$

Nonlinear first–order partial differential equation (20) for travel time τ is called the Hamilton–Jacobi equation. In wave propagation problems, it is also often referred to as the eikonal equation. The methods for solving the Hamilton–Jacobi equation are already mostly developed (Hamilton, 1837; Červený, 1972; Klimeš, 2002; 2010).

Hamilton–Jacobi equation (20) generates the equations of rays (Hamilton equations, equations of geodesics) and the related equations like the Hamiltonian equations of geodesic deviation (dynamic ray tracing equations).

The equations of rays may be expressed in various forms, e.g., as

$$\frac{\mathrm{d}x^i}{\mathrm{d}\gamma} = \frac{1}{2}\frac{\partial G}{\partial p_i} \quad , \tag{21}$$

$$\frac{\mathrm{d}p_i}{\mathrm{d}\gamma} = -\frac{1}{2}\frac{\partial G}{\partial x^i} \quad , \tag{22}$$

where parameter γ along rays coincides with the values of travel time τ.

Hereinafter, $\frac{\partial G}{\partial x^i}$ and $\frac{\partial G}{\partial p_i}$ denote the partial derivatives of function $G(x^m, p_n)$ of six phase–space coordinates x^m, p_n. Using this notation, the partial derivatives of any function $G\big(x^m, \tau_{,n}(x^a)\big)$ of three spatial coordinates read

$$G_{,j} = \frac{\partial G}{\partial x^j} + \frac{\partial G}{\partial p_k}\tau_{,kj} \quad . \tag{23}$$

Since

$$G(x^m, p_n) = \vec{g}_r(x^m, p_n)\Gamma^{rs}(x^a, p_b)g_s(x^c, p_d) \quad , \tag{24}$$

the first–order phase–space derivatives of eigenvalue $G(x^m, p_n)$ can be expressed as

$$\frac{\partial G}{\partial p_i}(x^m, p_n) = \vec{g}_r(x^m, p_n)\frac{\partial \Gamma^{rs}}{\partial p_i}(x^a, p_b)g_s(x^c, p_d) \tag{25}$$

and

$$\frac{\partial G}{\partial x^i}(x^m, p_n) = \vec{g}_r(x^m, p_n)\frac{\partial \Gamma^{rs}}{\partial x^i}(x^a, p_b)g_s(x^c, p_d) \quad . \tag{26}$$

We insert definition (12) into phase–space derivatives (25)–(26), and obtain expressions

$$\frac{\partial G}{\partial p_i}(x^m, p_n) = \vec{g}_r(x^m, p_n)\left[a^{riks}(x^a) + a^{rkis}(x^a)\right] p_k g_s(x^c, p_d) \tag{27}$$

and

$$\frac{\partial G}{\partial x^i}(x^m, p_n) = \vec{g}_r(x^m, p_n)\frac{\partial a^{rjks}}{\partial x^i}(x^a)\, p_j p_k g_s(x^c, p_d) \quad . \tag{28}$$

Note that

$$V^i(x^m, p_n) = \tfrac{1}{2}\vec{g}_r(x^m, p_n)\left[a^{riks}(x^a) + a^{rkis}(x^a)\right] p_k g_s(x^c, p_d) \tag{29}$$

is the ray–velocity vector.

4. Principal and additional amplitude components

We decompose vectorial amplitude coefficients $U_i^{[n]}$ in series (7) into the principal amplitude component $U_i^{[n]}$ and two additional amplitude components $U^{\perp[n]}$,

$$U_i^{[n]} = U^{[n]}g_i + \sum_{\perp} U^{\perp[n]}\, g_i^{\perp} \quad , \tag{30}$$

where $g_i^{\perp}$ are the other two eigenvectors of Christoffel matrix (12), indexed by $^{\perp}$ which takes two values. Considering expression (19), we assume that both $U^{\perp[0]} = 0$.

Definition (9) with decomposition (30) yields

$$N^i\big(U_m^{[n]}, \tau_{,n}\big) = \varrho \sum_{\perp} U^{\perp[n]}\left(G^{\perp} - 1\right) g_i^{\perp} \quad . \tag{31}$$

We multiply viscoelastodynamic equation (14) by left–hand eigenvector $\vec{g}_i^{\perp}$ of the Christoffel matrix, consider relation (31), and obtain expression

$$\boxed{U^{\perp[n]} = -\varrho^{-1}\left[\vec{g}_i^{\perp} M^i\big(U_k^{[n-1]}, \tau_{,n}\big) + \vec{g}_i^{\perp} L^i\big(U_k^{[n-2]}\big)\right]\left(G^{\perp} - 1\right)^{-1}} \tag{32}$$

for the additional amplitude components in terms of lower–order amplitudes. Expression (32) differs from the analogous expression for the symmetric stiffness matrix (Červený, 2001, eq. 5.7.13) just by left–hand eigenvectors $\vec{g}_i^{\perp}$.

5. Transport equation

We multiply viscoelastodynamic equation (14) by left–hand eigenvector $\vec{g}_i$ of the Christoffel matrix, consider relation (31), and obtain transport equation

$$\vec{g}_i M^i\big(U_k^{[n]}, \tau_{,n}\big) + \vec{g}_i L^i\big(U_k^{[n-1]}\big) = 0 \tag{33}$$

for the principal amplitude components. We separate the terms with higher–order principal amplitude components from the terms containing higher–order additional amplitude components and lower–order amplitude components,

$$\vec{g}_i M^i\big(U^{[n]} g_k, \tau_{,l}\big) = -\sum_{\perp} \vec{g}_i M^i\big(U^{\perp[n]} g_k^{\perp}, \tau_{,l}\big) - \vec{g}_i L^i\big(U_k^{[n-1]}\big) \quad . \tag{34}$$

Relations (33) and (34) differ from the analogous relations for the symmetric stiffness matrix (Červený, 2001, eqs. 5.7.19 and 5.7.20) just by left–hand eigenvector $\vec{g}_i$.

We express the left–hand side of transport equation (34) as

$$\vec{g}_i M^i(U^{[n]} g_m, \tau_{,l}) = 2\varrho V^j U_{,j}^{[n]} + (\varrho V^j)_{,j} U^{[n]} - 2\varrho\, S\, U^{[n]} \quad , \tag{35}$$

where ray–velocity vector V^j is given by definition (29), which can also be expressed as

$$\varrho(x^a) V^i(x^m, p_n) = \tfrac{1}{2} \vec{g}_r(x^m, p_n) \big[c^{riks}(x^a) + c^{rkis}(x^a)\big]\, p_k g_s(x^c, p_d) \quad . \tag{36}$$

The first two terms on the right–hand side of relation (35) are well known from the ray series with a symmetric stiffness matrix (Červený, 2001, eq. 5.7.23). Quantity S in the rightmost term of transport equation (35) can be determined using definition (10) as

$$S = \frac{1}{2\,\varrho} \left\{ \frac{1}{2} \Big[\vec{g}_i \left(c^{ijkl} + c^{ikjl}\right) \tau_{,k} g_l\Big]_{,j} - \vec{g}_i \left(c^{ijkl} \tau_{,k} g_l\right)_{,j} - \vec{g}_i c^{ikjl} \tau_{,k} g_{l,j} \right\} \quad , \tag{37}$$

see (36). Considering (35), we express transport equation (34) as

$$\sqrt{\varrho}\, V^j U_{,j}^{[n]} + \tfrac{1}{2\sqrt{\varrho}} (\varrho V^j)_{,j} U^{[n]} = \sqrt{\varrho}\, S\, U^{[n]} + Z^{[n-1]} \quad , \tag{38}$$

where

$$Z^{[n-1]} = -\frac{1}{2\sqrt{\varrho}} \left[\sum_{\perp} \vec{g}_i M^i\big(U^{\perp[n]} g_k^{\perp}, \tau_{,n}\big) + \vec{g}_i L^i\big(U_k^{[n-1]}\big) \right] \quad . \tag{39}$$

The solution of transport equation (38) for $n = 0$ reads

$$U^{[0]} = U_0^{[0]}\, (\varrho_0\, J_0)^{\frac{1}{2}}\, (\varrho\, J)^{-\frac{1}{2}}\, \exp\!\big(\textstyle\int_{\tau_0}^{\tau} \mathrm{d}\gamma\, S\big) \quad , \tag{40}$$

where subscript $_0$ denotes the initial conditions. Squared geometrical spreading

$$J = \det\!\left(\frac{\partial x^i}{\partial \gamma^a}\right) \tag{41}$$

(Babich, 1961, eq. 3.7; Červený, 2001, eq. 3.10.9) represents the Jacobian of transformation from ray coordinates $\gamma^1, \gamma^2, \gamma^3$ to spatial coordinates x^i. Here γ^1 and γ^2 are the ray parameters, and $\gamma^3 = \gamma$.

Factor $\exp\big(\int_{\tau_0}^{\tau} \mathrm{d}\gamma\, S\big)$ in (40) is present due to the skew part of the stiffness matrix, see the next section.

The solution of transport equation (38) for $n > 0$ reads (Červený, 2001, eq. 5.7.30)

$$U^{[n]} = U^{[0]} \left[\frac{U_0^{[n]}}{U_0^{[0]}} + \int_{\tau_0}^{\tau} \mathrm{d}\gamma\, \frac{Z^{[n-1]}}{U^{[0]}\,\sqrt{\varrho}} \right] \quad . \tag{42}$$

6. Difference between symmetric and non–symmetric stiffness matrices

The only difference of expressions (40) and (42) for the principal amplitudes from the analogous expressions derived for a symmetric stiffness matrix is exponential term $\exp\bigl(\int_{\tau_0}^{\tau} \mathrm{d}\gamma\, S\bigr)$ in expression (40). We shall now derive various expressions for quantity $S = S(x^m)$.

We express definition (37) as

$$S = \frac{1}{2\,\varrho}\left\{\frac{1}{2}\Big[\vec{g}_i\big(c^{ijkl}{+}c^{ikjl}\big)\tau_{,k}g_l\Big]_{,j} - \big(\vec{g}_i c^{ijkl}\tau_{,k}g_l\big)_{,j} + \vec{g}_{i,j}c^{ijkl}\tau_{,k}g_l - \vec{g}_i c^{ikjl}\tau_{,k}g_{l,j}\right\} \quad . \tag{43}$$

After summation, we obtain expression

$$S = \frac{1}{2}\Big[\vec{g}_{i,j}a^{ijkl}\tau_{,k}g_l - \vec{g}_i a^{ikjl}\tau_{,k}g_{l,j}\Big] - \frac{1}{4\,\varrho}\Big[\varrho\,\vec{g}_i\big(a^{ijkl} - a^{ikjl}\big)\tau_{,k}g_l\Big]_{,j} \quad , \tag{44}$$

where a^{ikjl} is the density–reduced stiffness tensor given by definition (13). The last term on the right–hand side differs from

$$\frac{1}{2\,\varrho}\big(\varrho V^j\big)_{,j} = \frac{1}{4\,\varrho}\Big[\varrho\,\vec{g}_i\big(a^{ijkl} + a^{ikjl}\big)\tau_{,k}g_l\Big]_{,j} \tag{45}$$

just by the subtraction. Note also that

$$\big[\varrho\big(a^{ijkl} - a^{ikjl}\big)\tau_{,k}\big]\ \tau_{,j} = 0 \quad . \tag{46}$$

We differentiate the product in the rightmost term of expression (44) and obtain expression

$$S = \frac{1}{4}\vec{g}_{i,j}\big(a^{ijkl}{+}a^{ikjl}\big)\tau_{,k}g_l - \frac{1}{4}\vec{g}_i\big(a^{ijkl}{+}a^{ikjl}\big)\tau_{,k}g_{l,j} - \frac{1}{4\,\varrho}\vec{g}_i\Big[\varrho\big(a^{ijkl}{-}a^{ikjl}\big)\Big]_{,j}\tau_{,k}g_l \ . \tag{47}$$

We differentiate characteristic equation (16) for the right–hand eigenvector with respect to spatial coordinates, consider birthonormality of the left–hand and right–hand eigenvectors, and obtain relation

$$g_{i,j} = \sum_{\perp} g_i^{\perp}\vec{g}_k^{\perp}\Gamma^{kl}_{,j}g_l\big(G{-}G^{\perp}\big)^{-1} + g_i\vec{g}_k g_{k,j} \tag{48}$$

for the spatial gradient of the right–hand eigenvector. The rightmost term in relation (48) accounts for the undefined changes of the length of the right–hand eigenvector g_i. We differentiate characteristic equation (17) for the left–hand eigenvector with respect to spatial coordinates, and obtain analogous relation

$$\vec{g}_{i,j} = \sum_{\perp} \vec{g}_k\Gamma^{kl}_{,j}g_l^{\perp}\vec{g}_i^{\perp}\big(G{-}G^{\perp}\big)^{-1} + \vec{g}_i g_k\vec{g}_{k,j} \tag{49}$$

for the spatial gradient of the left–hand eigenvector. The rightmost term in relation (49) accounts for the undefined changes of the length of the left–hand eigenvector $\vec{g}_i$, and satisfies identity

$$g_k\vec{g}_{k,j} = -\vec{g}_k g_{k,j} \tag{50}$$

obtained by differentiating normalization condition (18).

We insert the gradients (48) and (49) of the eigenvectors of the Christoffel matrix into expression (47), consider identity (50) and arrive at expression

$$S = \frac{1}{4}\sum_{\perp}\left(\vec{g}_k\Gamma^{kl}_{,j}g_l^{\perp}\vec{g}_r^{\perp}\frac{\partial\Gamma^{rs}}{\partial p_j}g_s - \vec{g}_r\frac{\partial\Gamma^{rs}}{\partial p_j}g_s^{\perp}\vec{g}_k^{\perp}\Gamma^{kl}_{,j}g_l\right)\Big(G{-}\,G^{\perp}\Big)^{-1}$$
$$- \frac{1}{4\,\varrho}\vec{g}_i\big(c^{ijkl} - c^{ikjl}\big)_{,j}\tau_{,k}g_l - \vec{g}_k g_{k,j}V^j \quad . \tag{51}$$

We now express the spatial derivatives $\Gamma^{kl}_{,j}$ of the Christoffel matrix in terms of its phase–space derivatives as

$$\Gamma^{kl}_{,j} = \frac{\partial \Gamma^{kl}}{\partial x^j} + \frac{\partial \Gamma^{kl}}{\partial p_s} \tau_{,sj} \quad . \tag{52}$$

The partial derivatives of the Kelvin–Christoffel matrix (12) with respect to phase–space coordinates x^m and p_n read

$$\frac{\partial \Gamma^{kl}}{\partial x^j}(x^m, \tau_{,n}) = a^{krsl}_{,j} \tau_{,r} \tau_{,r} \tag{53}$$

and

$$\frac{\partial \Gamma^{kl}}{\partial p_j}(x^m, \tau_{,n}) = \left(a^{kjrl} + a^{krjl}\right) \tau_{,r} \quad . \tag{54}$$

Quantity (51) with identity (52) finally reads

$$\begin{aligned} S = \frac{1}{4} \sum_{\perp} \left(\vec{g}_k \frac{\partial \Gamma^{kl}}{\partial x^j} g_l^{\perp} \vec{g}_r^{\perp} \frac{\partial \Gamma^{rs}}{\partial p_j} g_s - \vec{g}_k \frac{\partial \Gamma^{kl}}{\partial p_j} g_l^{\perp} \vec{g}_r^{\perp} \frac{\partial \Gamma^{rs}}{\partial x^j} g_s \right) \left(G - G^{\perp} \right)^{-1} \\ - \frac{1}{4\varrho} \vec{g}_i \left(c^{ijkl} - c^{ikjl} \right)_{,j} \tau_{,k} g_l - \vec{g}_i \frac{\mathrm{d} g_i}{\mathrm{d}\gamma} \quad . \end{aligned} \tag{55}$$

The last term $\vec{g}_i \frac{\mathrm{d}g_i}{\mathrm{d}\gamma}$ in expression (55) represents just the correction of principal amplitude $U^{[n]}$ in decomposition (30) due to the undefined length of right–hand eigenvector g_i, and vanish if we put

$$\vec{g}_i \frac{\mathrm{d} g_i}{\mathrm{d}\gamma} = 0 \tag{56}$$

along each ray.

Each element of matrix $(c^{ijkl} - c^{ikjl})_{,j}\tau_{,k}$ in the last but one term of relation (55) represents the divergence of a vector tangent to the wavefront, see identity (46). Note also that

$$\vec{g}_k \frac{\partial \Gamma^{kl}}{\partial p_j} g_l^{\perp} \tau_{,j} = 0 \tag{57}$$

and

$$\vec{g}_k^{\perp} \frac{\partial \Gamma^{kl}}{\partial p_j} g_l \, \tau_{,j} = 0 \quad . \tag{58}$$

Vectors

$$\vec{g}_k \frac{\partial \Gamma^{kl}}{\partial p_j} g_l^{\perp} \tag{59}$$

and

$$\vec{g}_k^{\perp} \frac{\partial \Gamma^{kl}}{\partial p_j} g_l \tag{60}$$

thus represent two sets of the contravariant basis vectors of the ray–centred coordinate system.

Expression (55) for quantity S may be singular at slowness–surface singularities, but is regular at spatial caustics.

Quantity S vanishes for a symmetric stiffness matrix. For a non–symmetric stiffness matrix, quantity S vanishes in a homogeneous medium.

Quantity S which expresses the difference between ray series for a symmetric and non–symmetric stiffness matrices is thus generated by a combination of a non–symmetric stiffness matrix and heterogeneities. Expression (55) with identities (46), (57) and (58) suggest that quantity S may mostly be influenced by the wavefront–tangent component of the gradient of the skew part of the stiffness matrix.

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