Longitudinal waves of a finite amplitude in nonlinear elastic media

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Summary

Asymptotic solution of equations of motion for a longitudinal wave propagating in a nonlinear elastic homogeneous medium is sought. The solution consists of the basic term known from the solution of linear problems, and from two corrections caused by nonlinearity of the medium. Nonlinearity leads to several interesting phenomena unknown in linear media. Examples are appearance of a wave with doubled frequency with respect to the basic mode, or appearance of transverse component of the displacement vector of the longitudinal wave propagating in a homogeneous medium.

1 Introduction

Contemporary theory of seismic wave propagation in realistic structures allows consideration of inhomogeneity, anisotropy or anelasticity of the structure through which the wave propagates. In this contribution, an attempt is made to study effects of another important and interesting phenomenon: nonlinearity of the medium. Nonlinearity has been studied intensively in various branches of physics, e.g., in electrodynamics. In contrast to linear problems, it is, however, not possible to simply transform results obtained in other fields to elasticity theory. The solutions of nonlinear problems in different branches of physics have different character. We start, therefore, with the simplest case of elastic wave propagation.

We consider longitudinal body waves in nonlinear elastic media. As it is common in acoustics (Rayleigh, 1910; Fay, 1931), we call them waves of a finite (but small) amplitudes in contrast to waves of infinitely small amplitudes considered in the linear theory. Because of the extreme complexity of the equations even for the simplest nonlinear Landau-Murnaghan (LM) medium (Landau, Rumer, 1937; Murnaghan, 1951), we concentrate on the case of a two-dimensional homogeneous unbounded medium. Let us mention that in case of body waves, nonlinearity generates only corrections (although important) of the solution of a linear problem while in case of surface waves nonlinearity affects the principal part of the solution, see Bataille and Lund (1982).

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We use the following notation: $\mathbf{u} = (u, v)$ is a displacement vector, the vector $\mathbf{x} = (x, y)$ specifies a point of observation, t is the time. The lower indices behind comma indicate partial differentiation with respect to appropriate variable, $u_{,x} = \partial u/\partial x$. We separate the multiplier $\exp(-\mathrm{i}\omega t)$, where ω is the circular frequency, and we concentrate on a stationary problem. Squares of velocities of longitudinal and transverse waves of the linear theory are given by formulae:

$$a^2 = (\lambda + 2\mu)/\rho, \qquad b^2 = \mu/\rho,$$

where λ and μ are Lamé's elastic parameters, ρ is the density.

Our goal is to show new physical phenomena which may arise due to the non-linearity of elastic media.

2 Equations of motion

The equations of motion of a nonlinear elastic LM medium have the following form (Landau and Lifshitz, 1969):

$$\rho\omega^2 u + F_1 + R_1 = 0, \tag{1}$$

$$\rho\omega^2 u + F_2 + R_2 = 0, \tag{2}$$

where

$$F_1 = (\lambda + 2\mu)u_{,xx} + \lambda v_{,xy} + \mu(u_{,yy} + v_{,xy}),$$

$$F_2 = \mu(u_{,xy} + v_{,xx}) + \lambda u_{,xy} + (\lambda + 2\mu)v_{,yy}.$$

In Eqs.(1) and (2), F_1 and F_2 are the expressions known from linear theory of elasticity. The terms R_1 and R_2 are relatively complicated nonlinear expressions, see Landau and Lifshitz (1969). We shall simplify them by introducing a small parameter of the problem.

Basic small parameter $\epsilon = \omega^{-1}$ of the considered problem is connected both with finite (but small) amplitudes of waves and with the high frequency of processes. We assume that the amplitude of the longitudinal displacement is proportional to ϵ^α and the amplitude of the transverse displacement to ϵ^β. Due to nonlinear terms in equations of motion (1), (2), solutions contain terms with multiple frequencies, see Whitham (1974), Molotkov and Vakulenko (1988). It means that waves with multiple frequencies are excited. Here we limit ourselves to double frequency terms only and assume, that the amplitude of longitudinal displacement with the double frequency is proportional to ϵ^γ. Moreover, we assume that

$$\alpha > 1, \qquad \beta > \alpha, \qquad \gamma > \alpha. \tag{3}$$

We assume that the principal propagation direction is along the x-axis. Therefore, we consider an eikonal, which depends primarily on x, and only weakly on y. We thus assume that its spatial dependence has the following form:

$$(\epsilon c_P)^{-1}x + \theta(x, y) = (\epsilon c_P)^{-1}x + c_P^{-1}x + b^{-1}y. \tag{4}$$

Here c_P is the velocity of propagation along the x-axis, b is the velocity of transverse waves in a nonlinear medium. Both velocities are constant because of the homogeneity of the medium.

Using the eikonal (4) and assumptions (3), it is possible to determine the order of individual terms in F_1, F_2, R_1 and R_2 with respect to the small parameter ϵ. Omitting small terms, the expressions for R_1, R_2 can be written as follows:

$$R_1 = c_1 u_{,x} u_{,xx}, \qquad R_2 = \mu(u_{,xx}u_{,y} + u_{,x}u_{,xy}) + (\lambda + 6C)u_{,x}u_{,xy}. \tag{5}$$

Here $c_1 = 6[(\lambda+2\mu)/2+C]$, C is the Landau elastic modulus of the 3-rd order, see Landau and Lifshitz (1969). Other Landau modules, A and B, do not play any role in the study of longitudinal waves. Let us note, that the terms R_1 and R_2 in Eqs.(1) and (2) are at least ϵ times less than terms F_1 and F_2, i.e., $R_i \sim \epsilon F_i$.

Neglecting small terms, Eq.(1) can be transformed into the form

$$u_{,xx} + \frac{1}{\epsilon^2 a^2}u + \frac{c_1}{\rho a^2}u_{,x}u_{,xx} + \frac{a^2 - b^2}{a^2}v_{,xy} + \frac{b^2}{a^2}u_{,yy} = 0. \tag{6}$$

3 Choice of the ansatz

In agreement with the above assumptions about the sizes of longitudinal and transverse amplitudes, the components of the displacement vector can be written in the following form:

$$u = \epsilon^{\alpha} U exp(\mathrm{i}\Psi) + \epsilon^{\gamma} W exp(2\mathrm{i}\Psi), \qquad v = \epsilon^{\beta} V exp(\mathrm{i}\Psi). \tag{7}$$

Here U and V are amplitudes of u and v components of the displacement vector, W is the amplitude of the double-frequency term, Ψ is a common phase, its principal part is the eikonal (4). Taking into account (3), we can see that the term with double frequency (with amplitude W) and the transverse component (with amplitude V) represent a correction to the basic term (with amplitude U). In the first equation in (7) we could also consider multiple-frequency terms starting with the term containing $\exp(3\mathrm{i}\Psi)$. We neglect them, however, because it is possible to show that the coefficient of the term $\exp(3\mathrm{i}\Psi)$ is already of the order of ϵ^4. The amplitudes U, W and V in (7) are generally functions of x and y.

Experience from solving nonlinear problems (see Molotkov and Vakulenko, 1988; Molotkov, 2003) suggests consideration of functions U, V, W and Ψ not as functions of x and y but of θ and x. They can be expanded into series in powers of the small parameter ϵ:

$$U = U_0(\theta, x) + \epsilon U_1(\theta, x) + ..., \qquad W = W_0(\theta, x) + \epsilon W_1(\theta, x) + ...,$$

$$V = V_0(\theta, x) + \epsilon V_1(\theta, x) + ...,$$
$$\Psi = (\epsilon a)^{-1}x + \Psi_0(\theta, x) + \epsilon \Psi_1(\theta, x) + ... \tag{8}$$

Equations (4), (7) and (8) represent the ansatz, i.e. the form, in which solution of Eqs.(1) and (2) is sought. The above ansatz represents a generalization of the ansatz used, e.g., in the standard ray method, see, e.g., Babich and Alekseev (1958), Červený, Molotkov and Pšenčík (1977).

4 The basic longitudinal wave

Inserting expressions for the displacement (7) and (8) with the eikonal (4) into Eq.(6), and equating to zero the terms with highest power of ϵ leads to the determination of the velocity c_P, $c_P = a$. It also indicates that α should be at least 2. In the following, we shall, therefore, consider that $\alpha = 2$. Let us note that the coefficients of $\exp(\mathrm{i}\Psi)$ and $\exp(2\mathrm{i}\Psi)$ cannot compensate each other and must, therefore, reduce to zero separately. By equating to zero real and imaginary parts of the terms of the order of ϵ, we find that

$$\frac{d\Psi_0}{dx} = 0 \tag{9}$$

and

$$\frac{dU_0}{dx} = 0. \tag{10}$$

Here d/dx denotes full derivative with respect to x, taking into account both direct dependence on x and dependence on x through θ. Eqs.(9) and (10) imply that the quantities Ψ_0 and U_0 can depend on the coordinate y only.

Equating to zero real and imaginary parts of the terms of the order ϵ^2 yields:

$$\frac{d\Psi_1}{dx} = -\frac{a}{2b^2}\Psi^2_{0,\theta}, \tag{11}$$

$$\frac{dU_1}{dx} = -\frac{b^2}{2a}(2U_{1,\theta}\Psi_{0,\theta} + U_0\Psi_{0,\theta\theta}). \tag{12}$$

In contrast to the formulae (9) and (10), which show independence of the quantities Ψ_0 and U_0 on the coordinate x, Eqs.(11) and (12) imply dependence of Ψ_1 and U_1 on x. Character of the variation of Ψ_1 and U_1 with x depends on the functions U_0 and Ψ_0.

5 Excitation of the transverse component and of the wave with double frequency

Excitation of transverse component of the displacement vector is described by Eq.(2). Similarly as in the previous section, we can find that $\beta = 3$ and

$$V_0 = ab^{-1}U_0\Psi_{0,\theta}, \tag{13}$$

$$U_{0,\theta} = 0 \tag{14}$$

and

$$U_{1,\theta} = 0. \tag{15}$$

The above equations have the following consequences. From Eqs.(10) and (14), we have $U_{0,x} = 0$. Taking into account Eq.(15) in Eq.(12), we get final form of Eq.(12):

$$\frac{dU_1}{dx} = -\frac{b^2}{2a}U_0\Psi_{0,\theta\theta}. \tag{16}$$

By equating to zero the terms with the same power of ϵ in the coefficient of the term $\exp(2i\Psi)$, we get results for the longitudinal wave with double frequency. We can find that $\gamma = 3$ and

$$W_0 = -\mathrm{i}c^{-1}U_0^2. \tag{17}$$

Here the coefficient $c = 3\rho a^3 c_1^{-1}$ has dimension of velocity. This is a specific velocity, depending on the modulus C and related to nonlinearity of the elastic medium. Obviously, $c_1 < a$. We note that the velocity c is close to the velocity a when the modulus C is small.

For obtaining final explicit formulae for the displacement (u, v), it is necessary to specify the initial conditions at $x = 0$ for Eqs.(1) and (2). This will be a subject of further study.

6 Conclusions

Asymptotic equations for longitudinal waves of finite amplitude propagating in nonlinear LM elastic media were derived. Equations for the leading terms of the ansatz specified by formulae (4), (7) and (8), were found.

By analysis of the obtained equations, it is possible to find some properties of nonlinear waves, which are unknown in the linear theory of elasticity. The most important are:

1) Excitation of the transverse component v of a longitudinal wave propagating in a homogeneous unbounded medium, see (4), (7), (8) and (13).

2) The wave field does not propagate only in the principal direction but also in a transverse direction. The transverse propagation is, however, much weaker, of higher order.

3) Appearance of a wave, propagating with doubled frequency. The amplitude of this wave is determined by Eq.(17).

As the next step, initial conditions relevant to realistic situations will be sought, and numerical experiments will be made, which should show what role the effects of nonlinearity could play in realistic problems of seismic wave propagation.

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