Perturbation methods for two-point ray tracing in anisotropic inhomogeneous media

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Summary

The problem of two-point ray tracing in a perturbed anisotropic inhomogeneous medium is discussed. It is assumed that the ray Ω in the perturbed medium is situated close to the ray Ω^0, constructed in the background medium. Simple expressions for the initial-value slowness vector of the perturbed ray are found. These expressions are explicit; their application does not require a solution of any system of equations. The solution needs only the 6×6 ray propagator matrix in Cartesian coordinates to be known, determined along the ray Ω^0 in the background medium. The results remain valid even for paraxial ray approximations (in which the perturbed and background media coincide, but the ray Ω is different from the ray Ω^0).

Key words
Anisotropic inhomogeneous media, ray perturbation methods, two-point ray tracing.

1 Introduction

In seismology, the problem of two-point ray tracing has traditionally played a very important role. Many procedures have been proposed to solve it. A detailed description of individual approaches to the problem can be found in Červený (2001), Section 3.11, where also many references are presented.

Here we shall discuss a two-point ray tracing method based on the structural perturbations; see also Červený (2001), Section 3.11.4. In general, we consider both the background medium $\mathcal{M}^0$ and the perturbed medium $\mathcal{M}$ to be anisotropic inhomogeneous. Both media $\mathcal{M}^0$ and $\mathcal{M}$ may also coincide. Then we speak of paraxial ray approximations. We assume that the ray $\mathcal{M}^0$, connecting points S and R in the background medium $\mathcal{M}^0$ is known, and wish to determine the ray Ω in the perturbed medium $\mathcal{M}$, connecting points S' and R', situated close to S and R, respectively.

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A simple compact expression for the initial slowness vector of the perturbed ray Ω at S' is found. The expression is explicit, no solution of a system of equations is required. Such explicit expressions can find useful applications in seismology and seismic exploration. As only first-order perturbations are used, the solution is only approximate. It can be, however, adjusted iteratively.

The proposed method requires the computation of the 6×6 ray propagator matrix $\mathbf{\Pi}^0(R, S)$ in Cartesian coordinates along the background ray Ω^0. The ray propagator matrix $\mathbf{\Pi}^0(R, S)$ can be calculated by dynamic ray tracing, along the background ray Ω^0. This is, however, a well-understood procedure.

The presented method uses certain ideas and equations proposed by Farra (1990, 1992, etc.). The final treatment is, however, different. The two-point ray tracing procedure proposed by Farra is not explicit; it requires to solve a system of equations to obtain the final result.

The computation of the perturbed ray Ω in the perturbed medium $\mathcal{M}$ may be also alternatively based on general equations for the second-order and higher-order perturbations of the travel times, proposed recently by Klimeš (2002).

2 Perturbation theory for seismic rays in anisotropic inhomogeneous media

We shall consider an inhomogeneous anisotropic medium $\mathcal{M}$, and any elementary seismic body wave propagating in this medium (qP, qS1 or qS2). The Hamiltonian corresponding to this wave is denoted $\mathcal{H}(x_i, p_i)$, where x_i are Cartesian coordinates, and p_i Cartesian components of the slowness vector. The relevant eikonal equation then reads:

$$\mathcal{H}(x_i, p_i) = 0 \ . \tag{1}$$

In addition, we shall also consider a pressumably simpler *background medium* $\mathcal{M}^0$, which differs only weakly from the medium $\mathcal{M}$, and denote the relevant Hamiltonian $\mathcal{H}^0(x_i, p_i)$. The Hamiltonian $\mathcal{H}(x_i, p_i)$ is related to the background Hamiltonian $\mathcal{H}^0(x_i, p_i)$ as follows:

$$\mathcal{H}(x_i, p_i) = \mathcal{H}^0(x_i, p_i) + \Delta\mathcal{H}(x_i, p_i) \ . \tag{2}$$

We shall call $\Delta\mathcal{H}(x_i, p_i)$ the first-order perturbation of the Hamiltonian, $\mathcal{H}(x_i, p_i)$ the perturbed Hamiltonian, and $\mathcal{M}$ the perturbed medium.

We shall now consider two fixed points, S and R, and a pressumably known reference ray Ω^0 of the elementary wave under consideration in the background medium $\mathcal{M}^0$, connecting these two points. The ray Ω^0 is parameterized by a monotonic parameter u, related to the Hamiltonian $\mathcal{H}^0(x_i, p_i)$. We also denote the values of paramater u at S and R as follows:

$$u_S = u(S) \ , \qquad u_R = u(R) \ . \tag{3}$$

The ray trajectory Ω^0 in the phase space x_i, p_i is specified by the relations

$$x_i = x_i^0(u) \ , \quad p_i = p_i^0(u) \ , \quad u_S \leq u \leq u_R \ . \tag{4}$$

Ray Ω^0 can be computed by well-known methods. Along Ω^0, we can also perform dynamic ray tracing, and determine the 6×6 ray propagator matrix $\mathbf{\Pi}^0(u, u_S)$ in Cartesian coordinates. We also introduce the 3×3 partition matrices $\mathbf{\Pi}^0_{IJ}(u, u_S)$ $(I, J = 1, 2)$, as follows:

$$\mathbf{\Pi}^0(u, u_S) = \begin{pmatrix} \mathbf{\Pi}^0_{11}(u, u_S) & \mathbf{\Pi}^0_{12}(u, u_S) \\ \mathbf{\Pi}^0_{21}(u, u_S) & \mathbf{\Pi}^0_{22}(u, u_S) \end{pmatrix} \ . \tag{5}$$

In the following, the 3×3 matrices $\mathbf{\Pi}^0_{IJ}(u, u_S)$ are assumed to be known along Ω^0.

Consider now any ray Ω in the perturbed medium $\mathcal{M}$, situated close to the background ray Ω^0. The ray may be described by the parameteric ray equations,

$$x_i(u) = x_i^0(u) + \Delta x_i(u) \ , \quad p_i(u) = p_i^0(u) + \Delta p_i(u) \ , \tag{6}$$

where the perturbations $\Delta x_i(u)$ and $\Delta p_i(u)$ should be determined.

We now introduce a point S', situated close to S, at which

$$x_i(S') = x_i^0(u_S) + \Delta x_i(u_S) \ , \quad p_i(S') = p_i^0(u_S) + \Delta p_i(u_S) \ . \tag{7}$$

Any initial-value ray Ω in the perturbed medium $\mathcal{M}$, with the initial point at S', can be then specified by the matrix relation,

$$\begin{pmatrix} \Delta\mathbf{x}(u) \\ \Delta\mathbf{p}(u) \end{pmatrix} = \mathbf{\Pi}^0(u, u_S)\begin{pmatrix} \Delta\mathbf{x}(u_S) \\ \Delta\mathbf{p}(u_S) \end{pmatrix} + \int_{u_S}^{u} \mathbf{\Pi}^0(u, u')\begin{pmatrix} \mathbf{E}(u') \\ \mathbf{F}(u') \end{pmatrix} \mathrm{d}u' \ , \tag{8}$$

where $\Delta\mathbf{x}$, $\Delta\mathbf{p}$, $\mathbf{E}$ and $\mathbf{F}$ are 3×1 column matrices. The Cartesian components of $\mathbf{E}$ and $\mathbf{F}$ are as follows:

$$E_i(u) = \frac{\partial \Delta\mathcal{H}(u)}{\partial p_i} \ , \quad F_i(u) = -\frac{\partial \Delta\mathcal{H}(u)}{\partial x_i} \ . \tag{9}$$

Along the whole ray, the constraint relation, following from the eikonal equation, must be satisfied:

$$\Delta\mathcal{H}(u) - \boldsymbol{\eta}^{0T}(u)\Delta\mathbf{x}(u) + \boldsymbol{\mathcal{U}}^{0T}(u)\Delta\mathbf{p}(u) = 0 \ , \tag{10}$$

where T denotes the transpose, and $\boldsymbol{\eta}^0(u)$ and $\boldsymbol{\mathcal{U}}^0(u)$ are given by relations

$$\mathcal{U}_i^0(u) = \frac{\partial \mathcal{H}^0(u)}{\partial p_i} \ , \quad \eta_i^0(u) = -\frac{\partial \mathcal{H}^0(u)}{\partial x_i} \ . \tag{11}$$

Here $\mathcal{U}_i^0(u)$ are Cartesian components of the vector tangent to the reference ray Ω^0 in the background medium $\mathcal{M}^0$. For the derivation of Equations (8)–(11), see, for example, Farra (1990, 1992), Červený (2001), Section 4.7.4, where many other references can be found. Equations (8) can be also simply generalized for the case of a layered medium, and for the case that Ω^0 represents merely a reference curve in $\mathcal{M}^0$, not the ray, see Farra (1992).

In this paper, we shall consider expression (8) for the initial-value ray tracing in the following form:

$$\begin{aligned}\Delta\mathbf{x}(u) &= \mathbf{\Pi}^0_{11}(u,u_S)\Delta\mathbf{x}(u_S)+\mathbf{\Pi}^0_{12}(u,u_S)\Delta\mathbf{p}(u_S)+\mathbf{C}(u,u_S)\ ,\\ \Delta\mathbf{p}(u) &= \mathbf{\Pi}^0_{21}(u,u_S)\Delta\mathbf{x}(u_S)+\mathbf{\Pi}^0_{22}(u,u_S)\Delta\mathbf{p}(u_S)+\mathbf{D}(u,u_S)\ ,\end{aligned} \tag{12}$$

where

$$\begin{aligned}\mathbf{C}(u,u_S) &= \int_{u_S}^{u}(\mathbf{\Pi}^0_{11}(u,u')\mathbf{E}(u')+\mathbf{\Pi}^0_{12}(u,u')\mathbf{F}(u'))\mathrm{d}u'\ ,\\ \mathbf{D}(u,u_S) &= \int_{u_S}^{u}(\mathbf{\Pi}^0_{21}(u,u')\mathbf{E}(u')+\mathbf{\Pi}^0_{22}(u,u')\mathbf{F}(u'))\mathrm{d}u'\ .\end{aligned} \tag{13}$$

Equations (12) and (13) represent a very effective tool for approximate initial-value ray tracing in a perturbed medium $\mathcal{M}$, in the vicinity of the background ray Ω^0. The simplicity consists in the fact that, for a specified perturbed medium, the functions $\mathbf{C}(u,u_S)$ and $\mathbf{D}(u,u_S)$ can be calculated in advance, and are the same for all perturbed rays.

Equations (12) and (13) remain valid even for $\Delta\mathcal{H}(x_i,p_i)=0$. Then $\mathbf{C}(u,u_S)=\mathbf{0}$ and $\mathbf{D}(u,u_S)=\mathbf{0}$, and we speak of paraxial ray approximation.

3 Two-point ray tracing in perturbed media

In this section, we shall discuss the problem of two-point ray tracing in the perturbed medium, in the vicinity of the background ray Ω^0, situated in the background medium. We shall consider two points S' and R', situated in the vicinity of S and R, respectively. We denote

$$\Delta x_i^S = x_i(S')-x_i(S)\ ,\qquad \Delta x_i^R = x_i(R')-x_i(R)\ . \tag{14}$$

Without any loss of generality, we can put $\Delta x_i^S=\Delta x_i(u_S)$. The problem is, however, with the point R, as the parameter u is, in general, different along Ω^0 in the background medium $\mathcal{M}^0$, and along Ω in the perturbed medium $\mathcal{M}$, $\Delta x_i^R\neq\Delta x_i(u_R)$. The relevant difference in parameters, Δu, is not yet known. We can express Δx_i^R as follows:

$$\Delta x_i^R = \Delta x_i(u_R)+\mathcal{U}_i^0(u_R)\Delta u\ . \tag{15}$$

Here $\mathcal{U}_i^0(u_R)$ represents the direction of the ray Ω^0 at $u=u_R$. Then, (12) and (15) yield, in matrix form,

$$\Delta\mathbf{x}^R = \mathbf{\Pi}^0_{11}(u_R,u_S)\Delta\mathbf{x}^S+\mathbf{\Pi}^0_{12}(u_R,u_S)\Delta\mathbf{p}^S+\mathbf{C}(u_R,u_S)+\boldsymbol{\mathcal{U}}^0(u_R)\Delta u\ . \tag{16}$$

This equation can be solved for $\Delta\mathbf{p}^S$:

$$\Delta\mathbf{p}^S = (\mathbf{\Pi}^0_{12})^{-1}(\Delta\mathbf{x}^R-\mathbf{\Pi}^0_{11}\Delta\mathbf{x}^S-\mathbf{C}-\boldsymbol{\mathcal{U}}^0(u_R)\Delta u)\ . \tag{17}$$

Here we have used the shorthand notations $\mathbf{\Pi}^0_{IJ}=\mathbf{\Pi}^0_{IJ}(u_R,u_S)$, $\mathbf{C}=\mathbf{C}(u_R,u_S)$. To determine Δu, we insert (17) into the constraint relation (10), specified for the initial point S of Ω^0:

$$\begin{aligned}&\boldsymbol{\mathcal{U}}^{0T}(u_S)(\mathbf{\Pi}^0_{12})^{-1}(\Delta\mathbf{x}^R-\mathbf{\Pi}^0_{11}\Delta\mathbf{x}^S-\mathbf{C}-\boldsymbol{\mathcal{U}}^0(u_R)\Delta u)\\ &\qquad\qquad = -\Delta\mathcal{H}(u_S)+\boldsymbol{\eta}^{0T}(u_S)\Delta\mathbf{x}^S\ .\end{aligned} \tag{18}$$

Equation (18) can be solved for Δu:

$$\Delta u = \frac{\Delta\mathcal{H}(u_S) - \boldsymbol{\mathcal{U}}^{0T}(u_S)(\boldsymbol{\Pi}_{12}^0)^{-1}(\mathbf{C} - \Delta\mathbf{x}^R + \boldsymbol{\Pi}_{11}^0\Delta\mathbf{x}^S) - \boldsymbol{\eta}^{0T}(u_S)\Delta\mathbf{x}^S}{\boldsymbol{\mathcal{U}}^{0T}(u_S)(\boldsymbol{\Pi}_{12}^0)^{-1}\boldsymbol{\mathcal{U}}^0(u_R)} \ . \tag{19}$$

Equation (17), with Δu given by (19), gives the final expression for $\Delta\mathbf{p}^S$, we have been looking for. Equations (12) and (13) can be then used to determine approximately the whole ray Ω, passing through points S' and R' in the perturbed medium.

The solution (17) for $\Delta\mathbf{p}^S$, with Δu given by (19), and with the relevant ray equations (12) and (13), are only approximate. If the accuracy of the solution at R' is not sufficient, it is necessary to solve the problem iteratively. In the first iteration, we compute $\Delta\mathbf{p}^S$ using (17), and perform exact ray tracing from S' with the initial-value slowness vector $\mathbf{p}(S') = \mathbf{p}_0(S) + \Delta\mathbf{p}^S$. The procedure should be then repeated until the required accuracy is achieved. The iterative procedure is not necessarily convergent; it may fail in certain situations. As we can see from (17), $\Delta\mathbf{p}^S$ is proportional to the inverse of the "geometrical spreading" matrix $\boldsymbol{\Pi}_{12}^0(u_R, u_S)$. This immediately implies that the procedure fails in the vicinity of caustic points.

It is straightforward to prove that $\Delta\mathbf{p}^S$, given by (17), satisfies the constraint relation (10) at $u = u_S$ (in the first-order accuracy). Consequently, the constraint relation (10) remains valid along the whole ray Ω^0.

The general equations (17) with (19) include both effects of structural perturbations (for $\Delta\mathbf{x}^R = \mathbf{0}$, $\Delta\mathbf{x}^S = \mathbf{0}$), and effects of paraxial ray approximations (for $\Delta\mathcal{H} = 0$). We shall now specify (17) with (19) for both these special cases.

1. **Structural perturbations**

Using $\Delta\mathbf{x}^R = \mathbf{0}$ and $\Delta\mathbf{x}^S = \mathbf{0}$ in (17) and (19), we obtain

$$\begin{aligned} \Delta\mathbf{p}^S &= -(\boldsymbol{\Pi}_{12}^0)^{-1}(\mathbf{C} + \boldsymbol{\mathcal{U}}^0(u_S)\Delta u) \ , \\ \Delta u &= \frac{\Delta\mathcal{H}(u_S) - \boldsymbol{\mathcal{U}}^{0T}(u_S)(\boldsymbol{\Pi}_{12}^0)^{-1}\mathbf{C}}{\boldsymbol{\mathcal{U}}^{0T}(u_S)(\boldsymbol{\Pi}_{12}^0)^{-1}\boldsymbol{\mathcal{U}}^0(u_R)} \ . \end{aligned} \tag{20}$$

Equations (20) correspond to the ray Ω in the perturbed medium, connecting directly points S and R.

2. **Paraxial ray approximations**

In this case, the model is not perturbed ($\Delta\mathcal{H} = 0$, $\mathbf{C} = \mathbf{0}$). We compute a new paraxial ray from S' to R' in the background medium. Equations (17) and (19) yield

$$\begin{aligned} \Delta\mathbf{p}^S &= (\boldsymbol{\Pi}_{12}^0)^{-1}(\Delta\mathbf{x}^R - \boldsymbol{\Pi}_{11}^0\Delta\mathbf{x}^S - \boldsymbol{\mathcal{U}}^0(u_R)\Delta u) \ , \\ \Delta u &= \frac{\boldsymbol{\mathcal{U}}^{0T}(u_S)(\boldsymbol{\Pi}_{12}^0)^{-1}(\Delta\mathbf{x}^R - \boldsymbol{\Pi}_{11}^0\Delta\mathbf{x}^S) - \boldsymbol{\eta}^{0T}(u_S)\Delta\mathbf{x}^S}{\boldsymbol{\mathcal{U}}^{0T}(u_S)(\boldsymbol{\Pi}_{12}^0)^{-1}\boldsymbol{\mathcal{U}}^0(u_R)} \ . \end{aligned} \tag{21}$$

4 Simple example

In this section, we shall discuss a simple example of an isotropic medium, in which the exact solution for $\Delta\mathbf{p}^S$ exists. In the medium under consideration, the gradient of the square of slowness is constant, see Červený (2001, pp. 165–167). We shall compare here the previously derived perturbation equations (17) and (19) for $\Delta\mathbf{p}^S$, with those following from the exact solutions. Without any loss of generality, we shall consider the gradient of the square of slowness oriented along the x_3 axis,

$$\frac{1}{V^2(x_i)} = \frac{1}{V_S^2} + b(x_3 - x_3^S) \ . \tag{22}$$

Here $V(x_i)$ denotes the velocity P or S waves, S any point with Cartesian coordinates x_i^S, and V_S the velocity at S.

4.1 Determination of $\Delta\mathbf{p}^S$ using the perturbation equations

The Hamiltonian corresponding to the model (22) can be suitably expressed as follows:

$$\mathcal{H}(x_i, p_i) = \frac{1}{2}\Big(p_i p_i - \frac{1}{V^2(x_i)}\Big) \ . \tag{23}$$

The relevant parameter u along the ray equals σ, related to the elementary arclength $\mathrm{d}s$ along the ray by the relation $\mathrm{d}\sigma = V\mathrm{d}s$.

We denote

$$\mathcal{H}(x_i, p_i) = \mathcal{H}^0(x_i, p_i) + \Delta\mathcal{H}(x_i, p_i) \ , \tag{24}$$

where

$$\mathcal{H}^0(x_i, p_i) = \tfrac{1}{2}(p_i p_i - 1/V_S^2) \ , \quad \Delta\mathcal{H}(x_i, p_i) = -\tfrac{1}{2}b(x_3 - x_3^S) \ . \tag{25}$$

Thus, the background medium is homogeneous, and $\mathcal{U}_i^0 = (\partial\mathcal{H}^0/\partial p_i)_{p_i=p_i^0} = p_i^0$. Moreover, $\Delta\mathcal{H}$ does not depend on slowness vector components p_i. The 6×6 propagator matrix $\mathbf{\Pi}^0(\sigma_R, \sigma_S)$ is given by a simple analytical expression

$$\mathbf{\Pi}^0(\sigma_R, \sigma_S) = \begin{pmatrix} \mathbf{I} & \mathbf{I}(\sigma_R - \sigma_S) \\ \mathbf{0} & \mathbf{I} \end{pmatrix} , \tag{26}$$

so that the 3×3 submatrices $\mathbf{\Pi}^0_{IJ}(\sigma^R, \sigma^S)$ read

$$\begin{aligned} \mathbf{\Pi}^0_{12}(\sigma_R, \sigma_S) &= \mathbf{I}(\sigma_R - \sigma_S) = \mathbf{I}l_0 V_S \ , \\ \mathbf{\Pi}^0_{11}(\sigma_R, \sigma_S) &= \mathbf{\Pi}^0_{22}(\sigma_R, \sigma_S) = \mathbf{I} \ , \quad \mathbf{\Pi}^0_{21}(\sigma_R, \sigma_S) = \mathbf{0} \ . \end{aligned} \tag{27}$$

Here $\mathbf{I}$ is 3×3 identity matrix, $\mathbf{0}$ the 3×3 null matrix, and l_0 the distance between S and R. See Červený (2001, Eq. (4.8.17)).

We further simply obtain,

$$\begin{aligned} E_i(\sigma) &= 0 \qquad (\text{for} \ \ i = 1, 2, 3) \ , \\ F_1(\sigma) &= F_2(\sigma) = 0 \ , \quad F_3(\sigma) = \tfrac{1}{2}b \ , \\ C_1(\sigma_R, \sigma_S) &= C_2(\sigma_R, \sigma_S) = D_1(\sigma_R, \sigma_S) = D_2(\sigma_R, \sigma_S) = 0 \ , \\ C_3(\sigma_R, \sigma_S) &= \tfrac{1}{4}bV_S^2 l_0^2 \ , \quad D_3(\sigma_R, \sigma_S) = \tfrac{1}{2}bV_S l_0 \ . \end{aligned}$$

We can further evaluate individual expressions in the equation (19) for Δu

$$\begin{aligned}
\boldsymbol{\mathcal{U}}^{0T}(\sigma_S)(\boldsymbol{\Pi}^0_{12})^{-1}\boldsymbol{\mathcal{U}}^0(\sigma_R) &= 1/V_S^3 l_0 \ , \\
\boldsymbol{\mathcal{U}}^{0T}(\sigma_S)(\boldsymbol{\Pi}^0_{12})^{-1}\mathbf{C}(\sigma_R,\sigma_S) &= \tfrac{1}{4} b V_S l_0 p_3^0 \ , \\
\boldsymbol{\mathcal{U}}^{0T}(\sigma_S)(\boldsymbol{\Pi}^0_{12})^{-1}\Delta\mathbf{x}^R &= \mathbf{p}^{0T}\Delta\mathbf{x}^R/V_S l_0 \ .
\end{aligned}$$

We shall again consider independently structural perturbations and paraxial ray approximations for $\Delta\mathbf{p}^S$.

1. **Structural perturbations**

   Using (20), we obtain

$$\begin{aligned}
\Delta u &= -\tfrac{1}{4}\, b\, V_S^4 l_0^2 p_3^0 \ , \\
\Delta\mathbf{p}^S &= -\tfrac{1}{4}\, b\, V_S l_0 (\mathbf{n} - V_S^2 p_3^0 \mathbf{p}^0) \ .
\end{aligned} \tag{28}$$

Here $\mathbf{n}$ is the unit vector along the x_3 axis. If we choose the axis x_2 perpendicular to the plane specified by $\mathbf{n}$ and $\boldsymbol{\mathcal{U}}^0$, we obtain

$$\begin{aligned}
\Delta p_1^S &= \tfrac{1}{4}\, b\, V_S^2 p_1^S (x_3^R - x_3^S) \ , \\
\Delta p_2^S &= 0 \ , \\
\Delta p_3^S &= -\tfrac{1}{4}\, \frac{b\, V_S}{l_0}\, (x_1^R - x_1^S)^2 \ .
\end{aligned} \tag{29}$$

2. **Paraxial ray approximations**

   Using (21), we obtain

$$\begin{aligned}
\Delta u &= V_S^2(\mathbf{p}^{0T}\Delta\mathbf{x}) \ , \\
\Delta\mathbf{p}^S &= \frac{1}{l_0 V_S}\Big[\Delta\mathbf{x} - V_S^2(\mathbf{p}^{0T}\Delta\mathbf{x})\mathbf{p}^0\Big] \ ,
\end{aligned} \tag{30}$$

where $\Delta\mathbf{x} = \Delta\mathbf{x}^R - \Delta\mathbf{x}^S$. If we choose $p_2^0 = 0$ and $\Delta x_2 = 0$, we obtain

$$\begin{aligned}
\Delta p_1^S &= V_S l_0^{-1} p_3^0 [p_3^0 \Delta x_1 - p_1^0 \Delta x_3] \ , \\
\Delta p_2^S &= 0 \ , \\
\Delta p_3^S &= V_S l_0^{-1} p_1^0 [-p_1^0 \Delta x_1 + p_3^0 \Delta x_3] \ .
\end{aligned} \tag{31}$$

Expressions (28)–(31) can be, of course, expressed in many other alternative forms.

4.2 Determination of $\Delta\mathbf{p}^S$ from the exact solutions

Consider a model (22), and two points, S and R, situated in this model. The exact solution for $p_1(S)$ is given by Červený, p. 166). It reads:

$$p_1^2(S) = \tfrac{1}{4}\, \frac{x^2}{l^2}\left[(V_R^{-2} + V_S^{-2}) + \sqrt{(V_R^{-2} + V_S^{-2})^2 - b^2 l^2}\right] \ . \tag{32}$$

Here l is the distance between S and R, and x is the horizontal distance between S and R. Considering the first-order expansion, we can neglect the quadratic term under the square root. Using also (22), we then obtain

$$p_1(S) = \frac{x}{V_S l}\Big[1 + \frac{bV_S^2}{2}(x_3^R - x_3^S)\Big]^{1/2} . \tag{33}$$

For small b, Equation (33) yields

$$p_1(S) = \frac{x}{V_S l}\Big(1 + \tfrac{1}{4}bV_S^2(x_3^R - x_3^S)\Big) . \tag{34}$$

This is the basic equation, which can be easily specified both for the two-point ray tracing in perturbed media, and for the paraxial ray approximation.

1. **Structural perturbations**

We assume that the points S and R are not shifted, $S' \equiv S$ and $R' \equiv R$. We take into account that $p_1(S) = p_1^S + \Delta p_1^S$, with $p_1^S = x/V_S l$. Then (34) yields,

$$\Delta p_1^S = \tfrac{1}{4}bV_S x(x_3^R - x_3^S)/l = \tfrac{1}{4}bp_1^S(x_3^R - x_3^S)V_S^2 . \tag{35}$$

This is exactly the same as in (29).

2. **Paraxial ray approximations**

In this case, we put $b = 0$ in (34), and obtain $p_1(S) = x/V_S l$. We further use

$$x_1 = x_1^0 + \Delta x_1 , \quad x_3 = x_3^0 + \Delta x_3 , \quad l^{-1} = l_0^{-1}\Big[1 - \frac{x_1^0 \Delta x_1}{l_0^2} - \frac{x_3^0 \Delta x_3}{l_0^2}\Big] ,$$

and obtain

$$\Delta p_1^S = V_S l_0^{-1} p_3^0 [p_3^0 \Delta x_1 - p_1^0 \Delta x_3] . \tag{36}$$

This is again the same as (31).

Thus, we have proved that the expansion of the exact solution gives the same result as the structural perturbation methods, in the simple case under consideration.

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