Calculation of the third and higher travel–time derivatives in isotropic and anisotropic media

Luděk Klimeš, Department of Geophysics, Charles University, Prague, Czech Republic

Summary

The explicit equations for the third and higher partial travel–time derivatives in both isotropic and anisotropic media are derived. The third and higher partial travel–time derivatives may be calculated by simple numerical quadratures along rays.

1 Introduction

In a general heterogeneous model, travel time and its first partial derivatives may be calculated by solving numerically a set of nonlinear ordinary differential equations called ray tracing equations. The second partial travel–time derivatives may be calculated by solving numerically a set of linear ordinary differential equations called dynamic ray tracing equations. The second partial travel–time derivatives, calculated by dynamic ray tracing, has found many geophysical applications. The third and higher partial travel–time derivatives may be computed by simple numerical quadratures along rays and, as the author believes, may also find some geophysical applications soon.

The general form of equations for the third and higher partial travel–time derivatives was suggested by Babich, Buldyrev & Molotkov (1985). In this paper, explicit equations for the third and higher partial travel–time derivatives in Cartesian coordinates are presented. The travel–time derivatives are expressed in the form of integrals along the ray, with the proper initial values. The integrands are composed of partial travel–time derivatives of orders lower than the computed, and of the derivatives of the Hamiltonian, which are often composed of partial slowness (or velocity) derivatives.

Although the partial travel–time derivatives in the ray–centred coordinate system could also be calculated, most applications are assumed to require the partial derivatives rather in global Cartesian coordinates or, possibly, in some other model or configuration coordinates. That is why principal attention is devoted to calculation in Cartesian coordinates. However, all equations expressed in terms of a general Hamiltonian are applicable to any coordinate system.

2 Equations for the third and higher partial travel–time derivatives

We shall use the Einstein summation convention, and denote the partial derivatives by indices following a comma. For instance,

$$\tau_{,ij\ldots n} = \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\cdots\frac{\partial}{\partial x_n}\tau \quad , \tag{1}$$

denotes the partial derivatives of travel time τ in coordinates x_i which may or may not be Cartesian. We also introduce the "multiindices" in order to make the equations more concise and easy to survey. For example,

$$\tau_{\begin{array}{l},rij\\,rik\\,rjk\end{array}}\ \tau_{\begin{array}{l},rk\\,rj\\,ri\end{array}} \tag{2}$$

stands for

$$\tau_{,rij}\tau_{,rk} + \tau_{,rik}\tau_{,rj} + \tau_{,rjk}\tau_{,ri} \quad . \tag{3}$$

In both isotropic and anisotropic media, we assume the travel time $\tau(x_i)$ determined by the eikonal equation of general form

$$H\left(x_i, \tau_{,j}(x_m)\right) = 0 \quad , \tag{4}$$

where function $H(x_i, p_j)$ of six phase–space coordinates is the Hamiltonian. The corresponding ray tracing equations than have the form of

$$\frac{\mathrm{d}}{\mathrm{d}\gamma_3}x_i = \frac{\partial}{\partial p_i}H(x_q,p_r) \ , \qquad \frac{\mathrm{d}}{\mathrm{d}\gamma_3}p_i = -\frac{\partial}{\partial x_i}H(x_q,p_r) \ , \tag{5}$$

where parameter γ_3 along rays is determined by the form of the Hamiltonian.

If $\tau(x_i)$ is the solution of the eikonal equation (4), function

$$h\left(x_k\right) = H\left(x_i, \tau_{,j}(x_m)\right) \tag{6}$$

is obviously constant (equal to zero) and all its partial derivatives thus vanish,

$$\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\ldots\frac{\partial}{\partial x_n}h\left(x_q\right) = 0 \quad . \tag{7}$$

In terms of the Hamiltonian, this equation reads

$$\left(\frac{\partial}{\partial x_i} + \tau_{,ia}\frac{\partial}{\partial p_a}\right)\left(\frac{\partial}{\partial x_j} + \tau_{,jb}\frac{\partial}{\partial p_b}\right)\ldots\left(\frac{\partial}{\partial x_n} + \tau_{,nf}\frac{\partial}{\partial p_f}\right)H = 0 \ . \tag{8}$$

Here the phase–space partial derivatives are computed at point $(x_q, p_r) = (x_q, \tau_{,r}(x_s))$.

We define the matrices of the partial derivatives of coordinates x_i and of the slowness vector components $\tau_{,i}$ with respect to 3 ray parameters: ray take–off parameters γ_1, γ_2, and parameter γ_3 along rays,

$$Q_{ia} = \frac{\partial x_i}{\partial \gamma_a} \quad , \qquad P_{ia} = \frac{\partial \tau_{,i}}{\partial \gamma_a} \quad . \tag{9}$$

Let us emphasize that the definition of matrices Q_{ia} and P_{ia} depends on the parameter γ_3 along rays, which is in turn determined by the form of the Hamiltonian. Usage of matrices Q_{ia} and P_{ia}, defined for a particular γ_3, in the equations for the travel–time derivatives derived from the Hamiltonian corresponding to different γ_3 must be avoided. Matrices Q_{ia} and P_{ia} are often calculated for $\gamma_3 = \tau$. The transformation of matrix Q_{ia} to the one defined for different γ_3 is described in Section 3.2.

Equation

$$P_{ia} = \tau_{,ij}\ Q_{ja} \tag{10}$$

is a direct consequence of the above definitions. The dynamic ray tracing system

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\gamma_3}Q_{ia} &= \frac{\partial}{\partial p_i}\frac{\partial}{\partial x_j}H(x_q,p_r)Q_{ja} + \frac{\partial}{\partial p_i}\frac{\partial}{\partial p_j}H(x_q,p_r)P_{ja} \\ \frac{\mathrm{d}}{\mathrm{d}\gamma_3}P_{ia} &= -\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}H(x_q,p_r)Q_{ja} - \frac{\partial}{\partial x_i}\frac{\partial}{\partial p_j}H(x_q,p_r)P_{ja}\end{aligned} \tag{11}$$

can be obtained by differentiating the ray tracing system (5) with respect to γ_a. After insertion of (10), the first equation of the dynamic ray tracing system (11) becomes

$$\frac{\mathrm{d}}{\mathrm{d}\gamma_3} Q_{ia} = Q_{ja} \left(\frac{\partial}{\partial x_j} + \tau_{,jk} \frac{\partial}{\partial p_k} \right) \frac{\partial}{\partial p_i} H \quad . \tag{12}$$

Let us now consider the covariant derivatives $\tau_{,ijk...mn} Q_{ia} Q_{jb} Q_{kc} \cdots Q_{me} Q_{nf}$ of the travel–time with respect to γ_a, differentiate them along a ray with respect to γ_3, and insert equations (5) and (12) for $\frac{\mathrm{d}}{\mathrm{d}\gamma_3} x_i$ and $\frac{\mathrm{d}}{\mathrm{d}\gamma_3} Q_{ia}$, respectively,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\gamma_3} [\tau_{,ijk...mn} Q_{ia} Q_{jb} Q_{kc} \cdots Q_{me} Q_{nf}] \\
&= \Bigg[\tau_{,rijk...mn} \frac{\partial}{\partial p_r} H \\
&\quad + \tau_{,rjk...mn} \left(\frac{\partial}{\partial x_i} + \tau_{,is} \frac{\partial}{\partial p_s} \right) \frac{\partial}{\partial p_r} H \\
&\quad + \tau_{,irk...mn} \left(\frac{\partial}{\partial x_j} + \tau_{,js} \frac{\partial}{\partial p_s} \right) \frac{\partial}{\partial p_r} H \\
&\quad \vdots \\
&\quad + \tau_{,ijk...mr} \left(\frac{\partial}{\partial x_n} + \tau_{,ns} \frac{\partial}{\partial p_s} \right) \frac{\partial}{\partial p_r} H \Bigg] \\
&\qquad \times Q_{ia} Q_{jb} Q_{kc} \cdots Q_{me} Q_{nf} \ .
\end{aligned} \tag{13}$$

The partial travel–time derivatives may then be expressed in the form of the integral along the ray,

$$\boxed{\begin{aligned}
&\tau_{,ijk...mn}(\gamma_3) = \tau_{,ijk...mn}(\gamma_3^0) \\
&+ Q^{-1}_{ai} Q^{-1}_{bj} Q^{-1}_{ck} \cdots Q^{-1}_{em} Q^{-1}_{fn} \int_{\gamma_3^0}^{\gamma_3} \mathrm{d}\gamma_3 \ L_{abc...ef} \ ,
\end{aligned}} \tag{14}$$

where

$$\boxed{L_{abc...ef} = K_{ijk...mn} Q_{ia} Q_{jb} Q_{kc} \cdots Q_{me} Q_{nf} \ ,} \tag{15}$$

with

$$\begin{aligned}
K_{ijk...mn} =& \tau_{,rijk...mn} \frac{\partial}{\partial p_r} H \\
&+ \tau_{,rjk...mn} \left(\frac{\partial}{\partial x_i} + \tau_{,is} \frac{\partial}{\partial p_s} \right) \frac{\partial}{\partial p_r} H \\
&+ \tau_{,irk...mn} \left(\frac{\partial}{\partial x_j} + \tau_{,js} \frac{\partial}{\partial p_s} \right) \frac{\partial}{\partial p_r} H \\
&\vdots \\
&+ \tau_{,ijk...mr} \left(\frac{\partial}{\partial x_n} + \tau_{,ns} \frac{\partial}{\partial p_s} \right) \frac{\partial}{\partial p_r} H \ .
\end{aligned} \tag{16}$$

In (14), we have used Q^{-1}_{ai} to denote the components of the matrix inverse to matrix Q_{ia}.

The travel–time derivatives on the right–hand side of (16) coincide with those terms on the left–hand side of equation (8) containing the highest–order travel–time derivatives, and are cancelled for the third and higher travel–time derivatives if subtracting, for the nonzeroth derivatives, (8) from (16),

$$\boxed{\begin{aligned}
K_{ijk...mn} =& \tau_{,rijk...mn} \frac{\partial}{\partial p_r} H \\
&+ \tau_{,rjk...mn} \left(\frac{\partial}{\partial x_i} + \tau_{,is} \frac{\partial}{\partial p_s} \right) \frac{\partial}{\partial p_r} H \\
&+ \tau_{,irk...mn} \left(\frac{\partial}{\partial x_j} + \tau_{,js} \frac{\partial}{\partial p_s} \right) \frac{\partial}{\partial p_r} H \\
&\vdots \\
&+ \tau_{,ijk...mr} \left(\frac{\partial}{\partial x_n} + \tau_{,ns} \frac{\partial}{\partial p_s} \right) \frac{\partial}{\partial p_r} H \\
-& \left(\frac{\partial}{\partial x_i} + \tau_{,ia} \frac{\partial}{\partial p_a} \right) \left(\frac{\partial}{\partial x_j} + \tau_{,jb} \frac{\partial}{\partial p_b} \right) \ldots \left(\frac{\partial}{\partial x_n} + \tau_{,nf} \frac{\partial}{\partial p_f} \right) H \ .
\end{aligned}} \tag{17}$$

The remaining travel–time derivatives in (17) are of lower orders than derivative (14) to be calculated, and should thus have been previously computed.

The last term in equation (17), present for the nonzeroth travel–time derivatives, contains all derivatives of Hamiltonian H from the first to the order equal to the number of indices of $K_{ij...n}$. It is composed of all possible terms of the form of

$$H^{,ab...f}_{,...} \tau_{,a...} \tau_{,b...} \cdots \tau_{,f...} \quad , \tag{18}$$

where we have put

$$H^{,ab...f}_{,ij\cdots n} = \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} \cdots \frac{\partial}{\partial x_n} \frac{\partial}{\partial p_a} \frac{\partial}{\partial p_b} \cdots \frac{\partial}{\partial p_f} H \quad . \tag{19}$$

The subscripts in these "terms" are represented by all possible combinations of the subscripts of $K_{ij...n}$, such that each $\tau_{,a...}$ has at least one of these subscripts in place of ..., whereas $H^{,ab...f}_{,...}$ need not have any subscript. $H^{,ab...f}_{,...}$ also need not have any superscript. Each possible summand is contained within the last term in equation (17) just once. The terms of forms $H^{,a}\tau_{,a...}$, $H^{,a}_{,i}\tau_{,a...}$ and $H^{,ab}\tau_{,ai}\tau_{,b...}$ in (17) cancel out for the third and higher travel–time derivatives, the other terms are retained.

Equations (14), (15) and (17) may be used to compute the third and higher partial travel–time derivatives in both isotropic and anisotropic models by numerical quadrature along the traced rays.

3 Examples of the integrands in general media

3.1 Zeroth, first and second travel–time derivatives

For the zeroth–order derivatives, equation (17) takes the form of

$$K = H^{,r} \tau_{,r} \quad , \tag{20}$$

which may be used to compute the travel time along rays for parameter γ_3 other than the travel time,

$$\frac{\partial \tau}{\partial \gamma_3} = K \quad . \tag{21}$$

Equation (17), specified for the first–order derivatives,

$$K_i = -H_{,i} + H_{,i}^{,r}\tau_{,r} + H^{,rs}\tau_{,r}\tau_{,si} \quad , \tag{22}$$

cannot be used for the calculation of the travel–time gradient, but will turn out to be valuable for the transformation of matrix Q_{ia} to the one defined for different γ_3.

On the other hand, equation (17), specified for the second–order travel–time derivatives,

$$K_{ij} = -H_{,ij} + H^{,ab}\tau_{,ai}\tau_{,bj} \quad , \tag{23}$$

has probably found no practical application yet.

3.2 Transformation of matrix Q

Denote Q_{ia}^{τ} matrix Q_{ia} defined for $\gamma_3 = \tau$. The transformation of the third column,

$$Q_{i3} = K\, Q_{i3}^{\tau} \quad , \tag{24}$$

is obvious from the definition of Q_{ia} and equation (21).

The first two columns Q_{iM}^{τ} of the matrix defined for $\gamma_3 = \tau$ are obviously tangent to the wavefront and equal to the columns Q_{iM} projected along the rays onto the wavefront. The difference between Q_{iM} and Q_{iM}^{τ} is thus parallel with vector $\frac{\mathrm{d}}{\mathrm{d}\gamma_3}x_i = H^{,i}$ tangent to the ray,

$$\boxed{Q_{iM} = Q_{iM}^{\tau} + H^{,i}\, g_M \quad .} \tag{25}$$

Multiplication of (25) with travel–time gradient $\tau_{,i}$ yields

$$g_M = \frac{\tau_{,i}Q_{iM}}{H^{,i}\tau_{,i}} = \frac{1}{K}\tau_{,i}Q_{iM} \quad , \tag{26}$$

see (20). The derivative of g_M along the ray is

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\gamma_3}g_M &= -\frac{1}{K^2}\frac{\partial K}{\partial\gamma_3}\tau_{,i}Q_{iM} + \frac{1}{K}\frac{\mathrm{d}}{\mathrm{d}\gamma_3}\left[\tau_{,i}Q_{iM}\right] \\ &= -\frac{1}{K}\frac{\partial K}{\partial\gamma_3}g_M + \frac{1}{K}K_iQ_{iM} \ ,\end{aligned} \tag{27}$$

see (14) with (15). Now we insert (25) on the right–hand side of (27),

$$\frac{\mathrm{d}}{\mathrm{d}\gamma_3}g_M = -\frac{1}{K}\frac{\partial K}{\partial\gamma_3}g_M + \frac{1}{K}K_iH^{,i}g_M + \frac{1}{K}K_iQ_{iM}^{\tau} \ . \tag{28}$$

The derivative of (20) along the ray is

$$\begin{aligned}\frac{\partial K}{\partial\gamma_3} &= H_{,r}^{,i}\frac{\partial x_r}{\partial\gamma_3}\tau_{,i} + H^{,ir}\tau_{,rs}\frac{\partial x_s}{\partial\gamma_3}\tau_{,i} + H^{,i}\frac{\partial\tau_{,i}}{\partial\gamma_3} \\ &= H_{,r}^{,i}H^{,r}\tau_{,i} + H^{,ir}\tau_{,rs}H^{,s}\tau_{,i} - H^{,i}H_{,i} = K_iH^{,i} \ ,\end{aligned} \tag{29}$$

see (22). The insertion of (29) into (28) yields

$$\boxed{g_M(\gamma_3) = g_M(\gamma_3^0) + \int_{\gamma_3^0}^{\gamma_3}\mathrm{d}\gamma_3\ \frac{1}{K}K_iQ_{iM}^{\tau} \quad .} \tag{30}$$

The initial values $g_M(\gamma_3^0)$ are determined through equation (26).

3.3 Restricting the form of the Hamiltonian

Assume that the Hamiltonian is chosen in the form of

$$H(x_q,p_r) = \overline{H}(x_q,p_r) - V(x_q) \quad , \tag{31}$$

where $\overline{H}(x_q,p_r)$ is a *homogeneous function of the second order* with respect to slowness vector p_k,

$$\overline{H}^{,a}p_a = 2\overline{H} \quad . \tag{32}$$

Then also

$$H_{,\dots}^{,a}\, p_a = 2\overline{H}_{,\dots} \quad . \tag{33}$$

Differentiating (33) with respect to p_b yields

$$H_{,\dots}^{,ab}p_a = \overline{H}_{,\dots}^{,b} \quad . \tag{34}$$

Insertion of (33), and subsequently (4) with (31), into (20) yields

$$K = 2\overline{H} = 2V \quad . \tag{35}$$

Insertion of (33) and (34), and subsequently (31), into (22) yields

$$K_i = -H_{,i} + 2\overline{H}_{,i} + \overline{H}^{,s}\tau_{,si} = 2V_{,i} + H_{,i} + H^{,s}\tau_{,si} = 2V_{,i} \ , \tag{36}$$

see (8).

3.4 Third and higher travel–time derivatives

Equation (17) is important especially for the calculation of the third travel–time partial derivatives,

$$K_{ijk} = -H_{,ijk} - H^{,a}_{\substack{,ij\\,ik\\,jk}}\ \tau_{\substack{,ak\\,aj\\,ai}} - H^{,ab}_{\substack{,i\\,j\\,k}}\tau_{\substack{,aj\\,ai\\,ai}}\ \tau_{\substack{,ak\\,ak\\,aj}} - H^{,abc}\tau_{,ai}\tau_{,bj}\tau_{,ck} \ , \tag{37}$$

the fourth travel–time partial derivatives,

$$\begin{aligned}K_{ijkl} = &- H_{,ijkl} - H^{,a}_{\substack{,ijk\\,ijl\\,ikl\\,jkl}}\ \tau_{\substack{,al\\,ak\\,aj\\,ai}} - H^{,a}_{\substack{,ij\\,ik\\,jk\\,il\\,jl\\,kl}}\ \tau_{\substack{,akl\\,ajl\\,ail\\,ajk\\,aik\\,aij}} - H^{,ab}_{\substack{,ij\\,ik\\,jk\\,il\\,jl\\,kl}}\ \tau_{\substack{,ak\\,aj\\,ai\\,aj\\,ai\\,ai}}\ \tau_{\substack{,bl\\,bl\\,bl\\,bk\\,bk\\,bj}} \\ &- H^{,ab}_{\substack{,i\\,j\\,i\\,k\\,j\\,k\\,i\\,l\\,j\\,l\\,k\\,l}}\tau_{\substack{,aj\\,ai\\,ak\\,ai\\,ak\\,aj\\,al\\,ai\\,al\\,aj\\,al\\,ak}}\ \tau_{\substack{,bkl\\,bkl\\,bjl\\,bjl\\,bil\\,bil\\,bjk\\,bjk\\,bik\\,bik\\,bij\\,bij}} - H^{,ab}\tau_{\substack{,aij\\,aik\\,ajk}}\ \tau_{\substack{,bkl\\,bjl\\,bil}} - H^{,abc}_{\substack{,i\\,j\\,k\\,l}}\tau_{\substack{,aj\\,ai\\,ai\\,ai}}\ \tau_{\substack{,bk\\,bk\\,bj\\,bj}}\ \tau_{\substack{,cl\\,cl\\,cl\\,ck}} \\ &- H^{,abc}\tau_{\substack{,aij\\,aik\\,ajk\\,ail\\,ajl\\,akl}}\ \tau_{\substack{,bk\\,bj\\,bi\\,bj\\,bi\\,bi}}\ \tau_{\substack{,cl\\,cl\\,cl\\,ck\\,ck\\,cj}} - H^{,abcd}\tau_{,ai}\,\tau_{,bj}\,\tau_{,ck}\,\tau_{,dl} \ ,\end{aligned} \tag{38}$$

and so on.

4 Examples of the integrands in isotropic media

In isotropic models, the Hamiltonian is a function of p_kp_k, but is independent of the direction of the slowness vector p_i. If the Hamiltonian is chosen in the form of

$$H(x_q,p_r) = \overline{H}(x_q,p_r) - V(x_q) \quad , \tag{39}$$

where $\overline{H}(x_q,p_r)$ is a *homogeneous function of the first order* with respect to $|p|^2 = p_kp_k$, all partial derivatives of the Hamiltonian containing the third or higher partial derivatives with respect to the slowness–vector components are zero,

$$H_{,\dots}^{,abc\dots} = 0 \quad , \tag{40}$$

which considerably simplifies equations such as (37) and (38),

$$K_{ijk} = -H_{,ijk} - H^{,a}_{\substack{,ij\\,ik\\,jk}}\ \tau_{\substack{,ak\\,aj\\,ai}} - H^{,ab}_{\substack{,i\\,j\\,k}}\tau_{\substack{,aj\\,ai\\,ai}}\ \tau_{\substack{,bk\\,bk\\,bj}} \quad , \tag{41}$$

$$K_{ijkl} = - H_{,ijkl} - H^{,a}_{\substack{,ijk\\,ijl\\,ikl\\,jkl}} \tau_{\substack{,al\\,ak\\,aj\\,ai}} - H^{,a}_{\substack{,ij\\,ik\\,jk\\,il\\,jl\\,kl}} \tau_{\substack{,akl\\,ajl\\,ail\\,ajk\\,aik\\,aij}} - H^{,ab}_{\substack{,ij\\,ik\\,jk\\,il\\,jl\\,kl}} \tau_{\substack{,ak\\,aj\\,ai\\,aj\\,ai\\,ai}} \tau_{\substack{,bl\\,bl\\,bl\\,bk\\,bk\\,bj}}$$

$$- H^{,ab}_{\substack{,i\\,j\\,i\\,k\\,j\\,k\\,i\\,l\\,j\\,l\\,k\\,l}} \tau_{\substack{,aj\\,ai\\,ak\\,ai\\,ak\\,aj\\,al\\,ai\\,al\\,aj\\,al\\,ak}} \tau_{\substack{,bkl\\,bkl\\,bjl\\,bjl\\,bil\\,bil\\,bjk\\,bjk\\,bik\\,bik\\,bij\\,bij}} - H^{,ab} \tau_{\substack{,aij\\,aik\\,ajk}} \tau_{\substack{,bkl\\,bjl\\,bil}} \ , \tag{42}$$

and so on.

4.1 Integration with respect to travel time

For example, if we choose the Hamiltonian in Cartesian coordinates in the special form of

$$H(x_q, p_r) = \tfrac{1}{2} v^2 p_k p_k - \tfrac{1}{2} \quad , \tag{43}$$

$v = v(x_i)$ being the phase velocity, then $\gamma_3 = \tau$ in (14), and only partial derivatives

$$H_{,\ldots} = \tfrac{1}{2}\left(v^2\right)_{,\ldots} p_k p_k \ , \quad H^{,a}_{,\ldots} = \left(v^2\right)_{,\ldots} p_a \ ,$$
$$H^{,ab}_{,\ldots} = \left(v^2\right)_{,\ldots} \delta_{ab} \tag{44}$$

of the Hamiltonian are nonzero. The travel–time derivatives on the right–hand side of equation (16) with (44) may then be cancelled, for the third and higher travel–time derivatives, by subtracting equation (7) with (43) from equation (16),

$$\boxed{K_{ijk\ldots mn} = v^2 \tau_{,r} \tau_{,rijk\ldots mn} + v^2 \tau_{\substack{,ri\\,rj\\\vdots\\,rn}} \tau_{\substack{,rjk\ldots mn\\,irk\ldots mn\\\vdots\\,ijk\ldots mr}} + (v^2)_{\substack{,i\\,j\\\vdots\\,n}} \tau_{\substack{,r\\,r\\\vdots\\,r}} \tau_{\substack{,rjk\ldots mn\\,irk\ldots mn\\\vdots\\,ijk\ldots mr}} - \left(\tfrac{1}{2} v^2 \tau_{,r} \tau_{,r} - \tfrac{1}{2}\right)_{,ijk\ldots mn} \ .} \tag{45}$$

Equations such as (41) or (42) for the third and higher travel–time derivatives then take the form

$$K_{ijk} = -\left(\tfrac{1}{2} v^2\right)_{,ijk} \tau_{,r} \tau_{,r} - \left(v^2\right)_{\substack{,ij\\,ik\\,jk}} \tau_{\substack{,r\\,r\\,r}} \tau_{\substack{,rk\\,rj\\,ri}} - \left(v^2\right)_{\substack{,i\\,j\\,k}} \tau_{\substack{,rj\\,ri\\,ri}} \tau_{\substack{,rk\\,rk\\,rj}} \ , \tag{46}$$

$$K_{ijkl} = -\left(\tfrac{1}{2} v^2\right)_{,ijkl} \tau_{,r} \tau_{,r} - \left(v^2\right)_{\substack{,ijk\\,ijl\\,ikl\\,jkl}} \tau_{\substack{,r\\,r\\,r\\,r}} \tau_{\substack{,rl\\,rk\\,rj\\,ri}} - \left(v^2\right)_{\substack{,ij\\,ik\\,jk\\,il\\,jl\\,kl}} \tau_{\substack{,r\\,r\\,r\\,r\\,r\\,r}} \tau_{\substack{,rkl\\,rjl\\,ril\\,rjk\\,rik\\,rij}}$$

$$- \left(v^2\right)_{\substack{,ij\\,ik\\,jk\\,il\\,jl\\,kl}} \tau_{\substack{,rk\\,rj\\,rk\\,rj\\,ri\\,ri}} \tau_{\substack{,rl\\,rl\\,rl\\,rk\\,rk\\,rj}} - \left(v^2\right)_{\substack{,i\\,j\\,i\\,k\\,j\\,k\\,i\\,l\\,j\\,l\\,k\\,l}} \tau_{\substack{,rj\\,ri\\,rk\\,ri\\,rk\\,rj\\,rl\\,ri\\,rl\\,rj\\,rl\\,rk}} \tau_{\substack{,rkl\\,rkl\\,rjl\\,rjl\\,ril\\,ril\\,rjk\\,rjk\\,rik\\,rik\\,rij\\,rij}} - v^2 \tau_{\substack{,rij\\,rik\\,rjk}} \tau_{\substack{,rkl\\,rjl\\,ril}} \ , \tag{47}$$

and so on.

4.2 Integration with respect to $\gamma_3 = \int v^2 \mathrm{d}\tau$

Moreover, if we choose the Hamiltonian in Cartesian coordinates in the special form of

$$H(x_q, p_r) = \tfrac{1}{2} p_k p_k - \tfrac{1}{2} v^{-2} \quad , \tag{48}$$

then $\mathrm{d}\gamma_3 = v^2 \mathrm{d}\tau$ in (14), and only partial derivatives

$$H_{,\ldots} = -\left(\tfrac{1}{2} v^{-2}\right)_{,\ldots} \ , \quad H^{,a} = p_a \ , \quad H^{,ab} = \delta_{ab} \tag{49}$$

of the Hamiltonian are nonzero. Equation (16) then simplifies to

$$K_{ijk\ldots mn} = \tau_{,r} \tau_{,rijk\ldots mn} + \tau_{\substack{,ri\\,rj\\\vdots\\,rn}} \tau_{\substack{,rjk\ldots mn\\,irk\ldots mn\\\vdots\\,ijk\ldots mr}} \ . \tag{50}$$

The travel–time derivatives on the right–hand side of (50) may then be cancelled for the third and higher travel–time derivatives by subtracting (7) with (48) from (50),

$$\boxed{K_{ijk\ldots mn} = \left(\tfrac{1}{2} v^{-2}\right)_{,ijk\ldots mn} + \tau_{,r} \tau_{,rijk\ldots mn} + \tau_{\substack{,ri\\,rj\\\vdots\\,rn}} \tau_{\substack{,rjk\ldots mn\\,irk\ldots mn\\\vdots\\,ijk\ldots mr}} - \left(\tfrac{1}{2} \tau_{,r} \tau_{,r}\right)_{,ijk\ldots mn} \ .} \tag{51}$$

Equations such as (41) or (42) for the third and higher travel–time derivatives then take the form

$$K_{ijk} = \left(\tfrac{1}{2} v^{-2}\right)_{,ijk} \quad , \tag{52}$$

$$K_{ijkl} = \left(\tfrac{1}{2} v^{-2}\right)_{,ijkl} - \tau_{\substack{,rij\\,rik\\,rjk}} \tau_{\substack{,rkl\\,rjl\\,ril}} \quad , \tag{53}$$

$$K_{ijklm} = \left(\tfrac{1}{2} v^{-2}\right)_{,ijklm} - \tau_{\substack{,rij\\,rik\\,rjk\\,ril\\,rjl\\,rkl\\,rim\\,rjm\\,rkm\\,rlm}} \tau_{\substack{,rklm\\,rjlm\\,rilm\\,rjkm\\,rikm\\,rijm\\,rjkl\\,rikl\\,rijl\\,rijk}} \quad , \tag{54}$$

and so on.

5 Conclusions

Equations (14), (15) and (17) may be used to calculate the third and higher partial travel–time derivatives in both isotropic and anisotropic models by numerical quadratures along rays. For a special choice of the Hamiltonian in isotropic models, equation (17) considerably simplifies to equations such as (45) or (51).

Reference

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Acknowledgements

The research has been supported by the Grant Agency of the Czech Republic under Contract 205/95/1465, and by consortium "Seismic Waves in Complex 3-D Structures", see "http:/seis.karlov.mff.cuni.cz/consort/main.htm".