Phase shift of the Green function due to caustics in anisotropic media

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Summary

The equations to determine the phase shift of the amplitude of the elastic Green function due to caustics in anisotropic media are presented.

Introduction

For the calculation of first–order anisotropic ray theory high–frequency asymptotic approximations of elastic wavefields or Green functions, the equations for ray tracing and dynamic ray tracing are required. The equations should also be supplemented with rules determining which ray to follow through singularities in ray tracing equations, and which of the square roots of the determinant of the matrix of geometrical spreading to select when crossing a caustic. For the Green function, the equations for the initial conditions corresponding to a unit force at the source point, including the initial value of the KMAH index, are also necessary. In layered and block models, the equations for reflection/transmission coefficients and for the transformation of the dynamic ray tracing at structural interfaces are also required.

The initial conditions for the Green function are the same as in the anisotropic medium locally homogeneous in a small vicinity of the point source. Refer, e.g., to Buchwald (1959). For the reflection/transmission coefficients refer, e.g., to Fedorov (1968). The dynamic ray tracing equations in the Cartesian coordinates, supplementing the ray tracing equations in anisotropic media, were derived by Červený (1972). For the transformation of the dynamic ray tracing at curved interfaces between anisotropic materials refer, e.g., to Gajewski & Pšenčík (1990).

In this paper, the rules determining the sign of the phase shift of the amplitude of the elastic Green function due to caustics in anisotropic media are presented. Unlike in isotropic media where the increment of the KMAH index determining the phase shift due to caustics is always positive, the increment of the KMAH index of S waves in anisotropic media may be either positive or negative. This is illustrated on a simple example.

The KMAH index of +1 means the phase shift of the complex–valued amplitude by $\frac{\pi}{2}$ in the direction corresponding to increasing time (or decreasing travel time) of time–harmonic wave. Hereinafter the time term of $\exp(-\mathrm{i}\omega t)$ is chosen for positive angular frequencies ω. Then the imaginary part of the matrix of the second travel–time derivatives of a Gaussian beam has to be positive–definite and the KMAH index of +1 corresponds to the amplitude multiplication factor of $\exp(-\mathrm{i}\frac{\pi}{2})$.

Simple (line) caustic

Assume simple (line) caustic at the ray bundle corresponding to a point source. At the simple caustic, submatrix $\mathbf{Q}_2$ of the paraxial–ray propagator matrix

$$\begin{pmatrix} \mathbf{Q}_1 & \mathbf{Q}_2 \\ \mathbf{P}_1 & \mathbf{P}_2 \end{pmatrix} \tag{1}$$

has just one zero eigenvalue. The ray–theory wavefield is the limiting case of the Gaussian beam infinitely narrow at the point source. We choose the real part of the matrix of the second travel–time derivatives of the Gaussian beam at the point source zero and denote the imaginary part by $\mathbf{Y}_0$. Inverse matrix $\mathbf{Y}_0^{-1}$ is thus positive–definite symmetric matrix limiting to the zero matrix. The matrix of geometrical spreading of the Gaussian beam is

$$\mathbf{Q} = [\mathbf{Q}_1 + \mathrm{i}\mathbf{Q}_2\mathbf{Y}_0]\mathbf{Q}_0 \quad , \tag{2}$$

where $\mathbf{Q}_0$ is the value of the matrix of geometrical spreading of the Gaussian beam at the point source. In the vicinity of the caustic we may approximate $\mathbf{Q}$ by the linear Taylor expansion with respect to increasing monotonic parameter γ_3 along the ray,

$$\mathbf{Q}(\Delta\gamma_3) \approx [\mathbf{Q}_1 + \mathrm{i}\mathbf{Q}_2\mathbf{Y}_0 + \mathbf{Q}_1'\Delta\gamma_3 + \mathrm{i}\mathbf{Q}_2'\mathbf{Y}_0\Delta\gamma_3]\mathbf{Q}_0 \quad , \tag{3}$$

centered at the caustic where $\Delta\gamma_3 = 0$. Here

$$\mathbf{Q}_1' = \frac{\mathrm{d}\mathbf{Q}_1}{\mathrm{d}\gamma_3} \quad , \qquad \mathbf{Q}_2' = \frac{\mathrm{d}\mathbf{Q}_2}{\mathrm{d}\gamma_3} \quad . \tag{4}$$

For very small $\mathbf{Y}_0^{-1}$ and $\Delta\gamma_3$ we may further approximate (3) by

$$\mathbf{Q}(\Delta\gamma_3) \approx [\mathbf{Q}_2 - \mathrm{i}\mathbf{Q}_1\mathbf{Y}_0^{-1} + \mathbf{Q}_2'\Delta\gamma_3]\ \mathrm{i}\mathbf{Y}_0\mathbf{Q}_0 \quad . \tag{5}$$

Introducing matrix

$$\mathbf{K} = \mathbf{Q}_2^{-1}\det\mathbf{Q}_2 \tag{6}$$

of rank 1 composed of the subdeterminants of $\mathbf{Q}_2$, determinant of (5) may be approximated by

$$\det\mathbf{Q}(\Delta\gamma_3) \approx [\mathrm{tr}(\mathbf{K}\mathbf{Q}_2')\Delta\gamma_3 - \mathrm{i}\ \mathrm{tr}(\mathbf{K}\mathbf{Q}_1\mathbf{Y}_0^{-1})]\ \det(\mathrm{i}\mathbf{Y}_0\mathbf{Q}_0) \quad . \tag{7}$$

If numbers

$$\overline{\overline{K}}_1 = \mathrm{tr}(\mathbf{K}\mathbf{Q}_1\mathbf{Y}_0^{-1}) \quad , \qquad K_2 = \mathrm{tr}(\mathbf{K}\mathbf{Q}_2') \tag{8}$$

have equal signs, determinant (7) passes for increasing $\Delta\gamma_3$ the origin of the complex plane counterclockwise and the increment of the KMAH index is $+1$, otherwise the increment of the KMAH index is -1.

Since $\mathbf{K}\mathbf{Q}_1$ is a symmetric matrix of rank 1 and $\mathbf{Y}_0^{-1}$ is positive–definite, sign of $\overline{\overline{K}}_1$ equals the sign of

$$\overline{K}_1 = \mathrm{tr}(\mathbf{K}\mathbf{Q}_1) \quad . \tag{9}$$

Because of the symplectic property of the paraxial–ray propagator matrix (1),

$$\mathrm{tr}(\mathbf{K}\mathbf{Q}_1)\ \mathrm{tr}(\mathbf{K}\mathbf{P}_2) = \mathrm{tr}(\mathbf{K}\mathbf{K}^{\mathrm{T}}) \quad , \tag{10}$$

see Appendix A, and the sign of $\overline{K}_1$ equals the sign of

$$K_1 = \mathrm{tr}(\mathbf{K}\mathbf{P}_2) \quad . \tag{11}$$

The sign of the increment of the KMAH index thus equals the sign of K_1K_2, where K_1 and K_2 are given by equations (11) and (8) with $\mathbf{K}$ defined by (6), no matter whether we consider 2×2 or 3×3 matrices in the ray–centred coordinates or 3×3 matrices in general coordinates.

Let us note that **(a)** K_2 always has the sign of $\det\mathbf{Q}_2$ beyond the caustic; **(b)** in an isotropic medium, both K_1 and K_2 have the same sign and the increment of the KMAH index is thus always positive there; **(c)** the increment of the KMAH index is negative if the wavefront curvature perpendicular to the caustic line has opposite sign than it always has in isotropic media.

Point caustic

Assume point caustic at the ray bundle corresponding to a point source. At the caustic, submatrix $\mathbf{Q}_2$ of the paraxial–ray propagator matrix (1) has two zero eigenvalues. If it is 2×2 matrix expressed in the ray–centred coordinates, it becomes the zero matrix.

Matrices 2×2 in ray–centred coordinates

For 2×2 matrices expressed in the ray–centred coordinates, determinant of (5) may be approximated by

$$\det\mathbf{Q}(\Delta\gamma_3) \approx \det[-\mathrm{i}\mathbf{Q}_1\mathbf{Y}_0^{-1}\mathbf{P}_2^{-1} + \mathbf{Q}_2'\mathbf{P}_2^{-1}\Delta\gamma_3]\ \det(\mathrm{i}\mathbf{P}_2\mathbf{Y}_0\mathbf{Q}_0) \quad . \tag{12}$$

Since

$$\mathbf{Q}_1\mathbf{Y}_0^{-1}\mathbf{P}_2^{-1} = (\mathbf{P}_2^{\mathrm{T}})^{-1}\mathbf{Y}_0^{-1}\mathbf{P}_2^{-1} \quad , \tag{13}$$

see ($A2$), is positive–definite symmetric 2×2 matrix, the phase shift due to caustics depends on the signs of the two eigenvalues of the symmetric 2×2 matrix $\mathbf{Q}_2'\mathbf{P}_2^{-1}$.

If

$$\det(\mathbf{Q}_2'\mathbf{P}_2^{-1}) > 0 \quad , \qquad \mathrm{tr}(\mathbf{Q}_2'\mathbf{P}_2^{-1}) > 0 \quad , \tag{14a}$$

both the eigenvalues of $-\mathrm{i}\mathbf{Q}_1\mathbf{Y}_0^{-1}\mathbf{P}_2^{-1} + \mathbf{Q}_2'\mathbf{P}_2^{-1}\Delta\gamma_3$ pass for increasing $\Delta\gamma_3$ the origin of the complex plane counterclockwise and the increment of the KMAH index is +2. If

$$\det(\mathbf{Q}_2'\mathbf{P}_2^{-1}) > 0 \quad , \qquad \mathrm{tr}(\mathbf{Q}_2'\mathbf{P}_2^{-1}) < 0 \quad , \tag{14b}$$

both the eigenvalues of $-\mathrm{i}\mathbf{Q}_1\mathbf{Y}_0^{-1}\mathbf{P}_2^{-1} + \mathbf{Q}_2'\mathbf{P}_2^{-1}\Delta\gamma_3$ pass for increasing $\Delta\gamma_3$ the origin of the complex plane clockwise and the increment of the KMAH index is −2. If

$$\det(\mathbf{Q}_2'\mathbf{P}_2^{-1}) < 0 \quad , \tag{14c}$$

one eigenvalue of $-\mathrm{i}\mathbf{Q}_1\mathbf{Y}_0^{-1}\mathbf{P}_2^{-1} + \mathbf{Q}_2'\mathbf{P}_2^{-1}\Delta\gamma_3$ passes for increasing $\Delta\gamma_3$ the origin of the complex plane counterclockwise whereas the other one passes the origin clockwise and the increment of the KMAH index is 0.

Matrices 3×3

Determinant $\det(\mathbf{Q}_2'\mathbf{P}_2^{-1})$ in conditions $(14a,b,c)$ may be replaced by the analogous determinant

$$\det(\widehat{\mathbf{Q}}_2'\widehat{\mathbf{P}}_2^{-1}) \tag{15}$$

of the corresponding 3×3 matrices in general coordinates because the third eigenvalue of 3×3 matrix $\widehat{\mathbf{Q}}_2'\widehat{\mathbf{P}}_2^{-1}$ is always positive (Klimeš 1994). Trace $\mathrm{tr}(\mathbf{Q}_2'\mathbf{P}_2^{-1})$ in conditions $(14a,b,c)$ should be replaced by

$$\mathrm{tr}(\widehat{\mathbf{Q}}_2'\widehat{\mathbf{P}}_2^{-1}) - \mathbf{p}^{\mathrm{T}}\widehat{\mathbf{Q}}_2'\widehat{\mathbf{P}}_2^{-1}\mathbf{p}\ (\mathbf{p}^{\mathrm{T}}\mathbf{p})^{-1} \quad , \tag{16}$$

where p is the slowness vector. Expression (16) is the trace of the projection of matrix $\widehat{\mathbf{Q}}_2'\widehat{\mathbf{P}}_2^{-1}$ onto the wavefront tangent plane.

Example

Assume 2-D homogeneous anisotropic medium with the triplication on the S–wave group–velocity surface. Let us select the ray perpendicular to the reverse branch of the group–velocity surface as the axial ray. Assume that the homogeneous anisotropic medium is separated from the homogeneous isotropic medium by the planar interface perpendicular to the axial ray. For the sake of simplicity we may assume that the (phase and group) velocities in the direction of the axial ray are similar (or equal) in both media.

Paraxial rays propagating in the anisotropic medium to one side from the axial ray have the slowness vectors pointing to the other side from the axial ray, see Figure 1. When entering the isotropic medium, the paraxial rays are refracted to the direction of the slowness vector and cross the axial ray at the caustic.

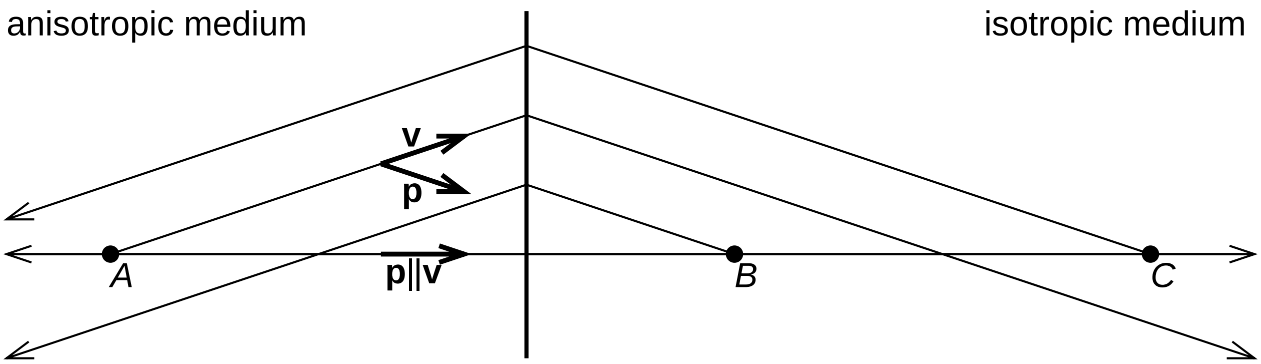

Figure 1. Sketch of axial ray and paraxial rays from points A, B, C in the discussed example. In the anisotropic halfspace, paraxial group–velocity vector **v** and slowness vector **p** are declined in opposite directions.

Phase shift in anisotropic media

Let us consider 3 points on the axial ray: Point A in the anisotropic medium, point B situated in the isotropic medium between the interface and the caustic corresponding to the point source at A, and point C situated in the isotropic medium beyond the caustic corresponding to the point source at A. The paraxial rays from all 3 points are depicted in Figure 1. The paraxial rays of the same perpendicular component of the slowness vector are parallel in each of the halfspaces.

The phase shift along the four rays between points A, B and C is accumulated in the following way:

$C{\rightarrow}A$: There is no caustic along the ray from point C to point A. If the KMAH index is defined zero at the point source in the isotropic medium, the resulting KMAH index at point A is 0.

$A{\rightarrow}C$: The initial KMAH index at point A on the reverse wavefront branch must be by 1 smaller than on the forward branch, i.e. -1 in our 2-D case with zero initial value of KMAH index in the isotropic medium. There is a single caustic along the ray from point A to point C, located in the isotropic medium, causing the KMAH index increment of $+1$. The resulting KMAH index at point C is then 0.

$A{\rightarrow}B$: The initial KMAH index at point A is again -1, but there is no caustic along the ray from point A to point B. The resulting KMAH index at point C is now -1.

$B{\rightarrow}A$: The initial KMAH index at point B is 0, as for $C{\rightarrow}A$. There is a single caustic along the ray from point B to point A, located in the anisotropic medium on the reverse branch of the group–velocity surface, causing the KMAH index increment of -1. The resulting KMAH index at point A is then -1.

Appendix A

Symplectic property of the paraxial–ray propagator matrix (1) may be expressed in the form of

$$\begin{pmatrix} \mathbf{Q}_1^{\mathrm{T}} & \mathbf{P}_1^{\mathrm{T}} \\ \mathbf{Q}_2^{\mathrm{T}} & \mathbf{P}_2^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{Q}_1 & \mathbf{Q}_2 \\ \mathbf{P}_1 & \mathbf{P}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{pmatrix} \quad . \tag{A1}$$

It implies, among others, symmetry of matrices $\mathbf{Q}_1\mathbf{Q}_2^{\mathrm{T}}$ and $\mathbf{Q}_2^{\mathrm{T}}\mathbf{P}_2$, consequently of matrices $\mathbf{K}\mathbf{Q}_1$ and $\mathbf{P}_2\mathbf{K}$, and identity

$$\mathbf{P}_2^{\mathrm{T}}\mathbf{Q}_1 - \mathbf{P}_1^{\mathrm{T}}\mathbf{Q}_2 = \mathbf{1} \quad . \tag{A2}$$

Multiplying ($A2$) with $\mathbf{K}^{\mathrm{T}}$ and $\mathbf{K}$ from left and right, respectively, we arrive at

$$\mathbf{K}^{\mathrm{T}}\mathbf{P}_2^{\mathrm{T}}\,\mathbf{Q}_1\mathbf{K} = \mathbf{K}^{\mathrm{T}}\mathbf{K} \quad . \tag{A3}$$

Since $\mathbf{P}_2\mathbf{K}$ is a symmetric matrix and $\mathbf{K}$ is of rank 1, equation (10) is the direct consequence of ($A3$).

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