Optimization of the shape of Gaussian beams

Karel Žáček, Department of Geophysics, Charles University, Prague, Czech Republic

Summary

The applicability and accuracy of the Gaussian beam method depend on the proper choice of the shape of beams. Gaussian beams become inaccurate solutions of the elastodynamic equation if the velocity field changes considerably within the beam width. We present a procedure of determinig the optimum initial shape of Gaussian beams based on minimizing the average widths of Gaussian beams and smoothing the distribution of the optimum parameters of Gaussian beams.

This procedure can increase the applicability of the Gaussian beam or packet method, especially in complex structures. Moreover, it can make the use of the Gaussian beam or packet method more comfortable. The presented procedure is suitable for the optimization of the shape of Gaussian beams for Gaussian beam or packet migration.

1 Introduction

Gaussian beams (GBs) represent high-frequency asymptotic time-harmonic solutions of the elastodynamic equation, which are concentrated close to rays (e.g., Červený et al., 1982, Červený & Pšenčík, 1983). The distribution of the amplitude of the principal component of the displacement in the profile perpendicular to the ray is Gaussian (bell-shaped). The great advantage of the Gaussian beam method is that GBs are regular along the whole ray, even at caustics.

However, the applicability and accuracy of the Gaussian beam method depend on the proper choice of the shape of beams. It is necessary to keep GBs narrow in relation to the velocity changes in the model, because GBs become inaccurate solutions of the elastodynamic equation if the velocity field changes considerably within the beam width. Unfortunately, if GBs are too narrow, they quickly increase in width as they propagate. Thus, we can use neither too narrow nor too wide beams as the initial choice of the shape of GBs. Furthermore, in a complex structure, we cannot judge solely from the final width of the beam whether the beam is or is not the correct solution of the elastodynamic equation. The beam must be sufficiently narrow along the whole ray path.

All these requirements force us to use a more sophisticated theory in the choice of the initial shape of GBs than just a wild guess or some kind of empirical rules. Klimeš (1989) proposed a procedure, which is followed in this paper, based on minimizing the integral of a certain expression along a fixed part of the beam's central ray. This approach allows us to minimize not only the width of GBs, but also the quadratic variations of the complex-valued phase along an arbitrary surface, along a structural interface or along a wavefront tangent plane.

In a complex structure, for various positions of the initial point of the beam's central ray (e.g., source), for various take-off parameters of the beam's central ray and for various travel times, the optimum initial parameters of GBs can vary considerably. This can bring about serious problems in the decomposition of the wave field into GBs or packets. Hence, we should be able to simultaneously optimize and smooth the distribution of the initial parameters of GBs.

For the sake of simplicity, the description of the whole procedure is restricted to a 2-D case in this expanded abstract.

2 The specification of some used quantities

In the case of the component notation, the upper-case indices take the values $I, J, ... = 1, 2$ and the lower-case indices take the values $i, j, ... = 1, 2, 3$. The Einstein summation over the pairs of identical indices is used.

We denote by A_J the amplitude and by θ the complex-valued phase of a frequency-domain Gaussian beam

$$g_J = A_J \exp(\mathrm{i}\omega\theta) \quad , \tag{1}$$

where i is the imaginary unit and ω is the circular frequency. In *ray-centred coordinates* q_J (e.g., Červený, 2001), where q_2 is an independent variable along the ray, the quadratic Taylor expansion of the phase has the form of

$$\theta(q_J) = \tau(q_2) + \frac{1}{2} M(q_2) q_1^2 \quad , \tag{2}$$

where τ is the travel time along the central ray and M is the complex-valued second differential of the phase along the plane tangent to the phase-front.

We are interested in the initial parameters of GBs. Let us consider that all the initial points of the rays are situated on a line. Hence, let us denote by M_0^{Σ} the initial parameters along the initial line (e.g., earth surface).

3 Phase-space ray coordinate system

The rays may be defined as the characteristic curves of the *eikonal equation* (e.g., Červený 2001). In smoothly inhomogeneous isotropic media, the *eikonal equation* reads

$$p_I p_I = v^{-2}(x_I) \quad , \tag{3}$$

where

$$p_I = \frac{\partial \tau}{\partial x_I} \quad , \tag{4}$$

x_I being the general Cartesian coordinates and v being the propagation velocity. In general, we shall write the *eikonal equation* as

$$H(x_I, p_I) = 0 \quad , \tag{5}$$

where the Hamiltonian function $H(x_I, p_I)$ may be specified in various ways.

We consider x_I and p_I to be independent coordinates in a four-dimensional *phase space*. The *eikonal equation* then defines a *Hamiltonian hypersurface* in the *phase space*.

Let us define a new coordinate system y_i in the three-dimensional *Hamiltonian hypersurface*. Coordinate y_1 corresponds to an independent variable along the ray, y_2 corresponds to take-off parameter of the ray and y_3 corresponds to the position of the initial point of the ray along the initial line. We shall call it the *phase-space ray coordinate system*. These coordinates are suitable for optimization of the shape of GBs for Gaussian beam or packet migration.

4 Algorithm

(a) First of all, we need to compute a sufficiently dense set of rays and store several important quantities along the rays. In dependency upon an independent variable along the ray (e.g., travel time), take-off parameter of the beam's central ray and position of the initial point of the ray (e.g., source), we have to store the *ray propagator matrix* (e.g., Červený 2001), the Cartesian components of the slowness vector and the transformation matrix, the columns of which constitute the local vector basis of the *ray-centred coordinates* expressed in Cartesian coordinates.

(b) In minimizing the objective function

$$T = \int_{q_2^{(1)}}^{q_2^{(2)}} \mathrm{tr}\{[\mathrm{Im}(M)]^{-1}\} \mathrm{d}q_2 \quad , \qquad (6)$$

where $\mathrm{Im}(M)$ is the imaginary part of M, we solve the ordinary differential equations proposed by Klimeš (1989) by numerical integration along the ray.

(c) The results of the numerical integration describe the optimum initial profiles of GBs perpendicularly to the central ray of GBs. We have to transform the results of the numerical integration to the analogous quantities describing the optimum shape of GBs along the initial surface.

(d) Finally, we can calculate the optimum initial parameters of GBs M_0^Σ. Parameters M_0^Σ are discretized in dependency upon an independent variable along the ray (e.g travel time), take-off parameters of the beam's central ray and position of the initial point of the ray,

$$M_0^\Sigma = M_0^\Sigma(y_i) \quad . \qquad (7)$$

Let us remind that y_i are the *phase-space ray coordinates* in the three-dimensional *Hamiltonian hypersurface*. Thus, each initial parameter of GBs is defined on the 3-D parameter grid, where each dimension is related to one of the quantities mentioned above.

(e) If necessary, we should smooth the distribution of the optimum parameters of GBs. In obtaining a smoother distribution, we minimize the squares of the relevant Sobolev norms of the parameters of GBs together with the mean squares of the widths of the corresponding GBs. The objective function to be minimized reads

$$O = \frac{T}{q_2^{(2)} - q_2^{(1)}} + \|\mathrm{Re}(M_0^\Sigma)\|^2 + \|\mathrm{Im}(M_0^\Sigma)\|^2 \quad , \qquad (8)$$

where $\|\cdot\|$ is the appropriate Sobolev norm. The Sobolev scalar product is a linear combination of the L2 Lebesgue scalar products of the zero, first, second or higher partial derivatives (Tarantola 1987). The partial derivatives are calculated with respect to the *phase-space ray coordinates* y_i. We have used only the first derivatives in the Sobolev norms in constructing the objective function O, see equation (8).

5 Numerical examples

We have decided to use the Marmousi model (Versteeg & Grau, 1991) as the velocity model. Since the original Marmousi model is too complex for ray methods, we have used the smoothed Marmousi model (Žáček, 2000) in the computations. The dimensions of the model are 9200 metres (length) by 3000 metres (depth). The grided values of velocity vary from 1520ms^{-1} (the lightest grey in Figure 1) to 4550ms^{-1} (the darkest grey in Figure 1).

Figure 1. The smoothed Marmousi model.

We have calculated the optimum initial parameters of GBs for nine different positions of the source. We have prepared two sets of initial parameters:

(a) a set smoothed to a constant value (the respective halfwidths of GBs are in Figure 2), $M_0^\Sigma = (0.19 + 0.31\mathrm{i}) \times 10^{-7}$, and

(b) a very little smoothed set (the respective halfwidths of GBs are in Figure 3), where the parameter $\mathrm{Re}(M_0^\Sigma)$ has varied from -0.72×10^{-6} to 0.53×10^{-7} and the parameter $\mathrm{Im}(M_0^\Sigma)$ has varied from 0.10×10^{-7} to 0.11×10^{-6} .

We have calculated the standard halfwidths of GBs for various initial values of parameters M_0^Σ. Standard halfwidth a of a Gaussian beam of crossection

$$\exp\left(-\frac{q_1^2}{2a^2}\right) \quad , \qquad (9)$$

multiplied by the square root of $(2\pi f)$, f being the frequency, has been interpolated between the rays and displayed (see Figure 2 and 3),

$$W = a\sqrt{2\pi f} \quad . \qquad (10)$$

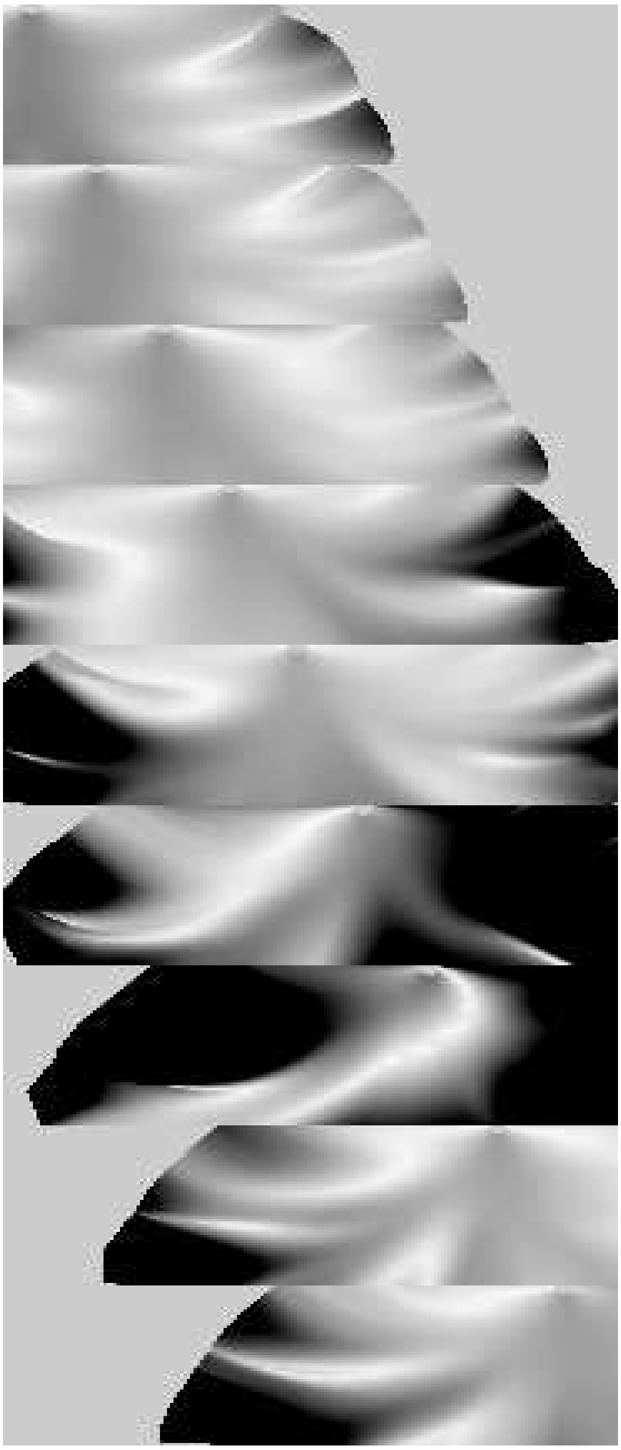

Figure 2. The halfwidths of Gaussian beams (quantity W, see equation (10)) with a constant value of the initial parameter of GBs. The rows correspond to various positions of the source.

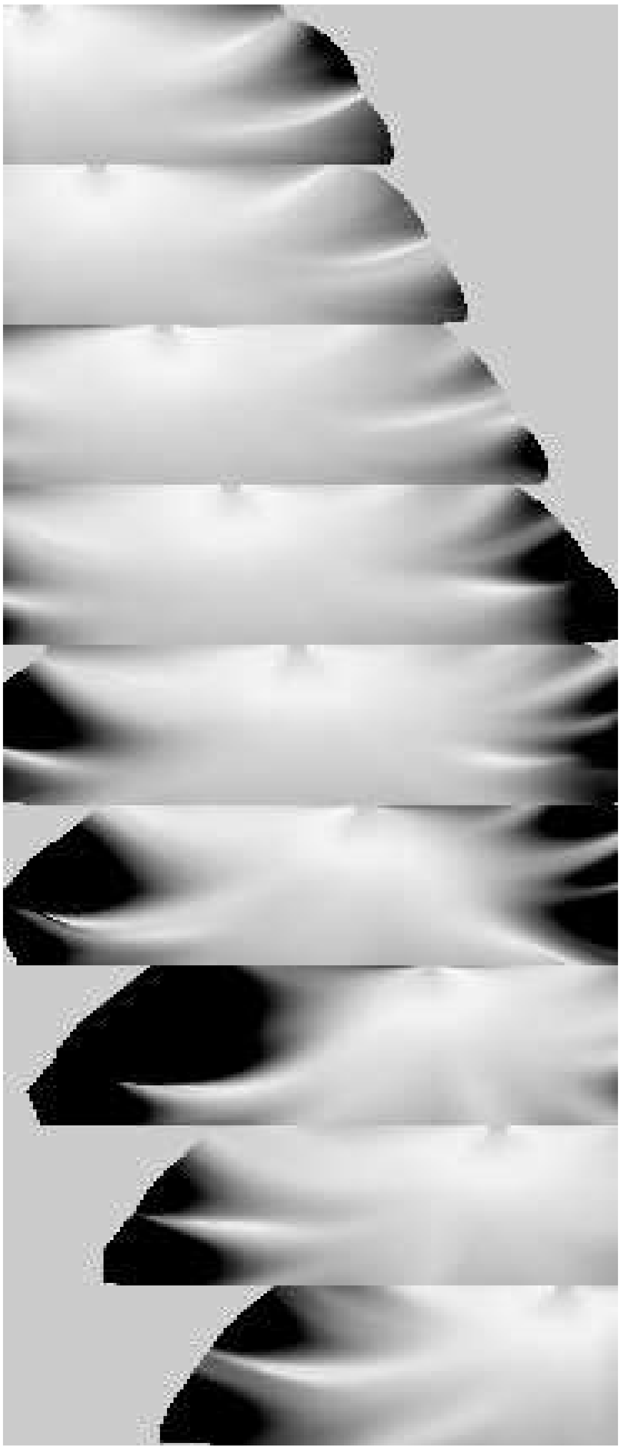

Figure 3. The halfwidths of Gaussian beams (the quantity W, see equation (10)), the optimum initial parameters of GBs with very little smoothing.

The gray-scale coded quantity W is displayed at the respective points along the central rays of GBs. The white colour corresponds to the GB halfwidth of 0 metres for all frequencies. The black colour corresponds to the GB halfwidths of 1010 metres and more for the frequency of 35Hz, and of 1890 metres and more for the frequency of 10Hz.

In Figure 2, the halfwidths of GBs have been calculated with a constant value of the initial parameter of GBs. As we can see, GBs become too quickly very wide for the position of the source close to the center of the profile. Although the model is quite smooth (see Figure 1), we would not be able to use the Gaussian beam method with the constant initial parameters.

In Figure 3, the optimum initall parameters of GBs are very little smoothed (just for numerical purposes). We can see significant differences between Figure 2 and Figure 3. Even for the position of the source close to the centre of the profile, GBs do not spread very quickly. The great advantage is the fact that we know this is the best result we can achieve. For instance, this can be a very important criterion in the choice of the smoothness of the velocity model for Gaussian beam or packet migration.

6 Conclusions

The presented procedure of determining the optimum initial shape of GBs can improve the applicability and accuracy of the Gaussian beam or packet method. We obtain the optimum parameters almost automatically. Only the smoothness of the distribution of the optimum parameters of GBs, which plays the key role in the decomposition of the wave field into Gaussian beams or packets, have to be chosen by the user.

Naturally, it is not necessary to use this procedure in simple structures. But the more complex is the model, the more important is to optimize the initial parameters of GBs. Furthemore, especially in simple structures, we can smooth the optimum initial parameters of GBs to make them constant in the whole *Hamiltonian hypersurface*. It is more comfortable than trying to obtain the constant optimum initial parameter of GBs just by chance.

In the future, we would like to study the detail properties of the smoothing – influence of various Sobolev norms, itterative smoothing, etc.

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