Weak–contrast reflection/transmission coefficients in a generally anisotropic background

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Summary

Explicit equations for approximate linearized reflection/transmission coefficients at a generally oriented weak–contrast interface separating generally anisotropic media are presented. The equations are derived also for all singular directions and are thus also valid in degenerate cases (e.g., in an isotropic background). The equations are expressed in general Cartesian coordinates, with arbitrary orientation of the interface. The explicit equations for linearized reflection/transmission coefficients have a very simple form, much simpler than the equations published previously. The equations for all reflection/transmission coefficients, with the exception of the unconverted transmitted wave, have a common form. The factors of proportionality of the contrasts of elastic parameters and density are expressed in terms of the slowness and polarization vectors of the corresponding generated wave and incident wave.

Keywords

Reflection/transmission coefficients, seismic anisotropy, ray theory, amplitude inversion, amplitude versus offset (AVO), amplitude versus azimuth (AVA).

1. Introduction

Approximate linearized reflection/transmission coefficients for weak–contrast interfaces are very useful in the structural inversion of seismic data, especially in amplitude–versus–offset (AVO) and amplitude–versus–azimuth (AVA) analyses. The linearized weak–contrast reflection/transmission coefficients are especially relevant for the analysis of the sensitivity of seismic data to discontinuities in the individual elastic parameters.

The linearized reflection/transmission coefficients for a weak–contrast interface between two isotropic media have been known for a long time (Bortfeld 1961; Richards & Frasier 1976; Aki & Richards 1980).

Banik (1987), Thomsen (1993), Rueger (1995, 1996, 1997) and Haugen & Ursin (1996) derived equations for the linearized reflection/transmission coefficients at a weak–contrast interface between two weakly anisotropic media of transversely isotropic symmetry, assuming several particular orientations of the crystal axes with respect to the interface and plane of incidence.

Vavryčuk & Pšenčík (1998) derived the equations for the linearized P–P reflection coefficient at a horizontal weak–contrast interface between general weakly anisotropic media. Zillmer, Gajewski & Kashtan (1998) derived equations for the linearized P–P, SV–SV and SH–SH reflection coefficients at a horizontal weak–contrast interface between general weakly anisotropic media. Pšenčík & Vavryčuk (1998) derived the equations for the linearized P–P reflection/transmission coefficients at a horizontal weak–contrast interface between general weakly anisotropic media.

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Vavryčuk (1999) derived the equations for the linearized reflection/transmission coefficients corresponding to a P wave incident at a generally oriented weak–contrast interface separating general weakly anisotropic media.

Ursin & Haugen (1996) derived explicit equations for the linearized reflection/transmission coefficients at a horizontal weak–contrast interface separating two transversely isotropic media with a vertical axis of symmetry.

We derive explicit equations for the linearized reflection/transmission coefficients at a generally oriented weak–contrast interface separating generally anisotropic media. The resulting equations are derived also for all singular directions. The equations are thus also valid in degenerate cases (e.g., in an isotropic background). The equations are expressed in general Cartesian coordinates, with arbitrary orientation of the interface. The resulting explicit equations for linearized reflection/transmission coefficients have a very simple form, much simpler than the equations published previously. The resulting equations for all reflection/transmission coefficients, with the exception of the unconverted transmitted wave, have a common form. The factors of proportionality of the contrasts of elastic parameters and density are expressed in terms of the slowness and polarization vectors of the corresponding generated wave and incident wave.

2. Specification of given quantities and reflection/transmission coefficients

Denote the elastic parameters (components of the stiffness tensor) and density at the incident side of a plane interface by

$$c_{ijkl} \ , \quad \varrho \quad , \tag{1}$$

and the elastic parameters and density at the other side of the interface by

$$c_{ijkl} + \Delta c_{ijkl} \ , \quad \varrho + \Delta\varrho \quad , \tag{2}$$

where Δc_{ijkl} and $\Delta\varrho$ are assumed to be small quantities. Denote the slowness vector and unit polarization vector of the incident plane wave and the normal to the interface by

$$P_i \ , \quad E_i \ , \quad n_i \quad . \tag{3}$$

Denote the slowness vectors and the unit polarization vectors of the *reflected waves* and the corresponding displacement reflection coefficients by

$$p_i^\alpha \ , \quad e_i^\alpha \ , \quad R^\alpha \ , \quad \alpha = 1,2,3 \quad . \tag{4}$$

Denote the slowness vectors and the unit polarization vectors of the *transmitted waves* and the corresponding displacement transmission coefficients by

$$p_i^\alpha \ , \quad e_i^\alpha \ , \quad R^\alpha \ , \quad \alpha = 4,5,6 \quad , \tag{5}$$

where index $\alpha = 6$ corresponds to the *unconverted wave*, also called the non–transformed wave (Červený & Ravindra 1971). The slowness vectors p_i^α of the six generated waves are determined by the slowness vector P_i of the incident wave through Snell's law, see Section 3.1. Unit polarization vectors E_i and e_i^α are determined by slowness vectors P_i and p_i^α through the corresponding Christoffel equations, see Section 3.4.

We speak about a *degenerate incident wave* if E_i is a linear combination of the eigenvectors corresponding to two or three identical or almost identical eigenvalues of the Christoffel matrix $\Gamma_{ik} = c_{ijkl}P_jP_l$ of the incident wave. Here we refer to the eigenvalues as almost identical if their difference is small, i.e. of the order of Δc_{ijkl} and $\Delta\varrho$. For a degenerate incident wave, the difference between vectors E_i and e_i^6 is small by definition.

We do not present the derivation in the degenerate case of three identical or almost identical eigenvalues of the Christoffel matrix of the incident wave. This case is completely improbable for seismic waves due to the elastic properties of rocks, while the derivation is analogous to the degenerate case of two almost identical eigenvalues.

We express the reflection/transmission coefficients in terms of new coefficients ΔR^α, $\alpha = 1, 2, 3, 4, 5, 6$, as

$$R^1 = \Delta R^1 \ , \ R^2 = \Delta R^2 \ , \ R^3 = \Delta R^3 \ , \ R^4 = -\Delta R^4 \ , \ R^5 = -\Delta R^5 \ , \ R^6 = 1 - \Delta R^6 \ . \tag{6}$$

All coefficients ΔR^α, $\alpha = 1, 2, 3, 4, 5, 6$, are small quantities of the order of Δc_{ijkl} and $\Delta\varrho$,

$$\Delta R^\alpha = O(\Delta c_{ijkl}, \Delta\varrho) \quad . \tag{7}$$

For $\alpha = 1, 2, 3, 4, 5$, $p_i^\alpha \neq P_i$, we may express definitions (6) as

$$R^\alpha = \Delta R^\alpha \ \mathrm{sgn}[V_i(p_i^\alpha - P_i)\, v_j^\alpha (P_j - p_j^\alpha)] \quad , \tag{8}$$

where

$$V_j = \varrho^{-1} c_{ijkl} E_i E_k P_l \ , \qquad v_j^\alpha = (\varrho^\alpha)^{-1} c_{ijkl}^\alpha e_i^\alpha e_k^\alpha p_l^\alpha \ , \tag{9}$$

with

$$\begin{aligned} c_{ijkl}^1 = c_{ijkl}^2 = c_{ijkl}^3 = c_{ijkl} \ , \qquad & \varrho^1 = \varrho^2 = \varrho^3 = \varrho \ , \\ c_{ijkl}^4 = c_{ijkl}^5 = c_{ijkl}^6 = c_{ijkl} + \Delta c_{ijkl} \ , \qquad & \varrho^4 = \varrho^5 = \varrho^6 = \varrho + \Delta\varrho \ , \end{aligned} \tag{10}$$

are the group velocity vectors. Note that the Einstein summation convention is applied to repeated lowercase subscripts throughout the paper. On the other hand, no summation is implied by repeated Greek superscripts indexing the generated waves.

3. Equations for weak–contrast displacement reflection/transmission coefficients

3.1. Snell's law

Snell's law for the generated waves may be expressed as

$$p_i^\alpha = P_i + n_i a^\alpha \ , \quad \alpha = 1, 2, 3, 4, 5, 6 \quad . \tag{11}$$

Note that the normal slowness–vector discontinuities a^α can be calculated as the relevant solutions of the sixth–order equations obtained by inserting Snell's law (11) into characteristic equations (29) and (30) for the Christoffel matrix. Normal slowness–vector discontinuities a^α, for $\alpha = 1, 2, 3, 4, 5$, are finite quantities for a non–degenerate incident wave, whereas a^6 is a small quantity,

$$a^6 = O(\Delta c_{ijkl}, \Delta\varrho) \quad . \tag{12}$$

For a degenerate incident wave, we shall assume that $\alpha = 5$ corresponds to the other degenerate polarization, approximately perpendicular to the incident polarization,

$$E_i e_i^5 = O(\Delta c_{ijkl}, \Delta\varrho) \quad . \tag{13}$$

In this case of a degenerate incident wave, a^5 is another small quantity,

$$a^5 = O(\Delta c_{ijkl}, \Delta\varrho) \quad . \tag{14}$$

Weakly anisotropic elastic parameters c_{ijkl}, differing from the isotropic elastic parameters only by small quantities $O(\Delta c_{ijkl}, \Delta\varrho)$, represent one important example of the degeneracy.

3.2. Continuity of displacement

The continuity of displacement across the interface is expressed by the equation

$$E_i + \sum_{\alpha=1}^{3} e_i^\alpha R^\alpha - \sum_{\alpha=4}^{6} e_i^\alpha R^\alpha = 0 \quad . \tag{15}$$

Inserting expressions (6) for the reflection/transmission coefficients into equation (15), the small discontinuity

$$\Delta e_i^6 = e_i^6 - E_i \tag{16}$$

in the unit polarization vector of the unconverted wave may be expressed in the form

$$\Delta e_i^6 = \sum_{\alpha=1}^{6} e_i^\alpha \Delta R^\alpha \quad . \tag{17}$$

Let us summarize that we have introduced ten small quantities ΔR^α, $\alpha = 1, 2, 3, 4, 5, 6$, a^6 and Δe_i^6 of the order of Δc_{ijkl} and $\Delta\varrho$ for a non–degenerate incident wave. For a degenerate incident wave, a^5 is the eleventh small quantity while ΔR^5 becomes a small *free parameter* parametrizing the projection of Δe_i^6 onto the second degenerate polarization vector e_i^5. Since the polarization vectors have unit lengths,

$$E_i \Delta e_i^6 = O^2(\Delta c_{ijkl}, \Delta\varrho) \quad . \tag{18}$$

Multiplying equation (17) by E_i and considering equation (18), we may express coefficient ΔR^6 in terms of five other coefficients,

$$\Delta R^6 = -\sum_{\alpha=1}^{5} E_i e_i^\alpha \Delta R^\alpha + O^2(\Delta c_{ijkl}, \Delta\varrho) \quad . \tag{19}$$

3.3. Continuity of traction

The continuity of traction across the interface is expressed by equation

$$c_{ijkl} n_j E_k P_l + \sum_{\alpha=1}^{3} c_{ijkl} n_j e_k^\alpha p_l^\alpha \Delta R^\alpha - \sum_{\alpha=4}^{6} (c_{ijkl} + \Delta c_{ijkl}) n_j e_k^\alpha p_l^\alpha R^\alpha = 0 \quad . \tag{20}$$

Inserting expressions (6) for the transmission coefficients, equation (20) reads

$$\sum_{\alpha=1}^{3} c_{ijkl} n_j e_k^\alpha p_l^\alpha \Delta R^\alpha + \sum_{\alpha=4}^{6} (c_{ijkl} + \Delta c_{ijkl}) n_j e_k^\alpha p_l^\alpha \Delta R^\alpha$$
$$+ c_{ijkl} n_j E_k P_l - (c_{ijkl} + \Delta c_{ijkl}) n_j e_k^6 p_l^6 = 0 \quad . \tag{21}$$

Keeping just the first–order small quantities and using Snell's law for p_i^6 results in

$$\sum_{\alpha=1}^{6} c_{ijkl} n_j e_k^\alpha p_l^\alpha \Delta R^\alpha - c_{ijkl} n_j (\Delta e_k^6 P_l + E_k n_l a^6) - \Delta c_{ijkl} n_j E_k P_l = O^2(\Delta c_{ijkl}, \Delta\varrho) \, . \tag{22}$$

Inserting equation (17) for Δe_k^6, we arrive at

$$\sum_{\alpha=1}^{6} c_{ijkl} n_j e_k^\alpha (p_l^\alpha - P_l) \Delta R^\alpha - c_{ijkl} n_j E_k n_l a^6 - \Delta c_{ijkl} n_j E_k P_l = O^2(\Delta c_{ijkl}, \Delta\varrho) \, . \tag{23}$$

Inserting Snell's law (11) for p_l^α and remembering that a^6 is a small quantity, equation (23) becomes

$$\sum_{\alpha=1}^{\underline{5}} A_{ik} e_k^\alpha a^\alpha \Delta R^\alpha - A_{ik} E_k a^6 - \Delta c_{ijkl} n_j E_k P_l = O^2(\Delta c_{ijkl}, \Delta\varrho) \quad , \tag{24}$$

where we have introduced notation

$$A_{ik} = c_{ijkl} n_j n_l \quad . \tag{25}$$

For a degenerate incident wave, both a^5 and ΔR^5 are small quantities and the summation in equation (24) is performed only over α from 1 to $\underline{4}$ (the last summation index is underlined to highlight the difference between a non–degenerate and degenerate incident wave).

3.4. Christoffel equations

The Christoffel equations for the incident wave and six generated waves are

$$c_{ijkl} P_j E_k P_l - \varrho E_i = 0 \quad , \tag{26}$$

$$c_{ijkl}\, p_j^\alpha e_k^\alpha p_l^\alpha - \varrho e_i^\alpha = 0 \ , \quad \alpha = 1,2,3 \quad , \tag{27}$$

and

$$(c_{ijkl} + \Delta c_{ijkl}) p_j^\alpha e_k^\alpha p_l^\alpha - (\varrho + \Delta\varrho) e_i^\alpha = 0 \ , \quad \alpha = 4,5,6 \quad . \tag{28}$$

Note that the characteristic equations for the Christoffel matrix, following from the Christoffel equations (27) and (28), are

$$\det\big[c_{ijkl} p_j^\alpha p_l^\alpha - \varrho\delta_{ik}\big] = 0 \ , \quad \alpha = 1,2,3 \quad , \tag{29}$$

and

$$\det\big[(c_{ijkl} + \Delta c_{ijkl}) p_j^\alpha p_l^\alpha - (\varrho + \Delta\varrho)\delta_{ik}\big] = 0 \ , \quad \alpha = 4,5,6 \quad . \tag{30}$$

Equations (29) and (30) with Snell's law (11) represent the sixth–order equations determining the slowness vectors p_i^α of the six generated waves.

Subtracting the Christoffel equation for the incident wave from the Christoffel equation for the unconverted wave and leaving only the first–order small quantities, we obtain

$$c_{ijkl}(n_j a^6 E_k P_l + P_j \Delta e_k^6 P_l + P_j E_k n_l a^6) - \varrho \Delta e_i^6 + \Delta c_{ijkl} P_j E_k P_l - \Delta\varrho E_i = O^2(\Delta c_{ijkl}, \Delta\varrho) \ , \tag{31}$$

which may be expressed as

$$(\Gamma_{ik} - \varrho\delta_{ik}) \Delta e_k^6 + B_{ik} E_k a^6 + \Delta c_{ijkl} P_j E_k P_l - \Delta\varrho E_i = O^2(\Delta c_{ijkl}, \Delta\varrho) \quad , \tag{32}$$

where

$$\Gamma_{ik} = c_{ijkl} P_j P_l \tag{33}$$

is the Christoffel matrix of the incident wave and

$$B_{ik} = c_{ijkl}(P_j n_l + n_j P_l) \quad . \tag{34}$$

Inserting equation (17) for Δe_k^6, equation (32) becomes

$$\sum_{\alpha=1}^{6} (\Gamma_{ik} - \varrho\delta_{ik}) e_k^\alpha \Delta R^\alpha + B_{ik} E_k a^6 + \Delta c_{ijkl} P_j E_k P_l - \Delta\varrho E_i = O^2(\Delta c_{ijkl}, \Delta\varrho) \ . \tag{35}$$

Since $(\Gamma_{ik} - \varrho\delta_{ik})e_k^6 = (\Gamma_{ik} - \varrho\delta_{ik})\Delta e_k^6$ is small,

$$\sum_{\alpha=1}^{5}(\Gamma_{ik} - \varrho\delta_{ik})e_k^\alpha \Delta R^\alpha + B_{ik}E_k a^6 + \Delta c_{ijkl}P_j E_k P_l - \Delta\varrho E_i = O^2(\Delta c_{ijkl}, \Delta\varrho) \quad . \quad (36)$$

For a degenerate incident wave, $(\Gamma_{ik} - \varrho\delta_{ik})e_k^5$ is also small and the summation in equation (36) is performed only over α from 1 to $\underline{4}$.

For a non–degenerate incident wave, equations (24) and (36) form a linear system of equations for six small quantities ΔR^α, $\alpha = 1, 2, 3, 4, \underline{5}$, and a^6.

For a degenerate incident wave, ΔR^5 is a small free parameter and equations (24) and (36) form a linear system of six equations for five small quantities ΔR^α, $\alpha = 1, 2, 3, \underline{4}$, and a^6. The redundant equation corresponds to the secular equation for E_i.

3.5. Approximate Christoffel equations

Leaving only finite quantities, Christoffel equations (27) and (28) for the generated waves may be approximated by

$$c_{ijkl}\, p_j^\alpha e_k^\alpha p_l^\alpha - \varrho e_k^\alpha = O(\Delta c_{ijkl}, \Delta\varrho) \ , \quad \alpha = 1, 2, 3, 4, 5, 6 \quad . \quad (37)$$

Expressing the approximate Christoffel matrices

$$\Gamma_{ik}^\alpha = c_{ijkl}\, p_j^\alpha p_l^\alpha = c_{ijkl}(P_j + n_j a^\alpha)(P_l + n_l a^\alpha) \quad (38)$$

of the generated waves in terms of matrices (25), (33) and (34),

$$\Gamma_{ik}^\alpha = \Gamma_{ik} + B_{ik}a^\alpha + A_{ik}a^\alpha a^\alpha \quad , \quad (39)$$

the approximate Christoffel equations (37) read

$$(\Gamma_{ik} - \varrho\delta_{ik} + B_{ik}a^\alpha + A_{ik}a^\alpha a^\alpha)e_k^\alpha = O(\Delta c_{ijkl}, \Delta\varrho) \quad . \quad (40)$$

4. Solution of the linear system of equations

Multiplying equation (24) by $e_i^\alpha a^\alpha$ and subtracting equation (36) multiplied by e_i^α, we arrive at the system

$$\sum_{\beta=1}^{5} e_i^\alpha(A_{ik}a^\alpha a^\beta - \Gamma_{ik} + \varrho\delta_{ik})e_k^\beta \Delta R^\beta - e_i^\alpha(A_{ik}a^\alpha + B_{ik})E_k a^6$$
$$-e_i^\alpha \Delta c_{ijkl}(P_j + n_j a^\alpha)E_k P_l + e_i^\alpha \Delta\varrho E_i = O^2(\Delta c_{ijkl}, \Delta\varrho) \quad (41)$$

of six equations, with $\alpha = 1, 2, 3, 4, 5, 6$.

For a non–degenerate incident wave and $\alpha = 6$, or for a degenerate incident wave and $\alpha = 5, 6$, equation (41) reads

$$-e_i^\alpha B_{ik}E_k a^6 - e_i^\alpha \Delta c_{ijkl}P_j E_k P_l + e_i^\alpha \Delta\varrho E_i = O^2(\Delta c_{ijkl}, \Delta\varrho) \quad . \quad (42)$$

For $\alpha = 6$, equation (42) can be used to calculate a^6,

$$E_i B_{ik} E_k a^6 = \Delta\varrho - \Delta c_{ijkl}E_i P_j E_k P_l + O^2(\Delta c_{ijkl}, \Delta\varrho) \quad . \quad (43)$$

For a degenerate incident wave, equation (42) for $\alpha = 5$ represents the secular equation for E_i,

$$e_i^5(B_{ik}a^6 + \Delta c_{ijkl}\, p_j^5 P_l)E_k = O^2(\Delta c_{ijkl}, \Delta\varrho) \quad , \quad (44)$$

where e_i^5 is approximately perpendicular to E_i, see relation (13). The approximate Christoffel equations (40) yield

$$e_i^\alpha (A_{ik} a^\alpha + B_{ik}) E_k \; a^\alpha = e_i^\alpha (\varrho \delta_{ik} - \Gamma_{ik}) E_k + O(\Delta c_{ijkl}, \Delta\varrho) \quad , \tag{45}$$

where $(\varrho\delta_{ik} - \Gamma_{ik})E_k$ is identical to zero due to Christoffel equation (26),

$$e_i^\alpha (A_{ik} a^\alpha + B_{ik}) E_k \; a^\alpha = O(\Delta c_{ijkl}, \Delta\varrho) \quad . \tag{46}$$

Note that the right–hand side of equation (46) is identical to zero for reflected waves, i.e. for $\alpha = 1, 2, 3$ (Fryer & Frazer 1984, eq. 3.20). For a non–degenerate incident wave and $\alpha = 1, 2, 3, 4, \underline{5}$, or for a degenerate incident wave and $\alpha = 1, 2, 3, \underline{4}$, equation (46) yields

$$e_i^\alpha (A_{ik} a^\alpha + B_{ik}) E_k = O(\Delta c_{ijkl}, \Delta\varrho) \quad . \tag{47}$$

Considering equations (11) and (47), the system of equations (41) for coefficients ΔR^α reads

$$\sum_{\beta=1}^{\underline{5}} e_i^\alpha (A_{ik} a^\alpha a^\beta - \Gamma_{ik} + \varrho\delta_{ik}) e_k^\beta \Delta R^\beta = \Delta c_{ijkl} \, e_i^\alpha p_j^\alpha E_k P_l - \Delta\varrho \, e_i^\alpha E_i + O^2(\Delta c_{ijkl}, \Delta\varrho) \; , \tag{48}$$

where $\alpha = 1, 2, 3, 4, \underline{5}$ for a non–degenerate incident wave. For a degenerate incident wave (small a^5), $\alpha = 1, 2, 3, \underline{4}$ and the summation in equation (48) is performed only over β from 1 to $\underline{4}$.

We now demonstrate that the system of equations (48) for coefficients ΔR^α is approximately decoupled, i.e.

$$e_i^\alpha (A_{ik} a^\alpha a^\beta - \Gamma_{ik} + \varrho\delta_{ik}) e_k^\beta = O(\Delta c_{ijkl}, \Delta\varrho) \tag{49}$$

for $\alpha \neq \beta$.

4.1. Decoupling for sufficiently different slowness vectors of generated waves

Subtracting equation (40) for e_i^β multiplied by e_i^α and equation (40) for e_i^α multiplied by e_i^β, we obtain the approximate identity

$$e_i^\alpha [B_{ik}(a^\beta - a^\alpha) + A_{ik}(a^\beta a^\beta - a^\alpha a^\alpha)] e_k^\beta = O(\Delta c_{ijkl}, \Delta\varrho) \quad . \tag{50}$$

For $a^\beta - a^\alpha$ being a finite (i.e. not small) quantity, equation (50) may be divided by $a^\beta - a^\alpha$. We then obtain

$$e_i^\alpha [B_{ik} + A_{ik}(a^\beta + a^\alpha)] e_k^\beta = O(\Delta c_{ijkl}, \Delta\varrho) \quad . \tag{51}$$

We now multiply equation (51) by a^β, and obtain

$$e_i^\alpha [B_{ik} a^\beta + A_{ik}(a^\beta + a^\alpha) a^\beta] e_k^\beta = O(\Delta c_{ijkl}, \Delta\varrho) \quad . \tag{52}$$

Subtracting equation (40) for e_i^β multiplied by e_i^α from equation (52), we arrive at equation (49). Equation (49) has thus been derived for finite $a^\beta - a^\alpha$.

4.2. Decoupling for almost identical slowness vectors of generated waves

We shall now derive equation (49) also for zero or small $a^\beta - a^\alpha$, i.e. for

$$a^\beta - a^\alpha = O(\Delta c_{ijkl}, \Delta\varrho) \quad . \tag{53}$$

In this case, we may write

$$e_i^\alpha[A_{ik}a^\alpha a^\beta - \Gamma_{ik} + \varrho\delta_{ik}]e_k^\beta = e_i^\alpha[A_{ik}a^\alpha a^\alpha - \Gamma_{ik} + \varrho\delta_{ik}]e_k^\beta + O(\Delta c_{ijkl}, \Delta\varrho) \quad , \tag{54}$$

where

$$e_i^\alpha[A_{ik}a^\alpha a^\alpha - \Gamma_{ik} + \varrho\delta_{ik}]e_k^\beta = e_i^\alpha\{c_{ijkl}[(p_j^\alpha - P_j)(p_l^\alpha - P_l) - P_jP_l] + \varrho\delta_{ik}\}e_k^\beta \; . \tag{55}$$

Using the approximate Christoffel equations (37), we obtain

$$\begin{aligned} &e_i^\alpha\{c_{ijkl}[(p_j^\alpha - P_j)(p_l^\alpha - P_l) - P_jP_l] + \varrho\delta_{ik}\}e_k^\beta \\ &\qquad = c_{ijkl}(p_j^\alpha - P_j)p_l^\alpha(e_i^\beta e_k^\alpha + e_i^\alpha e_k^\beta) + O(\Delta c_{ijkl}, \Delta\varrho) \quad . \end{aligned} \tag{56}$$

For zero or small $a^\beta - a^\alpha$, equations (54), (55) and (56) yield

$$e_i^\alpha[A_{ik}a^\alpha a^\beta - \Gamma_{ik} + \varrho\delta_{ik}]e_k^\beta = c_{ijkl}(p_j^\alpha - P_j)p_l^\alpha(e_i^\beta e_k^\alpha + e_i^\alpha e_k^\beta) + O(\Delta c_{ijkl}, \Delta\varrho) \; . \tag{57}$$

Considering the equation

$$p_i^\beta - p_i^\alpha = O(\Delta c_{ijkl}, \Delta\varrho) \tag{58}$$

which follows from equation (53), the approximate Christoffel matrix (37) for e_i^β may be approximated by

$$c_{ijkl}\, p_j^\alpha e_k^\beta p_l^\alpha - \varrho e_k^\beta = O(\Delta c_{ijkl}, \Delta\varrho) \quad . \tag{59}$$

For $\alpha \neq \beta$, e_i^α and e_i^β are thus approximately two mutually perpendicular eigenvectors of the same Christoffel equation (59),

$$e_i^\alpha e_i^\beta = O(\Delta c_{ijkl}, \Delta\varrho) \quad . \tag{60}$$

Then

$$c_{ijkl}\, e_i^\beta p_j^\alpha e_k^\alpha p_l^\alpha = O(\Delta c_{ijkl}, \Delta\varrho) \tag{61}$$

for $\alpha \neq \beta$.

At an *intersection singularity* (curvilinear intersection of slowness sheets), the polarization vectors e_k^α and e_k^β are continuous and differentiable with respect to p_j^α along the individual slowness sheets. The derivative of equation (61) with respect to p_j^α is

$$c_{ijkl}\left(e_i^\beta e_k^\alpha + e_i^\alpha e_k^\beta\right)p_l^\alpha + c_{imkl}\, p_m^\alpha p_l^\alpha \left(\frac{\partial e_i^\beta}{\partial p_j}e_k^\alpha + \frac{\partial e_i^\alpha}{\partial p_j}e_k^\beta\right) = O(\Delta c_{ijkl}, \Delta\varrho) \tag{62}$$

for $\alpha \neq \beta$. The approximate Christoffel equations (37) and (59) yield

$$c_{imkl}\, p_m^\alpha p_l^\alpha \left(\frac{\partial e_i^\beta}{\partial p_j}e_k^\alpha + \frac{\partial e_i^\alpha}{\partial p_j}e_k^\beta\right) = \varrho\, \left(\frac{\partial e_i^\beta}{\partial p_j}e_i^\alpha + \frac{\partial e_i^\alpha}{\partial p_j}e_i^\beta\right) + O(\Delta c_{ijkl}, \Delta\varrho) \quad . \tag{63}$$

Equations (60), (62) and (63) yield

$$c_{ijkl}\left(e_i^\beta e_k^\alpha + e_i^\alpha e_k^\beta\right)p_l^\alpha = O(\Delta c_{ijkl}, \Delta\varrho) \tag{64}$$

for $\alpha \neq \beta$. Equations (57) and (64) yield equation (49) for the intersection singularity. Note that the equation

$$c^{\alpha}_{ijkl}\left(e^{\beta}_i e^{\alpha}_k + e^{\alpha}_i e^{\beta}_k\right) p^{\alpha}_l = 0 \quad , \tag{65}$$

derived similarly as the approximate equation (64) but from the exact Christoffel matrix, is suitable for defining the polarization vectors at an intersection singularity.

At an *isotropic singularity* (identical slowness sheets) or a *kiss singularity* (touching slowness sheets), we may choose any couple of mutually perpendicular polarization vectors e^{α}_i and e^{β}_i. Equation (64) then follows directly from equation (60). Equations (57) and (64) then yield equation (49) for touching slowness sheets.

At a *conical singularity* (a conical intersection of slowness sheets at a single point), we define the polarization vectors for finite $p^{\alpha}_j - P_j$ as the eigenvectors e^{α}_i and e^{β}_i of the Christoffel equation corresponding to p^{α}_j, satisfying the additional condition

$$c^{\alpha}_{ijkl}\left(e^{\beta}_i e^{\alpha}_k + e^{\alpha}_i e^{\beta}_k\right) P_j \; p^{\alpha}_l = 0 \quad . \tag{66}$$

Equations (57) and (66) yield equation (49) for a conical singularity.

For an appropriate choice of polarization vectors e^{α}_i and e^{β}_i at the contact of two slowness sheets, discussed above in this section, equation (49) thus also holds for zero or small $a^{\beta} - a^{\alpha}$, $\alpha \neq \beta$.

A *contact of three slowness sheets* is completely improbable for seismic waves due to the elastic properties of rocks. We thus only briefly outline the selection of polarization vectors and derivation of equation (49) at the corresponding singularities. Equation (65) applies to touching or intersecting slowness sheets. If the polarization vector is continuous along a slowness sheet intersecting a conical singularity, equation (65) relates the continuous polarization vector to the polarization vectors corresponding to the conical singularity, while the two perpendicular polarization vectors corresponding to the conical singularity are selected according to equation (66). If the polarization vectors are discontinuous along all three slowness sheets at a conical singularity, the three perpendicular polarization vectors are determined using equation (66).

4.3. Decoupled linear equations

We have demonstrated in Sections 4.1 and 4.2 that equation (49) is valid for all $\alpha \neq \beta$. Equations (48) for coefficients ΔR^{α} are thus decoupled,

$$e^{\alpha}_i(A_{ik}a^{\alpha}a^{\alpha} - \Gamma_{ik} + \varrho\delta_{ik})\, e^{\alpha}_k \Delta R^{\alpha} = \Delta c_{ijkl}\, e^{\alpha}_i p^{\alpha}_j E_k P_l - \Delta\varrho\, e^{\alpha}_i E_i + O^2(\Delta c_{ijkl}, \Delta\varrho) \ , \tag{67}$$

where $\alpha = 1, 2, 3, 4, \underline{5}$ for a non–degenerate incident wave, and $\alpha = 1, 2, 3, \underline{4}$ for a degenerate incident wave.

The factor at ΔR^{α} in equation (67) may be converted into a simpler form. For $\alpha = \beta$, equation (57) reads

$$e^{\alpha}_i(A_{ik}a^{\alpha}a^{\alpha} - \Gamma_{ik} + \varrho\delta_{ik})e^{\alpha}_k = 2\; c_{ijkl}\; e^{\alpha}_i\,(p^{\alpha}_j - P_j)e^{\alpha}_k p^{\alpha}_l + O(\Delta c_{ijkl}, \Delta\varrho) \quad . \tag{68}$$

Equation (68) may be expressed as

$$e^{\alpha}_i(A_{ik}a^{\alpha}a^{\beta} - \Gamma_{ik} + \varrho\delta_{ik})e^{\alpha}_k = 2\varrho^{\alpha}v^{\alpha}_j(p^{\alpha}_j - P_j) + O(\Delta c_{ijkl}, \Delta\varrho) \quad , \tag{69}$$

where the group velocity vector v^{α}_j of the generated wave is defined by equation (9). Factor (69) is finite for a non–degenerate incident wave and $\alpha = 1, 2, 3, 4, \underline{5}$, or for a degenerate incident wave and $\alpha = 1, 2, 3, \underline{4}$. Then

$$\Delta R^\alpha = \frac{\Delta\varrho\, e_i^\alpha E_i - \Delta c_{ijkl}\, e_i^\alpha p_j^\alpha E_k P_l}{2\varrho^\alpha v_i^\alpha\, (P_i - p_i^\alpha)} + O^2(\Delta c_{ijkl}, \Delta\varrho) \quad , \tag{70}$$

which may be simply converted using equation (8) into the weak–contrast reflection/ transmission coefficients

$$\boxed{R^\alpha = \frac{\Delta\varrho\, e_i^\alpha E_i - \Delta c_{ijkl}\, e_i^\alpha p_j^\alpha E_k P_l}{2\varrho^\alpha |v_i^\alpha\, (P_i - p_i^\alpha)|} \operatorname{sgn}[V_i(p_i^\alpha - P_i)] + O^2(\Delta c_{ijkl}, \Delta\varrho) \quad , \qquad (71)}$$

where $\alpha = 1, 2, 3, 4, \underline{5}$ for a non–degenerate incident wave, and $\alpha = 1, 2, 3, \underline{4}$ for a degenerate incident wave. Although equations (70) and (71) have been derived for incident polarization E_i satisfying secular equation (44) for a degenerate incident wave, equations (70) and (71) are linear in E_i and are thus valid for general incident polarization E_i. We thus ignore secular equation (44), because it might contradict definition (65) of the polarization vectors at an intersection singularity or definition (66) of the polarization vectors at a conical singularity of the slowness surface at p_i^α. Note that the sign factor of $\operatorname{sgn}[V_i(p_i^\alpha - P_i)]$ is present in equation (71) because the sign of $\Delta\varrho$ and Δc_{ijkl} is defined with respect to the direction V_i of incidence and not with respect to the discontinuity $p_i^\alpha - P_i$ in the slowness vector.

The weak–contrast displacement transmission coefficient of the unconverted wave is given by the equation

$$R^6 = 1 + \sum_{\alpha=1}^{5} E_i e_i^\alpha \Delta R^\alpha + O^2(\Delta c_{ijkl}, \Delta\varrho) \quad , \tag{72}$$

which follows from equation (19). For a degenerate incident wave, both $E_i e_i^5$ and ΔR^5 are small quantities and the summation in equation (72) is performed only over α from 1 to $\underline{4}$.

5. Reciprocal normalized weak–contrast reflection/transmission coefficients

The reciprocal *normalized displacement reflection/transmission coefficients* $\overline{R}^\alpha$ are defined by the relation (Červený 2001, eq. 5.4.13)

$$\overline{R}^\alpha = R^\alpha \sqrt{\frac{|\varrho^\alpha v_i^\alpha n_i|}{|\varrho V_j\, n_j|}} \quad , \tag{73}$$

where the group velocity vectors and ϱ^α are given by equations (9) and (10).

Equations (70) and (71), specified for normalized coefficients $\overline{R}^\alpha$, read

$$\Delta\overline{R}^\alpha = \frac{\Delta\varrho e_i^\alpha E_i - \Delta c_{ijkl}\, e_i^\alpha p_j^\alpha E_k P_l}{2\sqrt{|\varrho^\alpha v_i^\alpha (P_i - p_i^\alpha)\ \varrho V_j (p_j^\alpha - P_j)|}} \operatorname{sgn}[v_i^\alpha (P_i - p_i^\alpha)] + O^2(\Delta c_{ijkl}, \Delta\varrho) \quad , \tag{74}$$

and

$$\overline{R}^\alpha = \frac{\Delta\varrho e_i^\alpha E_i - \Delta c_{ijkl}\, e_i^\alpha p_j^\alpha E_k P_l}{2\sqrt{|\varrho^\alpha v_i^\alpha (P_i - p_i^\alpha)\ \varrho V_j (p_j^\alpha - P_j)|}} \operatorname{sgn}[V_i (p_i^\alpha - P_i)] + O^2(\Delta c_{ijkl}, \Delta\varrho) \quad , \tag{75}$$

where $\alpha = 1, 2, 3, 4, \underline{5}$ for a non–degenerate incident wave, and $\alpha = 1, 2, 3, \underline{4}$ for a degenerate incident wave. Note that the terms $\Delta\varrho \operatorname{sgn}[V_i(p_i^\alpha - P_i)]$ and $\Delta c_{ijkl} \operatorname{sgn}[V_i(p_i^\alpha - P_i)]$

are reciprocal, unlike $\Delta\varrho$ and Δc_{ijkl}, because the sign of $\Delta\varrho$ and Δc_{ijkl} is defined with respect to V_i.

The reciprocal normalized weak–contrast transmission coefficient $\overline{R}^6$ of the unconverted wave is approximately unity (Chapman 1994, eq. 27; Vavryčuk 1999, eq. 15),

$$\overline{R}^6 = 1 + O^2(\Delta c_{ijkl}, \Delta\varrho) \quad . \tag{76}$$

For a degenerate incident wave, the reciprocal normalized weak–contrast transmission coefficient $\overline{R}^5$, corresponding to the second degenerate polarization vector e_i^5, is a small free parameter related to the small parameter R^5 through equation (73).

Definition (73) and equation (76) enable the weak–contrast displacement transmission coefficient of the unconverted wave to be expressed as

$$R^6 = \sqrt{\frac{|\varrho V_j(p_j^6 - P_j)|}{|\varrho^6 v_i^6(P_i - p_i^6)|}} + O^2(\Delta c_{ijkl}, \Delta\varrho) \quad . \tag{77}$$

Unlike the more complex expression (72), equation (77) does not explicitly express the approximately linear dependence of the weak–contrast displacement transmission coefficient of the unconverted wave on the contrasts Δc_{ijkl} and $\Delta\varrho$.

6. Conclusions

The linearized weak–contrast displacement reflection/transmission coefficients are given by equations (71) and (72). The equations hold for general anisotropic media, including all singular directions or an isotropic background. The equations are expressed in general Cartesian coordinates, for an arbitrary orientation of the interface. Equation (71) for linearized reflection/transmission coefficients has a very simple form, much simpler than the equations published previously, including the equations for weakly anisotropic or isotropic media. The form of equation (71) is particularly relevant for the analysis of the sensitivity of seismic data to discontinuities in individual elastic parameters. The resulting equations for linearized reflection/transmission coefficients, applied to a finite–contrast interface, are independent on the selection of the background. Note that the same equations for the linearized weak–contrast displacement reflection/transmission coefficients may also be derived by means of the Born approximation.

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