RELATION OF THE WAVE–PROPAGATION METRIC TENSOR TO THE CURVATURES OF THE SLOWNESS AND RAY–VELOCITY SURFACES

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ABSTRACT

The contravariant components of the wave–propagation metric tensor equal half the second–order partial derivatives of the selected eigenvalue of the Christoffel matrix with respect to the slowness–vector components. The relations of the wave–propagation metric tensor to the curvature matrix and Gaussian curvature of the slowness surface and to the curvature matrix and Gaussian curvature of the ray–velocity surface are demonstrated with the help of ray–centred coordinates.

Keywords: Ray theory, elastic anisotropy, Finsler space, metric tensor, slowness surface, ray–velocity surface, ray–centred coordinates.

1. INTRODUCTION

Rays corresponding to the ray theory for anisotropic media represent geodesics in Finsler space with the Finslerian metric determined by the *wave–propagation metric tensor* (*Rund, 1959*; *Babich, 1961*; *Hanyga, 1984*; *Červený, 2002*). The contravariant components of the wave–propagation metric tensor equal half the second–order partial derivatives of the selected eigenvalue of the Christoffel matrix with respect to the slowness–vector components. The covariant wave–propagation metric tensor is the matrix inverse to the contravariant wave–propagation metric tensor.

The contravariant components of the wave–propagation metric tensor are closely related to the curvature matrix of the slowness surface. The covariant components of the wave–propagation metric tensor are closely related to the curvature matrix of the ray–velocity surface. We shall demonstrate these relations by transforming the contravariant and covariant wave–propagation metric tensors into ray–centred coordinates.

In this paper, lower–case Roman indices take values $i, j, ... = 1, 2, 3$, upper–case Roman indices take values $I, J, ... = 1, 2$, and the Einstein summation convention applies to pairs of identical indices.

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2. FINSLERIAN WAVE–PROPAGATION METRIC TENSOR

Let $G(x^m, p_n)$ be a selected eigenvalue of the Christoffel matrix

$$\Gamma_{jk}(x^m, p_n) = p_i\, a_{ijkl}(x^m)\, p_l \quad . \tag{1}$$

Here x^m are Cartesian coordinates, $a_{ijkl}(x_m)$ are density–normalized elastic moduli, and p_i are the components of the slowness vector, defined as partial derivatives of travel time (action) τ,

$$p_i = \frac{\partial \tau}{\partial x^i} \quad . \tag{2}$$

Refer, e.g., to the book by *Červený (2001)* for a more detailed description.

All geometrical objects considered in this paper, such as the sheets of the slowness surface, sheets of the ray–velocity surface or phase–space rays, satisfy equation

$$G(x^m, p_n) = 1 \quad . \tag{3}$$

The *ray–velocity vector* (group–velocity vector) $V^i(x^m, p_n)$ is defined by equation

$$V^i = \frac{1}{2}\frac{\partial G}{\partial p_i} \quad . \tag{4}$$

The *contravariant wave–propagation metric tensor* is defined by equation

$$G^{ij}_{(x)} = \frac{1}{2}\frac{\partial^2 G}{\partial p_i \partial p_j} \tag{5}$$

(*Rund, 1959,* eq. 1.5.4; *Hanyga, 1984,* eq. 2.3.76; *Asanov, 1985,* eq. 6.20; *Klimeš, 1994,* sec. 6; *Červený, 2002,* eq. 54). This contravariant tensor is also used in the dynamic ray tracing equations for anisotropic media (*Červený, 1972,* eq. 32). Tensor (5) may also be expressed as the partial derivative of the ray–velocity vector (4),

$$G^{ij}_{(x)} = \frac{\partial V^i}{\partial p_j} \quad . \tag{6}$$

Eigenvalues $G(x^m, p_n)$ of the Christoffel matrix are homogeneous functions of the second degree with respect to p_i, which is the direct consequence of equation (1) and of the definition of eigenvalues. Euler's theorem on homogeneous functions then yields $V^i p_i = G$ and

$$G^{ij}_{(x)}\, p_j = V^i \quad . \tag{7}$$

Relation $V^i p_i = G$ reads

$$V^i p_i = 1 \tag{8}$$

on all geometrical objects which satisfy equation (3).

The *covariant wave–propagation metric tensor* $G^{(x)}_{ij}$ is the inverse matrix to the contravariant wave–propagation metric tensor,

$$G^{(x)}_{ij} G^{jk}_{(x)} = \delta^k_i \tag{9}$$

(*Rund, 1959,* eq. 1.5.9). Equation (7) then yields

$$p_i = G^{(x)}_{ij} V^j \quad . \tag{10}$$

Anisotropic ray theory rays are the geodesics in Finsler space with the Finslerian metric determined by the wave–propagation metric tensor. Refer, e.g., to the books by *Rund (1959)* or *Asanov (1985)* for the Finsler geometry.

Mapping $V^i = V^i(x^m, p_n)$ defined by equation (4) may not be uniquely invertible with respect to p_i. Inverse mapping $p_i = p_i(x^m, V^n)$ can thus be defined only locally, in the vicinity of a chosen ray–velocity vector V^l. Inverting equation (6), we may express the covariant wave–propagation metric tensor locally as the partial derivative of slowness vector $p_i = p_i(x^m, V^n)$ with respect to the components of the ray–velocity vector,

$$G^{(x)}_{ij} = \frac{\partial p_i}{\partial V^j} \quad . \tag{11}$$

The local expression

$$\frac{1}{2}\frac{\partial G}{\partial V^i} = V^j G^{(x)}_{ji} = p_i \tag{12}$$

for the first derivatives of $G(x^m, p_n(x^k, V^l))$ with respect to V^i follows from equations (4), (10) and (11). The local expression

$$\frac{1}{2}\frac{\partial^2 G}{\partial V^i \partial V^j} = \frac{\partial p_i}{\partial V^j} = G^{(x)}_{ij} \tag{13}$$

(*Rund, 1959,* eq. 1.3.1) for the second derivatives of $G(x^m, p_n(x^k, V^l))$ with respect to V^i follows from equations (11) and (12).

3. RAY–CENTRED COORDINATES FOR ANISOTROPIC MEDIA

The q^3–coordinate line of ray–centred coordinates q^m is a selected ray, parametrized by travel time $q^3 = \tau$. Coordinates q^1 and q^2 lie, for fixed q^3, in the plane tangent to the wavefront (*Hanyga, 1982*; *Kendall, Guest and Thomson, 1992*; *Klimeš, 1994*). The Cartesian coordinates corresponding to q^m are

$$x^i = x^i_0(q^3) + h^i_M(q^3)\, q^M \quad , \tag{14}$$

where $x^i_0(\tau)$ are points of the selected ray. Relation (14) is both the definition of the ray–centred coordinates, and the transformation equation from ray–centred to Cartesian coordinates. Basis vectors $h^i_M(\tau)$ are chosen arbitrarily in the plane tangent to the wavefront,

$$p_i\, h^i_M = 0 \quad . \tag{15}$$

The transformation matrices, taken at the central ray, are

$$h^i_m = \frac{\partial x^i}{\partial q^m} \quad , \tag{16}$$

and

$$\hat{h}^m_i = \frac{\partial q^m}{\partial x^i} \quad , \tag{17}$$

where "contravariant basis vectors" h^i_1, h^i_2, h^i_3 of the ray–centred coordinate system and "covariant basis vectors" $\hat{h}^1_i$, $\hat{h}^2_i$, $\hat{h}^3_i$ of the ray–centred coordinate system are mutually biorthonormal, i.e. satisfy relations

$$h^i_m\,\hat{h}^m_j = \delta^i_j \quad , \qquad \hat{h}^m_i\,h^i_n = \delta^m_n \quad . \tag{18}$$

Vector h^i_3 is tangent to the ray,

$$h^i_3 = V^i \quad , \tag{19}$$

which follows from the ray tracing equations (*Červený, 1972,* eq. 13). Vector $\hat{h}^3_i$ is normal to the wavefront,

$$\hat{h}^3_i = p_i \quad , \tag{20}$$

see (2). Vectors $\hat{h}^1_i$ and $\hat{h}^2_i$ are normal to the ray,

$$V^i\,\hat{h}^M_i = 0 \quad , \tag{21}$$

as follows from (18) and (19). Equations (18), (19) and (20) can serve as the definition of transformation matrices h^i_m and $\hat{h}^m_i$. The ray–centred coordinates for anisotropic media may or may not converge to the ray–centred coordinates (Riemannian normal coordinates) for isotropic media used by *Luneburg (1944)*, *Babich and Buldyrev (1972)* or *Popov and Pšenčík (1978a, 1978b)*. The isotropic limit depends on additional constraints imposed on vectors h^i_1 and h^i_2 (*Klimeš, 1994*).

Vectors h^i_1 and h^i_2 need not be orthonormal. The non–orthonormality of vectors h^i_1 and h^i_2 may be quantified by positive–definite symmetric 2×2 matrix

$$h_{MN} = h^i_M\,h^i_N \quad . \tag{22}$$

Analogously, the non–orthonormality of vectors $\hat{h}^1_i$ and $\hat{h}^2_i$ may be quantified by positive–definite symmetric 2×2 matrix

$$\hat{h}^{MN} = \hat{h}^M_i\,\hat{h}^N_i \quad . \tag{23}$$

Vectors h^i_1 and h^i_2 would be orthonormal if $h_{MN} = \delta_{MN}$, whereas vectors $\hat{h}^i_1$ and $\hat{h}^i_2$ would be orthonormal if $\hat{h}^{MN} = \delta^{MN}$. Equations (8), (15), (19) and (22) yield

$$\det(h^i_m) = \varepsilon_{ijk}\,h^i_1 h^j_2 h^k_3 = \pm\sqrt{\det(h_{MN})}\;v p_k V^k = \pm v\sqrt{\det(h_{MN})} \quad , \tag{24}$$

where

$$v = (p_i p_i)^{-\frac{1}{2}} \tag{25}$$

is the *phase velocity.* Here $\varepsilon_{ijk}=\varepsilon^{ijk}$ is completely skew Levi–Civita's symbol, with $\varepsilon_{123}=\varepsilon_{231}=\varepsilon_{312}=1$. Equations (8), (20), (21) and (23) yield

$$\det(\hat{h}^m_i) = \varepsilon^{ijk}\,\hat{h}^1_i\hat{h}^2_j\hat{h}^3_k = \pm\sqrt{\det(\hat{h}^{MN})}\;V^{-1}V^k p_k = \pm V^{-1}\sqrt{\det(\hat{h}^{MN})} \quad , \tag{26}$$

where

$$V = \sqrt{V^i V^i} \tag{27}$$

is the *ray velocity* (group velocity). Equations (18), (24) and (26) yield

$$\det(h_{MN})\,\det(\hat{h}^{KL}) = \left(\frac{V}{v}\right)^2 \quad . \tag{28}$$

Note that if we choose vectors h^i_1 and h^i_2 orthonormal, $h_{MN} = \delta_{MN}$, we cannot choose vectors $\hat{h}^1_i$ and $\hat{h}^2_i$ orthonormal, and vice versa.

The contravariant wave–propagation metric tensor transformed into ray–centred coordinates,

$$G^{mn}_{(q)} = G^{ij}_{(x)}\, \hat{h}^m_i\, \hat{h}^n_j \quad , \tag{29}$$

is used in the dynamic ray tracing equations in ray–centred coordinates. Equations (7), (8), (20) and (21) yield

$$G^{M3}_{(q)} = G^{3M}_{(q)} = 0 \tag{30}$$

and

$$G^{33}_{(q)} = 1 \tag{31}$$

(*Klimeš, 1994,* eqs. 49e, 49f). The 2×2 submatrix $G^{MN}_{(q)}$ depends on the choice of vectors h^i_1 and h^i_2 in the plane tangent to the wavefront. Due to equations (30) and (31), the determinant of the 2×2 submatrix $G^{MN}_{(q)}$ equals the determinant of the 3×3 matrix $G^{mn}_{(q)}$,

$$\det\big(G^{MN}_{(q)}\big) = \det\big(G^{mn}_{(q)}\big) \quad . \tag{32}$$

Equations (26) and (29) yield

$$\det\big(G^{mn}_{(q)}\big) = V^{-2} \det\big(\hat{h}^{MN}\big)\, \det\big(G^{ij}_{(x)}\big) \quad . \tag{33}$$

The covariant wave–propagation metric tensor transformed into ray–centred coordinates is

$$G^{(q)}_{mn} = G^{(x)}_{ij}\, h^i_m\, h^j_n \quad . \tag{34}$$

Unlike contravariant wave–propagation metric tensor $G^{mn}_{(q)}$, covariant wave–propagation metric tensor $G^{(q)}_{mn}$ in ray–centred coordinates has found no significant application yet. Equations (8), (10), (15) and (19) yield

$$G^{(q)}_{M3} = G^{(q)}_{3M} = 0 \tag{35}$$

and

$$G^{(q)}_{33} = 1 \quad . \tag{36}$$

Identities (35) and (36) may also be obtained by inverting matrix $G^{mn}_{(q)}$ which satisfies (30) and (31). Due to equations (35) and (36), the determinant of 2×2 submatrix $G^{(q)}_{MN}$ equals the determinant of 3×3 matrix $G^{(q)}_{mn}$,

$$\det\big(G^{(q)}_{MN}\big) = \det\big(G^{(q)}_{mn}\big) \quad . \tag{37}$$

Equations (24) and (34) yield

$$\det\big(G^{(q)}_{mn}\big) = v^2 \det\big(h_{MN}\big)\, \det\big(G^{(x)}_{ij}\big) \quad . \tag{38}$$

4. CURVATURE OF THE SLOWNESS SURFACE

The *slowness surface* (phase–slowness surface, index surface) at spatial point x^m is composed of three sheets corresponding to the three eigenvalues of the Christoffel matrix. The sheet, corresponding to a selected eigenvalue G of the Christoffel matrix, is the set of slowness vectors p_i satisfying equation (3). Each of the three sheets of the slowness surface may be parametrized by unit vector vp_i.

Curvature matrix K^{MN} *of the slowness surface* (matrix of normal curvatures of the slowness surface) is defined with respect to two orthonormal vectors $\hat{e}_i^1$ and $\hat{e}_i^2$,

$$\hat{e}_i^M \, \hat{e}_i^N = \delta^{MN} \quad , \tag{39}$$

tangent to the slowness surface and thus perpendicular to the ray–velocity vector V^i,

$$\hat{e}_i^M \, V^i = 0 \quad . \tag{40}$$

The curvature matrix of the slowness surface is determined by the second derivatives of G, expressed with respect to orthonormal vectors $\hat{e}_i^M$, divided by the norm of the gradient of G, see (4), (5) and (27),

$$K^{MN} = V^{-1} \hat{e}_i^M \, G_{(x)}^{ij} \, \hat{e}_j^N \quad . \tag{41}$$

The orthonormal vectors may be chosen as

$$\hat{e}_i^M = \hat{h}_i^N \, [\hat{h}^{-\frac{1}{2}}]^{NM} \quad , \tag{42}$$

where $[\hat{h}^{-\frac{1}{2}}]^{MN}$ stands for the components of the $-\frac{1}{2}$th power of $2{\times}2$ matrix $\hat{h}^{MN}$. Note that the functions of symmetric matrices may conveniently be defined in terms of the eigenvectors and the functions of the eigenvalues. For option (42), equations (29) and (41) yield

$$K^{MN} = V^{-1} [\hat{h}^{-\frac{1}{2}}]^{MK} \, G_{(q)}^{KL} \, [\hat{h}^{-\frac{1}{2}}]^{LN} \quad . \tag{43}$$

Note that equation (43) would simplify to $K^{MN} = V^{-1} G_{(q)}^{MN}$ if vectors $\hat{h}_i^1$ and $\hat{h}_i^2$ were chosen orthonormal.

The *Gaussian curvature of the slowness surface* (total curvature of the slowness surface) is then

$$\det\bigl(K^{MN}\bigr) = \frac{\det\bigl(G_{(q)}^{MN}\bigr)}{V^2 \det\bigl(\hat{h}^{KL}\bigr)} \quad . \tag{44}$$

Equations (32), (33) and (44) yield the Gaussian curvature of the slowness surface in the form

$$\boxed{\det\bigl(K^{MN}\bigr) = V^{-4} \det\bigl(G_{(x)}^{ij}\bigr) \quad . \qquad (45)}$$

Equations (27), (39) and (40) yield

$$\hat{e}_i^M \, \hat{e}_j^M = \delta_{ij} - V^{-2} V^i V^j \quad . \tag{46}$$

Inserting (46) into (41), we obtain expression

$$\boxed{\tfrac{1}{2} K^{MM} = \tfrac{1}{2}\bigl(V^{-1} G_{(x)}^{ii} - V^{-3} V^i G_{(x)}^{ij} V^j\bigr) \qquad (47)}$$

for the *mean curvature of the slowness surface.* Equations (45) and (47) may also be used to calculate eigenvalues

$$\tfrac{1}{2}K^{MM} \pm \sqrt{(\tfrac{1}{2}K^{MM})^2 - \det(K^{MN})} \tag{48}$$

of the curvature matrix K^{MN} of the slowness surface (*principal normal curvatures of the slowness surface*).

5. CURVATURE OF THE RAY–VELOCITY SURFACE

The *ray–velocity surface* (group–velocity surface, Fresnel wave surface) at spatial point x^m is composed of three sheets corresponding to the three eigenvalues of the Christoffel matrix. The sheets of the ray–velocity surface are images of the corresponding sheets of the slowness surface with mapping

$$V^i = V^i(x^m, p_n) \tag{49}$$

defined by equation (4). Each sheet of the ray–velocity surface at spatial point x^m is thus a set of ray–velocity vectors (49), where slowness vector p_i satisfies equation (3). Each of the three sheets of the ray–velocity surface may be parametrized by unit vector vp_i, analogously to the sheets of the slowness surface, but cannot be uniquely parametrized by unit vector $V^{-1}V^i$.

The normal to the ray–velocity surface is given by slowness vector p_i, see (12). *Curvature matrix* K_{MN} *of the ray–velocity surface* (matrix of normal curvatures of the ray–velocity surface) is defined with respect to two orthonormal vectors e^i_1 and e^i_2,

$$e^i_M\, e^i_N = \delta_{MN} \quad , \tag{50}$$

tangent to the ray–velocity surface and thus perpendicular to slowness vector p_i,

$$e^i_M\, p_i = 0 \quad . \tag{51}$$

The curvature matrix of the ray–velocity surface is determined by the second derivatives of $G(x^m, p_n(V^k))$ with respect to V^k, transformed into orthonormal vectors e^i_M, and divided by the norm of the gradient of G with respect to V^k (*Beem, 1970*),

$$K_{MN} = v\, e^i_M\, G^{(x)}_{ij}\, e^j_N \quad , \tag{52}$$

see (12), (13) and (25). The orthonormal vectors may be chosen as

$$e^i_M = h^i_N\, [h^{-\frac{1}{2}}]_{NM} \quad , \tag{53}$$

where $[h^{-\frac{1}{2}}]_{MN}$ stands for the components of the $-\frac{1}{2}$th power of $2{\times}2$ matrix h_{MN}. For option (53), equations (34) and (52) yield

$$K_{MN} = v\, [h^{-\frac{1}{2}}]_{MK}\, G^{(q)}_{KL}\, [h^{-\frac{1}{2}}]_{LN} \quad . \tag{54}$$

Note that equation (54) would simplify to $K_{MN} = vG^{(q)}_{MN}$ if vectors h^i_1 and h^i_2 were chosen orthonormal.

The *Gaussian curvature of the ray–velocity surface* (total curvature of the ray–velocity surface) is then

$$\det\left(K^{MN}\right) = \frac{v^2\ \det\left(G^{(q)}_{KL}\right)}{\det\left(h_{MN}\right)} \quad . \tag{55}$$

Equations (37), (38) and (55) yield the Gaussian curvature of the ray–velocity surface in the form

$$\boxed{\det\left(K_{MN}\right) = v^4 \det\left(G^{(x)}_{ij}\right) \quad .} \tag{56}$$

Equations (25), (50) and (51) yield

$$e^i_M\, e^j_M = \delta^{ij} - v^2 p_i p_j \quad . \tag{57}$$

Inserting (57) into (52), we obtain expression

$$\boxed{\tfrac{1}{2} K_{MM} = \tfrac{1}{2}\left(v\, G^{(x)}_{ii} - v^3 p_i\, G^{(x)}_{ij}\, p_j\right)} \tag{58}$$

for the *mean curvature of the ray–velocity surface*. Equations (56) and (58) may also be used to calculate eigenvalues

$$\tfrac{1}{2} K_{MM} \pm \sqrt{\left(\tfrac{1}{2} K_{MM}\right)^2 - \det\left(K_{MN}\right)} \tag{59}$$

of the curvature matrix K_{MN} of the ray–velocity surface (*principal normal curvatures of the ray–velocity surface*).

The product of equations (45) and (56) yields relation

$$\det\left(K^{MN}\right)\ \det\left(K_{KL}\right) = \left(\frac{v}{V}\right)^4 \tag{60}$$

derived by *Grechka and Obolentseva* (*1993,* eq. 17) and *Vavryčuk and Yomogida* (*1996,* eq. B.6).

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