SUMMATION OF GAUSSIAN BEAMS AND PACKETS

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Abstract: Gaussian beams and packets may serve as building blocks of a wave field. The summation of Gaussian beams and packets is considerably comprehensive and flexible. It may be formulated in many ways, including the Maslov method and its various generalizations as special cases. The form of the summation depends primarily on the specification of the wave field, comprising also the system of Gaussian packets scattered from Gabor functions forming medium perturbations.

INTRODUCTION

The summation of Gaussian beams (Červený et al. [4]; Červený [3]) and packets overcomes the problems of the standard ray theory with caustics. We concentrate here on the representation of non–directional wave fields without pronounced diffractions rather than on the decomposition of directional beams or of diffracted waves. We thus omit the higher–order Gaussian beams (Bessel–Gaussian beams, Hermite–Gaussian beams, Laguerre–Gaussian beams) and the diffracted Gaussian beams.

GAUSSIAN BEAMS AND PACKETS

Under the "wave equation" we understand here a non–dissipative hyperbolic second–order partial differential equation (e.g., elastodynamic equation, system of Maxwell equations). The basic principles of the summation of Gaussian beams and packets are independent of a particular selection of the wave equation.

Gaussian beams. A Gaussian beam is a high–frequency asymptotic time–harmonic solution of the wave equation, with an approximately Gaussian profile perpendicularly to the central ray. It represents a bundle of complex–valued rays concentrated in a vicinity of a real–valued central ray. Evolution of the shape of a Gaussian beam along the central ray is determined by the respective equations.

Gaussian packets. Gaussian packets, also called (space-time) Gaussian beams (Ralston [17]) or quasi-photons (Babich et al. [2]), are high–frequency asymptotic space–time solutions of the wave equation. Their envelopes at any given time are nearly Gaussian functions. A Gaussian packet is concentrated to a real–valued space–time ray, like a Gaussian beam to a spatial ray. In a stationary medium, a Gaussian packet propagates along its real–valued spatial central ray. Evolution of the shape of a Gaussian packet is determined by the respective equations.

Optimization of the shape. Gaussian beams and packets are high–frequency approximate solutions of the wave equation. The accuracy of these approximate solutions depends on their shape, especially on their width. The "optimum" shape depends on the distance from the source (Klimeš [13]). To preserve the accuracy of the approximate decomposition of a wave field into Gaussian beams or packets, optimization of the shape of individual beams or packets should be supplemented with smoothing the dependence of the shape on summation parameters (Žáček [19]).

ASYMPTOTIC SUMMATION OF GAUSSIAN BEAMS AND PACKETS

Asymptotic decomposition into Gaussian beams. A general time–harmonic wave field given along a surface, decomposed into individual polarizations and expressed in terms of the amplitude and travel time, can be asymptotically expressed as the two–parametric integral superposition of Gaussian beams (Klimeš [10]). If the wave field is propagated by ray theory towards another surface, the integral superposition is independent on the selection of the surface for decomposition. That is why the ray–theory wavefield may be decomposed into Gaussian beams locally, even in its singular regions. The accuracy of the integral superposition of Gaussian beams thus depends on the optimization of the shape of beams only, not on the surface for decomposition. The variation of the optimum shape of beams with the distance from the source is quite different from the evolution of individual Gaussian beams along the same central ray. In another words, we decompose the wave field into different Gaussian beams for different propagation distances.

Asymptotic decomposition into Gaussian packets. A time–harmonic Gaussian beam may be expressed as a one–parametric integral superposition of space–time Gaussian packets. A general time–harmonic wave field, specified in terms of the amplitude and travel time, can be asymptotically expressed as the three–parametric integral superposition of space–time Gaussian packets (Klimeš [11]; [14]). The properties of the integral superposition of space–time Gaussian packets follow from the properties of the integral superposition of Gaussian beams.

Discretization error. The integral superposition of Gaussian beams is discretized into the summation in numerical algorithms. The error due to the discretization depends on frequency and on the shape of Gaussian beams and can be controlled by the selection of a discretization step (Klimeš [12]). The error due to the discretization of the integral superposition of Gaussian packets can be controlled analogously (Klimeš [14]).

Maslov methods. The standard ray theory is derived and expressed in the "coordinate representation", i.e., with respect to the spatial coordinates. In order to obtain various special cases of the summation of Gaussian beams and packets, the high–frequency approximation may be developed with respect to 3 appreciably general "representation coordinates" chosen in 6-D phase space, and then transformed to the coordinate representation.

The original Maslov method (Maslov [16]; Chapman et al. [5]) consists in weighted combination of the standard ray–theory approximation with the Maslov methods of the first, second and third order. The Maslov method of the first (second, third) order corresponds to one (two, three) spatial coordinate(s) replaced by the respective momentum coordinate(s). The travel time in this representation is obtained by the Legendre transform with respect to 1, 2 or 3 coordinates. The high–frequency asymptotic approximation of the wave field is then transformed to the coordinate representation by the 1-D, 2-D or 3-D Fourier transform, respectively. The Maslov method of the first (second) order represents the one(two)–parametric superposition of infinitely broad Gaussian beams with second–order derivatives of travel time vanishing along the summation lines (surfaces). The Maslov method of the third order represents the superposition of infinitely broad Gaussian packets with vanishing second–order derivatives of travel time.

Alonso et al. [1] selected each representation coordinate as a real–valued linear combination of a spatial coordinate and the corresponding momentum coordinate. The travel time in this representation is obtained by the fractional Legendre transform. The high–frequency asymptotic approximation of the wave field is then transformed to the coordinate representation by the 1-D, 2-D or 3-D fractional Fourier transform (Condon [6]). The resulting approximation represents the superposition of infinitely broad Gaussian beams (1-D, 2-D) or packets (3-D). Equivalent results have been achieved by application of the Maslov method in local curvilinear coordinates or with respect to "reference travel time" (Kendall et al. [9]). This approximation may artificially be supplemented with Gaussian windowing through the Gaussian–windowed Fourier transform (Alonso et al. [1]; Forbes et al. [7]; Kravtsov et al. [15]). Analogous Gaussian windowing may be introduced using the coherent state transform (Foster et al. [8]).

The Maslov method yields general superpositions of Gaussian beams or packets if the representation coordinates are sufficiently general complex–valued linear combinations of phase–space coordinates (Klimeš [11], eq. 24). The travel time in the new representation is obtained by the generalized Legendre transform (Klimeš [11], eq. 39). The high–frequency asymptotic approximation of the wave field is then transformed to the coordinate representation by the weighted Fourier transform (Klimeš [11], eq. 27), which is a generalization of both the fractional Fourier transform and Gaussian–windowed Fourier transform. Analogous (or even more general?) superpositions may also be obtained by means of the coherent state transform (Thomson [18]).

DECOMPOSITION OF A GENERAL WAVEFIELD INTO GAUSSIAN PACKETS

Assume a time–dependent wavefield specified along a given surface, and call it "time section". The trace of a Gaussian packet in the time section is approximately a Gabor function. The widths of the envelopes of Gabor functions are inversely proportional to the square root of frequency, not constant as for the Gabor transforms (discrete, integral) nor inversely proportional to frequency as for the wavelet transforms. Moreover, the shape of the packets has to be to some extent optimized and is thus often dependent on time and on the coordinates and wavenumbers along the surface. Žáček [20] generalized the integral Gabor transform towards the approximate integral expansion of a time section into the Gabor functions of varying shape.

SENSITIVITY OF WAVES TO HETEROGENEITIES

We decompose perturbations of the coefficients of the wave equation (e.g., elastic moduli and density in the elastodynamic equation) into Gabor functions. We consider a short–duration incident wavefield with a smooth broad–band frequency spectrum. The wavefield scattered by the perturbations is then composed of waves scattered by individual Gabor functions. Each Gabor function usually generates only few narrow–band space–time Gaussian packets propagating in specific directions (Žáček et al. [21]). Each Gaussian packet is sensitive to just a single linear combination of the coefficients of the wave equation. This information about the Gabor function is lost if the scattered Gaussian packet does not fall into the aperture covered by the receivers and into the legible frequency band.

MIGRATIONS

A "prestack depth migration" is a simple back–projection of a wave field, roughly approximating the inversion of wide–angle scattering. It often includes even additional rough approximations. The back–propagated wavefield is compared with the incident wavefield, forming an "image" (convolutional transform) of the gradient of a particular linear combination of the coefficients of the wave equation.

Gaussian packet migrations. The recorded wave field (time section) is decomposed into the Gaussian packets as described above. Individual Gaussian packets are back–propagated and compared with the incident wave field. The image of each back–propagated Gaussian packet is approximately formed by one or few Gabor functions. We thus obtain not only the image of small–scale structural heterogeneities, but also the relation between the time section and the heterogeneities.

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