First-order S-wave ray computations in inhomogeneous weakly anisotropic media

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Summary

We propose an extension for S waves of the first-order ray tracing (FORT) and dynamic ray tracing (FODRT), which was originally developed for P waves. We present approximate ray-tracing and dynamic ray-tracing equations and approximate solution of the transport equation for coupled S waves propagating in laterally varying, weakly anisotropic media. For their derivation we use the so-called common S-wave ray-tracing concept. In it ray equations are governed by the Hamiltonian formed by averaging eigenvalues of the Christoffel matrix, corresponding to the two S-wave modes. Solution of the transport equation assuming the common S wave leads to the system of two coupled frequency-dependent, ordinary differential equations for S-wave amplitude coefficients, which should be solved along a common ray.

In contrast to the standard common S-wave ray tracing, we work with the first-order approximations of the exact S-wave eigenvalues. We use the perturbation theory, in which deviations of anisotropy from isotropy are considered to be first-order perturbations, of order $O(\omega^{-1})$, where ω is the circular frequency. This makes possible to interprete the coefficients of the coupled differential equations for S-wave amplitudes in terms of the geometrical spreading and other quantities related to the common ray. First-order approximation also leads to considerably simpler ray tracing and dynamic ray tracing equations than the exact ones. Although it is not shown explicitly here, for anisotropic media of higher-symmetry than monoclinic, all equations involved differ only slightly from the corresponding equations for isotropic media. For vanishing anisotropy, the equations reduce to standard, exact ray-tracing and dynamic ray-tracing equations for isotropic media.

The proposed ray-tracing and dynamic ray-tracing equations, corresponding travel-times and the geometrical spreading are all valid to the first order. Accuracy of the traveltimes can be simply increased by calculating a second-order correction along first-order common S-wave rays. This extends considerably the applicability of the proposed procedure.

Keywords: inhomogeneous media, common S wave, coupling, perturbation methods, weak anisotropy

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1 Introduction

In our two successive recent papers, Pšenčík & Farra (2005, 2007), we proposed the so-called first-order ray tracing (FORT) and first-order dynamic ray tracing (FODRT) for P waves propagating in smoothly varying weakly anisotropic media. We used the perturbation theory, in which deviations of anisotropy from isotropy were considered to be the first-order quantities. In this contribution, we propose an extension of the above procedures for S waves propagating in smooth, laterally varying, weakly anisotropic structures.

Farra (2005) has made an attempt in this respect by writing first-order ray-tracing and dynamic ray-tracing equations for each of the two S waves propagating in anisotropic media. In inhomogeneous weakly anisotropic media however, the S waves are usually coupled. Coupling can be described by the coupling ray theory (Coates & Chapman, 1990; Bulant & Klimeš, 2002). There are many simplified versions of the coupling ray theory, one of them being the quasiisotropic approximation (Kravtsov 1968, Pšenčík, 1998). In the quasiisotropic approximation, a common ray for both S waves is constructed in a reference isotropic medium. Along this common S-wave ray, coupled amplitude terms are evaluated. This approach has several limitations leading to its lower accuracy. The most important are the common S-wave ray in the isotropic reference medium (which may differ from actual rays along which the two coupled S waves propagate in the anisotropic medium), inaccuracy of the calculated traveltime (only first-order accuracy) and the geometrical spreading calculated in the reference isotropic medium. In order to remove these limitations, Bakker (2002) proposed the use of the Hamiltonian obtained as an average of the Hamiltonians of the two S waves propagating in an anisotropic medium. This approach leads to a common S-wave ray, which is traced in the studied anisotropic medium. In contrast to rays of individual S waves (obtained from their corresponding Hamiltonians), the common S-wave ray traced in an anisotropic medium removes problems with S-wave singularities. In a way, such a common S-wave ray behaves like a ray of a P wave. Klimeš (2006b) provides a detailed review and analysis of common S-wave ray tracing and dynamic ray tracing.

In this contribution, we follow the Bakker's (2002) approach. We construct an artificial Hamiltonian by averaging the S-wave eigenvalues of the Christoffel matrix. Instead of exact eigenvalues (Bakker, 2002), we use their first-order approximations as in Pšenčík & Farra (2005, 2007). We use the perturbation theory, in which deviations of anisotropy from isotropy are considered to be first-order perturbations, of order $O(\omega^{-1})$, where ω is the circular frequency. In this way, we derive the first-order coupled equations for the amplitudes and ray tracing and dynamic ray tracing equations related to the first-order common S-wave rays. As in the case of P waves, we are able to improve the accuracy of the first-order traveltimes calculated along the first-order rays by the second-order correction. Use of the first-order approximation allows us to write formulae even for most general anisotropic symmetry in an explicit form. Anisotropy is, however, assumed to be weak.

In Sec.2, we review the basic equations of the ray theory, which are used in derivations in the following sections. In Sec.3, the mean value of the first-order S-wave eigenvalues of the Christoffel matrix is introduced. An explicit expression for the first-order mean value is given. The mean value is used in the derivation of the first-order S-wave common-ray tracing. Formula for the second-order traveltime correction, which can be evaluated by quadratures along the first-order common ray is also given in Sec.3. Sec.4 is devoted to the first-order dynamic ray tracing along the first-order S-wave common ray. It is shown how the results of the dynamic ray tracing can be used to determine the first-order relative geometrical spreading. Sec.5 closely follows Sec.4 of Bakker (2002), and gives the solution of the transport equation for a specific ansatz containing amplitude coefficients of both S waves. Sec.5 ends with the system of two coupled ordinary differential equations for the amplitude coefficients of the coupled S waves. The contribution ends with a short conclusive section.

The lower-case indices $i, j, k, l, ...$ take the values of 1,2,3, the upper-case indices $I, J, K, L, ...$ take the values of 1,2. The upper index $[\mathcal{M}]$ is used to denote quantities related to the S-wave common ray. The Einstein summation convention over repeated indices is used. Index following comma in the subscript denotes partial derivative with respect to the relevant Cartesian coordinate.

2 Basic equations

In the frequency domain, the homogeneous elastodynamic equation (without a source term) for an inhomogeneous anisotropic medium reads

$$(c_{ijkl}u_{k,l})_{,j} + \rho\omega^2 u_i = 0. \tag{1}$$

In (1), $c_{ijkl} = c_{ijkl}(x_m)$ are elements of the tensor of the elastic moduli, $\rho = \rho(x_m)$ is the density and ω is the circular frequency. The symbol $u_i = u_i(x_m, \omega)$ denotes components of the sought displacement vector $\mathbf{u}$. Standard zero-order ray solution of the elastodynamic equation is sought in the form of the following ansatz

$$u_i(x_m, \omega) = U_i(x_m)\mathrm{exp}[\mathrm{i}\omega\tau(x_m)]. \tag{2}$$

Here i is the imaginary unit, $\tau(x_m)$ is the eikonal, which also serves as traveltime and $U_i(x_m)$ are components of the zero-order vectorial amplitude coefficient $\mathbf{U}(x_m)$. By inserting (2) into (1), we arrive at

$$(-\mathrm{i}\omega)^2 N_i(U_k) + \mathrm{i}\omega M_i(U_k) + L_i(U_k) = 0. \tag{3}$$

The differential vectorial operators $\mathbf{N}$, $\mathbf{M}$ and $\mathbf{L}$ are given by the relations:

$$N_i(U_k) = a_{ijkl}U_k\tau_{,l}\tau_{,j} - U_i, \tag{4}$$

$$M_i(U_k) = \rho^{-1}(\rho a_{ijkl}U_k\tau_{,l})_{,j} + a_{ijkl}U_{k,l}\tau_{,j}, \tag{5}$$

$$L_i(U_k) = \rho^{-1}(\rho a_{ijkl}U_{k,l})_{,j}, \tag{6}$$

see Červený (2001). In (4) – (6), a_{ijkl} are density normalized elastic moduli,

$$a_{ijkl} = c_{ijkl}/\rho\ . \tag{7}$$

Since (3) should be satisfied for any high frequency ω, the operators $\mathbf{N}$, $\mathbf{M}$ and $\mathbf{L}$ should be zero. From $\mathbf{N} = \mathbf{0}$ we get the Christoffel equation

$$(\Gamma_{ik} - \delta_{ik})U_k = 0\ , \tag{8}$$

where

$$\Gamma_{ik}(x_m, p_m) = a_{ijkl}(x_m) p_j p_l \ . \tag{9}$$

In (9), Γ_{ik} is the generalized Christoffel matrix. We call it generalized because it contains components $p_j = \tau_{,j} = n_j/c$ of the slowness vector $\mathbf{p}$ instead of components n_j of a unit directional vector $\mathbf{n}$ appearing in the standard Christoffel matrix; c is the phase velocity. For the sake of brevity, we call Γ_{ik} the Christoffel matrix in what follows. The three eigenvalues of the Christoffel matrix (9) can be used in the eikonal equations for each of the three waves which can propagate in inhomogeneous anisotropic media. The eikonal equations can be solved using the ray tracing equations, which provide traveltimes of considered waves. Equation $\mathbf{M} = \mathbf{0}$ with $\mathbf{M}$ given in (5), yields the transport equations, from which ray amplitudes along corresponding rays can be calculated.

In the following, we first introduce the first-order ray tracing and dynamic ray tracing for common S-wave rays with which, we can determine the first-order common S-wave traveltime $\tau^{[\mathcal{M}]}$ and the corresponding relative geometrical spreading $\mathcal{L}^{[\mathcal{M}]}$, quantity important for the determination of the ray amplitude coefficients.

3 First-order common S-wave ray tracing

In order to calculate the first-order common S-wave rays and ray amplitudes, we use a procedure similar to that used by Pšenčík & Farra (2005, 2007), who obtained the first-order ray tracing and dynamic ray tracing equations for P waves. Instead of using the greatest of the three first-order eigenvalues $G^{[3]} = G^{[3]}(x_m, p_m)$ (which belongs to P waves) of the generalized Christoffel matrix (9), we use the mean value $G^{[\mathcal{M}]} = G^{[\mathcal{M}]}(x_m, p_m)$ of the smaller first-order eigenvalues $G^{[1]}$ and $G^{[2]}$ of Γ_{ik} (which belong to S waves). The first-order S-wave mean value $G^{[\mathcal{M}]}$ reads

$$G^{[\mathcal{M}]} = \frac{1}{2}(G^{[1]} + G^{[2]}). \tag{10}$$

The first-order ray-tracing (FORT) equations for the common S-wave rays can be then written as follows:

$$\frac{dx_i}{d\tau} = \frac{1}{2}\frac{\partial G^{[\mathcal{M}]}}{\partial p_i}, \qquad \frac{dp_i}{d\tau} = -\frac{1}{2}\frac{\partial G^{[\mathcal{M}]}}{\partial x_i}, \tag{11}$$

where $G^{[\mathcal{M}]}$ is given by (10). In (11), x_i are Cartesian coordinates of the trajectory of the first-order common S-wave ray. The symbols p_i denote now components of the corresponding first-order slowness vectors, $p_i = n_i/c^{[\mathcal{M}]}$. Here n_i are components of the unit normal $\mathbf{n}$ to the wavefront and $c^{[\mathcal{M}]} = c^{[\mathcal{M}]}(x_m, n_m)$ is the first-order common S-wave phase velocity corresponding to $G^{[\mathcal{M}]}$, defined as the square root of $G^{[\mathcal{M}]}(x_m, n_m)$. The parameter along the ray, τ, is the first-order traveltime $\tau^{[\mathcal{M}]}$. At each point along the ray, the first-order slowness vector determined from equations (11) should satisfy the first-order eikonal equation:

$$G^{[\mathcal{M}]}(x_m, p_m) = (c^{[\mathcal{M}]})^{-2} G^{[\mathcal{M}]}(x_m, n_m) = 1. \tag{12}$$

The explicit expression for $G^{[\mathcal{M}]}$ could be obtained from (10) and from explicit expressions for $G^{[1]}$ and $G^{[2]}$, given in Farra & Pšenčík (2003). We use here, however, the property of conservation of the trace of a matrix during its rotation, and we express the first-order S-wave mean value $G^{[\mathcal{M}]}(x_m, p_m)$ in the form

$$G^{[\mathcal{M}]} = \frac{1}{2}[Tr(\mathbf{\Gamma}) - G^{[3]}]. \tag{13}$$

The explicit formulae for the trace of the Christoffel matrix $Tr(\mathbf{\Gamma})$ and its derivatives with respect to x_i and p_i can be found in Appendix A. Explicit formulae for the first-order P-wave eigenvalue $G^{[3]}$ of the Christoffel matrix (9) together with its first and second derivatives with respect to x_i and p_i can be found in Pšenčík & Farra (2005, 2007). For $G^{[\mathcal{M}]}$, combination of both formulae yields

$$\begin{aligned} G^{[\mathcal{M}]} = \beta^2[&p_ip_i + \epsilon_{45}p_1p_2 + \epsilon_{46}p_1p_3 + \epsilon_{56}p_2p_3 \\ &+\gamma_x(p_1^2+p_3^2) + \gamma_y(p_2^2+p_3^2) + \gamma_z(p_1^2+p_2^2)] - \alpha^2(p_ip_i)^{-1}[\eta_xp_2^2p_3^2 + \eta_yp_1^2p_3^2 + \eta_zp_1^2p_2^2 \\ &+(\chi_z - \epsilon_{16})p_1p_2(p_2^2+p_3^2-p_1^2) + (\chi_z - \epsilon_{26})p_1p_2(p_1^2+p_3^2-p_2^2) \\ &+(\chi_x - \epsilon_{24})p_2p_3(p_1^2+p_3^2-p_2^2) + (\chi_y - \epsilon_{15})p_1p_3(p_2^2+p_3^2-p_1^2) \\ &+(\chi_x - \epsilon_{34})p_2p_3(p_1^2+p_2^2-p_3^2) + (\chi_y - \epsilon_{35})p_1p_3(p_1^2+p_2^2-p_3^2)]. \end{aligned} \tag{14}$$

The coefficients of eq. (14) contain linear combinations of weak anisotropy (WA) parameters. For the definition of the WA parameters and the parameters η_x, η_y and η_z, see Appendix B. The quantities α and β in eq. (14) are reference P- and S-wave velocities used in the definition of the WA parameters. It can be easily shown that $G^{[\mathcal{M}]}(x_m, p_m)$ and its derivatives are independent of α and β. The reference velocities can thus be chosen as arbitrary non-zero constants. Another interesting feature of eq. (14) is that, like in the P-wave case, see Pšenčík & Farra (2005), $G^{[\mathcal{M}]}$ is expressed in terms of 15 independent coefficients, which are formed by 21 WA or elastic parameters, see Farra & Pšenčík (2003).

The initial conditions for the ray-tracing equations (11) at the source, for $\tau = 0$, read:

$$x_i(0) = x_i^0, \qquad p_i(0) = p_i^0. \tag{15}$$

Here, x_i^0 are the coordinates of the source point and p_i^0 are the components of the first-order slowness vector at the source specified as $p_i^0 = n_i^0/c_0^{[\mathcal{M}]}$. The symbol $c_0^{[\mathcal{M}]}$ denotes the first-order approximation of the common S-wave phase velocity in the direction $\mathbf{n}^0$ at the source point $\mathbf{x}^0$. The velocity $c_0^{[\mathcal{M}]}$ is given by the square root of $G^{[\mathcal{M}]}(x_m^0, n_m^0)$, see eq. (12). The vector $\mathbf{n^0}$ is specified by two ray parameters $\gamma^{(J)}$, chosen as two take-off angles, ϕ_0 and δ_0:

$$n_1^0 = \cos\phi_0\cos\delta_0, \quad n_2^0 = \sin\phi_0\cos\delta_0, \quad n_3^0 = \sin\delta_0. \tag{16}$$

For vanishing anisotropy, eqs. (11) reduce to the exact ray-tracing equations for an S wave in an inhomogeneous isotropic medium.

By integrating FORT in eq. (11), we obtain the first-order traveltime $\tau^{[\mathcal{M}]}$ along a ray. Its accuracy can be simply increased by calculating the second-order traveltime correction along the first-order rays (11):

$$\Delta\tau^{[\mathcal{M}]} = -\frac{1}{2}\int_{\Omega_0} \Delta G^{[\mathcal{M}]}(x_m, p_m)d\tau. \tag{17}$$

Here Ω_0 denotes the first-order common S-wave ray, x_m are coordinates and p_m components of the first-order slowness vectors along the ray Ω_0. The symbol $\Delta G^{[\mathcal{M}]}$ denotes the difference between the exact common S-wave value $G_{ex}^{[\mathcal{M}]}(x_m, p_m)$ and its approximation $G^{[\mathcal{M}]}(x_m, p_m)$ along Ω_0. For the estimation of $G_{ex}^{[\mathcal{M}]}$, we use the mean of the second-order expressions for the S-wave eigenvalues. Expressions for the second-order S-wave eigenvalues given by Farra & Pšenčík (2003) lead to

$$\Delta G^{[\mathcal{M}]}(x_m, p_m) = -[c^{[\mathcal{M}]}(x_m, n_m)]^{-2} \ \frac{B_{13}^2(x_m, n_m) + B_{23}^2(x_m, n_m)}{V_P^2 - V_S^2}. \tag{18}$$

Inserting (18) into (17), we arrive at the final expression for the second-order traveltime correction

$$\Delta \tau^{[\mathcal{M}]} = \frac{1}{2} \int_{\Omega_0} [c^{[\mathcal{M}]}(x_m, n_m)]^{-2} \frac{B_{13}^2(x_m, n_m) + B_{23}^2(x_m, n_m)}{V_P^2(x_m) - V_S^2(x_m)} d\tau. \tag{19}$$

The quantities B_{13} and B_{23} in eq. (19) are elements of the matrix

$$B_{mn}(x_m, n_m) = \Gamma_{jk}(x_m, n_m) e_j^m e_k^n, \tag{20}$$

see Farra & Pšenčík (2003). The symbols e_i^j denote components of three mutually perpendicular unit vectors $\mathbf{e^1}$, $\mathbf{e^2}$ and $\mathbf{e^3}$. The vector $\mathbf{e^3}$ is chosen so that $\mathbf{e^3} = \mathbf{n}$, where $\mathbf{n}$ is a unit vector perpendicular to the wavefront. The vectors $\mathbf{e^1}$ and $\mathbf{e^2}$ can be chosen arbitrarily in the plane perpendicular to $\mathbf{n}$. In (20), B_{mn} are elements of a matrix $\mathbf{B}$, which can be obtained by a simple rotation

$$\boldsymbol{B} = \boldsymbol{R}^T \bar{\boldsymbol{B}} \boldsymbol{R} \tag{21}$$

from the matrix $\bar{\mathbf{B}}$, whose elements are given in Appendix B. The rotation matrix $\boldsymbol{R}$ has the following form:

$$\boldsymbol{R} = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{22}$$

In (22), the angle ϕ is the acute angle between the basis vectors $\mathbf{e}^{[K]}$ introduced above and the vectors $\bar{\mathbf{e}}^{[K]}$ used in Appendix B, see eq. (B4).

The matrix B_{mn} in (20) is independent of the choice of the reference velocities α and β used in Appendix B for the definition of the WA parameters. It is easy to show that the term $B_{13}^2 + B_{23}^2$ is independent of the choice of the vectors $\mathbf{e^1}$ and $\mathbf{e^2}$. In eq. (19), we can thus use directly the elements B_{13} and B_{23} from eq. (B3).

The symbols V_P and V_S in (19) are the P- and S-wave velocities of a reference isotropic medium closely approximating the studied weakly anisotropic medium along the ray Ω_0. Similarly to the case of P waves, we use $V_S^2 = (p_k p_k)^{-1}$ and $V_P^2 = 3V_S^2$, where p_k are the components of the first-order slowness vector obtained during the integration of the FORT system (11). The values of V_P and V_S determined in this way vary along the ray. The above specification proved to be most accurate in the case of P waves.

Note the difference in signs between eq. (19) and the corresponding second-order correction for P-wave traveltimes (Pšenčík & Farra, 2005, 2007). It is simple to see that the common S-wave traveltime correction is always positive. It is in agreement with the observation of Farra & Pšenčík (2003) that the first-order S-wave eigenvalues are always larger than or equal to the exact ones.

4 First-order dynamic ray tracing along the common S wave ray

For the calculation of amplitudes of coupled S waves, see Sec.5, we need to know the first-order relative geometrical spreading $\mathcal{L}^{[\mathcal{M}]}(R,S)$ along a first-order ray from the source S to the receiver R:

$$\mathcal{L}^{[\mathcal{M}]}(R,S) = |\mathbf{X}^{(1)} \times \mathbf{X}^{(2)}|^{1/2}. \tag{23}$$

Eq. (23) holds for the vectors $\mathbf{X}^{(I)}$ and other important vectors $\mathbf{Y}^{(I)}$, defined below, determined from the first-order dynamic ray-tracing system with especially chosen initial conditions, given below, see also Pšenčík & Teles (1996). For the sake of brevity, we call $\mathcal{L}^{[\mathcal{M}]}(R,S)$ the geometrical spreading in the following. If we denote the ray parameters specifying the common S-wave ray (i.e., the initial angles, ϕ_0 or δ_0, specifying the orientation of the vector $\mathbf{n}^0$, see eq. (16)) by $\gamma^{(I)}$, the vectors $\mathbf{X}^{(I)}$ and $\mathbf{Y}^{(I)}$ can be expressed as

$$X_i^{(I)} = [\frac{\partial x_i}{\partial \gamma^{(I)}}]_{\tau=const}, \quad Y_i^{(I)} = [\frac{\partial p_i}{\partial \gamma^{(I)}}]_{\tau=const}. \tag{24}$$

The quantities $X_i^{(I)}$ and $Y_i^{(I)}$ describe variations along the wavefront of the coordinates x_i and of the components p_i of the slowness vector due to the variations of the parameters $\gamma^{(I)}$. The values of $X_i^{(I)}$ and $Y_i^{(I)}$ can be found from a system of linear differential equations obtained by the differentiation of the FORT equations (11) with respect to $\gamma^{(I)}$. The resulting system is the first-order dynamic ray tracing system (FODRT) for the common S wave. It can be developed in the same way as for P waves, see Pšenčík & Farra (2007):

$$\begin{aligned}
\frac{dX_i^{(I)}}{d\tau} &= \frac{1}{2}\Big(\frac{\partial^2 G^{[\mathcal{M}]}(x_m,p_m)}{\partial p_i \partial x_j} X_j^{(I)} + \frac{\partial^2 G^{[\mathcal{M}]}(x_m,p_m)}{\partial p_i \partial p_j} Y_j^{(I)}\Big), \\
\frac{dY_i^{(I)}}{d\tau} &= -\frac{1}{2}\Big(\frac{\partial^2 G^{[\mathcal{M}]}(x_m,p_m)}{\partial x_i \partial x_j} X_j^{(I)} + \frac{\partial^2 G^{[\mathcal{M}]}(x_m,p_m)}{\partial x_i \partial p_j} Y_j^{(I)}\Big).
\end{aligned} \tag{25}$$

Here, $G^{[\mathcal{M}]}(x_m,p_m)$ is given in eqs (10), (13) or (14). Important part of the second derivatives of $G^{[\mathcal{M}]}$ are derivatives of the trace of the Christoffel matrix, see (13). Explicit expressions for the second derivatives of the trace of the Christoffel matrix are given in Appendix A. Explicit expressions for the second derivatives of the first-order P-wave eigenvalue $G^{[3]}(x_m,p_m)$, which also appears in (13), can be found in Pšenčík & Farra (2007). For vanishing anisotropy, the equations reduce to the corresponding exact equations for an S wave propagating in an isotropic medium.

We obtain the quantities $X_i^{(J)}$ required for the calculation of the relative geometrical spreading, see eq. (23), by specifying the initial conditions for the FODRT equations (25) in the following way:

$$X_i^{(J)} = 0, \quad Y_i^{(J)} = (c_0^{[\mathcal{M}]})^{-1}(Z_{iJ} - p_i^0 {v_0}_k^{[\mathcal{M}]} Z_{kJ}), \tag{26}$$

where

$$\begin{aligned}
Z_{11} &= -\sin\phi_0\cos\delta_0, \quad Z_{21} = \cos\phi_0\cos\delta_0, \quad Z_{31} = 0, \\
Z_{12} &= -\cos\phi_0\sin\delta_0, \quad Z_{22} = -\sin\phi_0\sin\delta_0, \quad Z_{32} = \cos\delta_0,
\end{aligned} \tag{27}$$

see Pšenčík & Teles (1996). In eqs (26), $c_0^{[\mathcal{M}]}$ denotes, as before, the first-order common S-wave phase velocity at the point S, $v_{0i}^{[\mathcal{M}]}$ denotes the components of the first-order ray-velocity vector of the common S wave, $v_i^{[\mathcal{M}]} = dx_i/d\tau$, at the same point. The symbols ϕ_0 and δ_0 in eqs (27) denote again the take-off angles introduced in eq. (16). As in the case of P waves, the solution of FODRT equations (25) can be effectively used for the two-point ray tracing. The quantities $X_i^{(J)}$ can be used for estimation of the ray parameters $\gamma^{(J)}$ (take-off angles ϕ_0 and δ_0) from the distance between termination points of rays shot at a receiver surface (surface containing receivers) and the receiver, at which the two-point ray should terminate.

5 Solution of the transport equation along the common ray

In this contribution, we are making a slightly different ansatz than in eq. (2). We follow the procedure proposed by Bakker (2002) based on the common S-wave ray specified by the mean value $G^{[\mathcal{M}]}$, see eqs (10), (13) or (14). Throughout this section, we assume that the deviations of the eigenvalues $G^{[1]}$ and $G^{[2]}$ of the S1 and S2 waves from their mean value $G^{[\mathcal{M}]}$, corresponding to the common S wave, are of the order $O(\omega^{-1})$. Along the common S-wave ray, we use the ansatz in the following form:

$$\mathbf{u}(x_m,\omega) = [\mathcal{A}(x_m)\mathbf{f}^{[1]}(x_m,p_m) + \mathcal{B}(x_m)\mathbf{f}^{[2]}(x_m,p_m)]\exp[i\omega\tau^{[\mathcal{M}]}(x_m)]. \qquad (28)$$

Note that we do not include the term parallel to the P-wave eigenvector $\mathbf{g}^{[3]}$ of the Christoffel matrix Γ_{ik} into (28) as Bakker (2002) did. The term does not play any role in the following formulae anyway, see below. In eq. (28), $\tau^{[\mathcal{M}]}(x_m)$ is the first-order or second-order traveltime along the common S-wave ray, $\mathbf{p} = \nabla\tau^{[\mathcal{M}]}$, and $\mathbf{f}^{[1]}, \mathbf{f}^{[2]}$ are two mutually perpendicular unit vectors situated in the plane perpendicular to the P-wave eigenvector $\mathbf{g}^{[3]}$.

Inserting (28) into (1) and neglecting terms of order O(1), we arrive at

$$(i\omega)^2(\boldsymbol{I}-\boldsymbol{\Gamma})(\mathcal{A}\mathbf{f}^{[1]}+\mathcal{B}\mathbf{f}^{[2]}) - i\omega\ \mathbf{M}(\mathcal{A}\mathbf{f}^{[1]}+\mathcal{B}\mathbf{f}^{[2]}) = 0. \qquad (29)$$

Let us emphasize again that in the first term on the left-hand side of eq. (29), we neglect the term proportional to $\mathbf{g}^{[3]}$ since it would be cancelled in the next step anyway. In (29), the symbol $\mathbf{M}$ denotes the vectorial differential operator introduced in (5). Multiplying eq. (29) successively by $\mathbf{f}^{[1]}$ and $\mathbf{f}^{[2]}$, and taking into account that the terms $\mathbf{f}^{[1]T}\boldsymbol{\Gamma}\mathbf{f}^{[1]} - \mathbf{f}^{[2]T}\boldsymbol{\Gamma}\mathbf{f}^{[2]}$ and $\mathbf{f}^{[1]T}\boldsymbol{\Gamma}\mathbf{f}^{[2]}$ are of the order $O(\omega^{-1})$, see Appendix C, we get the following system of coupled equations:

$$i\omega\begin{pmatrix}\mathbf{f}^{[1]T}(\boldsymbol{I}-\boldsymbol{\Gamma})\mathbf{f}^{[1]} & \mathbf{f}^{[1]T}(\boldsymbol{I}-\boldsymbol{\Gamma})\mathbf{f}^{[2]}\\ \mathbf{f}^{[2]T}(\boldsymbol{I}-\boldsymbol{\Gamma})\mathbf{f}^{[1]} & \mathbf{f}^{[2]T}(\boldsymbol{I}-\boldsymbol{\Gamma})\mathbf{f}^{[2]}\end{pmatrix}\begin{bmatrix}\mathcal{A}\\ \mathcal{B}\end{bmatrix} - \begin{bmatrix}\mathbf{f}^{[1]T}\mathbf{M}\mathbf{f}^{[1]} & \mathbf{f}^{[1]T}\mathbf{M}\mathbf{f}^{[2]}\\ \mathbf{f}^{[2]T}\mathbf{M}\mathbf{f}^{[1]} & \mathbf{f}^{[2]T}\mathbf{M}\mathbf{f}^{[2]}\end{bmatrix}\begin{bmatrix}\mathcal{A}\\ \mathcal{B}\end{bmatrix}$$
$$= \begin{bmatrix}\mathbf{f}^{[1]T}\partial\boldsymbol{\Gamma}/\partial p_j\mathbf{f}^{[1]} & \mathbf{f}^{[1]T}\partial\boldsymbol{\Gamma}/\partial p_j\mathbf{f}^{[2]}\\ \mathbf{f}^{[2]T}\partial\boldsymbol{\Gamma}/\partial p_j\mathbf{f}^{[1]} & \mathbf{f}^{[2]T}\partial\boldsymbol{\Gamma}/\partial p_j\mathbf{f}^{[2]}\end{bmatrix}\begin{bmatrix}\mathcal{A}_{,j}\\ \mathcal{B}_{,j}\end{bmatrix}. \qquad (30)$$

When deriving eq. (30), we took into account that

$$\mathbf{f}^{[K]T}\mathbf{M}(\mathcal{C}\mathbf{f}^{[L]}) = \mathcal{C}\mathbf{f}^{[K]T}\mathbf{M}\mathbf{f}^{[L]} + \mathbf{f}^{[K]T}\partial\boldsymbol{\Gamma}/\partial p_j\mathbf{f}^{[L]}\ \mathcal{C}_{,j}. \qquad (31)$$

As terms of order less than $O(1)$ are neglected in (30), the terms without $\mathrm{i}\omega$ in (30) can be evaluated to $O(1)$ and the terms with $\mathrm{i}\omega$ to $O(\omega^{-1})$. With accuracy to $O(1)$, we can write:

$$\begin{aligned}
\mathbf{f}^{[K]T}\partial\mathbf{\Gamma}/\partial p_j\mathbf{f}^{[L]} &= 2\ \delta_{KL}\ v_j^{[\mathcal{M}]},\\
\mathbf{f}^{[K]T}\mathbf{M}\mathbf{f}^{[K]} &= \rho^{-1}\ \mathrm{div}(\rho\mathbf{v}^{[\mathcal{M}]}),\\
\mathbf{f}^{[K]T}\mathbf{M}\mathbf{f}^{[L]} &= 2\mathbf{f}^{[K]}\cdot d\mathbf{f}^{[L]}/d\tau, \qquad K\neq L.
\end{aligned} \tag{32}$$

Here $v_j^{[\mathcal{M}]}$ denotes again the components of the first-order ray-velocity vector $\mathbf{v}^{[\mathcal{M}]}$.

With (32), the system (30) can be rewritten as:

$$\begin{aligned}
\begin{bmatrix} d\mathcal{A}/d\tau \\ d\mathcal{B}/d\tau \end{bmatrix} = &-\tfrac{i\omega}{2}\begin{pmatrix} \mathbf{f}^{[1]T}(\mathbf{\Gamma}-\boldsymbol{I})\mathbf{f}^{[1]} & \mathbf{f}^{[1]T}(\mathbf{\Gamma}-\boldsymbol{I})\mathbf{f}^{[2]} \\ \mathbf{f}^{[2]T}(\mathbf{\Gamma}-\boldsymbol{I})\mathbf{f}^{[1]} & \mathbf{f}^{[2]T}(\mathbf{\Gamma}-\boldsymbol{I})\mathbf{f}^{[2]} \end{pmatrix}\begin{bmatrix} \mathcal{A} \\ \mathcal{B} \end{bmatrix}\\
&-\tfrac{1}{2}\begin{bmatrix} \rho^{-1}\mathrm{div}(\rho\mathbf{v}^{[\mathcal{M}]}) & 2\mathbf{f}^{[1]}\cdot d\mathbf{f}^{[2]}/d\tau \\ 2\mathbf{f}^{[2]}\cdot d\mathbf{f}^{[1]}/d\tau & \rho^{-1}\mathrm{div}(\rho\mathbf{v}^{[\mathcal{M}]}) \end{bmatrix}\begin{bmatrix} \mathcal{A} \\ \mathcal{B} \end{bmatrix}.
\end{aligned} \tag{33}$$

As Pšenčík (1998; eq. 37), we can seek the solution of eq. (33) in the form

$$\mathcal{A} = \frac{\mathcal{A}_0}{\sqrt{\rho c^{[\mathcal{M}]}\mathcal{L}^{[\mathcal{M}]}}}, \qquad \mathcal{B} = \frac{\mathcal{B}_0}{\sqrt{\rho c^{[\mathcal{M}]}\mathcal{L}^{[\mathcal{M}]}}}. \tag{34}$$

Here $c^{[\mathcal{M}]}$ and $\mathcal{L}^{[\mathcal{M}]}$ are the phase velocity, see (12), and the geometrical spreading, see (23), associated with the first-order mean value $G^{[\mathcal{M}]}$. Inserting (34) into (33), we get the following system of coupled linear differential equations:

$$\begin{aligned}
\begin{bmatrix} d\mathcal{A}_0/d\tau \\ d\mathcal{B}_0/d\tau \end{bmatrix} = &-\tfrac{i\omega}{2}\begin{pmatrix} \mathbf{f}^{[1]T}(\mathbf{\Gamma}-\boldsymbol{I})\mathbf{f}^{[1]} & \mathbf{f}^{[1]T}(\mathbf{\Gamma}-\boldsymbol{I})\mathbf{f}^{[2]} \\ \mathbf{f}^{[2]T}(\mathbf{\Gamma}-\boldsymbol{I})\mathbf{f}^{[1]} & \mathbf{f}^{[2]T}(\mathbf{\Gamma}-\boldsymbol{I})\mathbf{f}^{[2]} \end{pmatrix}\begin{bmatrix} \mathcal{A}_0 \\ \mathcal{B}_0 \end{bmatrix}\\
&-\begin{bmatrix} 0 & \mathbf{f}^{[1]}\cdot d\mathbf{f}^{[2]}/d\tau \\ \mathbf{f}^{[2]}\cdot d\mathbf{f}^{[1]}/d\tau & 0 \end{bmatrix}\begin{bmatrix} \mathcal{A}_0 \\ \mathcal{B}_0 \end{bmatrix}.
\end{aligned} \tag{35}$$

We can see that the coefficients $\mathcal{A}_0$ and $\mathcal{B}_0$ are frequency dependent. To solve the coupled system (35), it is necessary to specify its initial conditions and to specify how to evaluate the vectors $\mathbf{f}^{[K]}$ along the ray. The initial conditions can be specified in the very same way as in Pšenčík (1998; eq. 39). The specification of the vectors $\mathbf{f}^{[K]}$ can be done in many ways, see Klimeš (2006a). One possibility is to approximate the vectors $\mathbf{f}^{[K]}$ by the unit vectors $\mathbf{e}^{[K]}$ situated in the plane tangent to the wavefront. It is easy to show that the differences between the vectors $\mathbf{f}^{[K]}$ and the vectors $\mathbf{e}^{[K]}$ are $O(\omega^{-1})$. The vectors $\mathbf{e}^{[K]}$ and the vector $\mathbf{n}$ form a basis of the so-called wavefront orthonormal coordinate system, see Červený (2001; sec.4.2.2), Klimeš (2006a). Červený (2001; eq. 4.2.17) gives differential equations

$$\frac{de_i^{[K]}}{d\tau} = -V_S^2(e_k^{[K]}\cdot\frac{dp_k}{d\tau})p_i, \tag{36}$$

whose solution along a selected ray gives the vectors $\mathbf{e}^{[K]}$ at any of its point.

Substitution of the vectors $\mathbf{f}^{[K]}$ by the basis vectors of the wavefront orthogonal coordinate system, $\mathbf{e}^{[K]}$, see (36), in eq. (35) leads to the coupled system of differential equations whose form reminds quasiisotropic coupled equations of Pšenčík (1998; eq 38):

$$\left[\begin{array}{c} d\mathcal{A}_0/d\tau \\ d\mathcal{B}_0/d\tau \end{array}\right] = -\frac{i\omega}{2}\left(\begin{array}{cc} B_{11}-1 & B_{12} \\ B_{12} & B_{22}-1 \end{array}\right)\left[\begin{array}{c} \mathcal{A}_0 \\ \mathcal{B}_0 \end{array}\right] . \tag{37}$$

In (37), B_{mn} are elements of the matrix $\mathbf{B}$, see (20), which can be obtained by the rotation (21) from the matrix $\bar{\mathbf{B}}$ in Appendix B. The vectors $\mathbf{e}^{[K]}$ used in (37) are perpendicular to the first-order slowness vector $\mathbf{p}$, and not to the ray as in Pšenčík (1998).

6 Discussion and conclusions

The preceding sections should be considered as a preliminary proposal of an approximate procedure for computation of S waves in inhomogeneous weakly anisotropic media. In such media, the two S waves are coupled and must be, therefore, calculated together. We used the so-called common S-wave ray tracing and dynamic ray tracing (Bakker, 2002; Klimeš, 2006a) with the common S-wave ray situated in the studied weakly anisotropic medium. We combined the common S-wave ray tracing with the perturbation approach, which we applied recently to P waves (Pšenčík & Farra, 2005, 2007) and obtained an approximate procedure, which has similar advantages as FORT and FODRT for P waves:

1) FORT and FODRT equations are expressed in terms of the WA parameters, which represent a more natural description of a weakly anisotropic medium than standard elastic moduli.

2) FORT and FODRT equations have a considerably simpler structure than the exact ones, which leads to a significant reduction of the number of algebraic operations involved.

3) In the most general case, FORT and FODRT equations depend only on the 15 independent coefficients (which differ from coefficients of P waves).

4) The FORT and FODRT equations allow S waves to be treated separately from P waves, as in isotropic media.

5) The accuracy of the calculated traveltimes can be easily increased by introducing the second-order traveltime correction (19).

6) The FORT and FODRT equations are applicable to inhomogeneous isotropic as well as anisotropic media. In isotropic media they yield exact rays and exact dynamic-ray-tracing result, in anisotropic media their first-order approximations.

7) Last, but not least, the proposed procedure incorporates coupling of S waves.

The authors are aware that the preceding text needs further work. Several steps need more detailed description, the basic equations valid for general, but weak, anisotropy should be specified for the most frequently used types of anisotropy like transverse isotropy and orthorhombic symmetry.

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Appendix A: Formulae for $Tr(\mathbf{\Gamma})$ and its derivatives

The explicit expression for $Tr(\mathbf{\Gamma})$ has the following form:

$$Tr(\mathbf{\Gamma}) = \alpha^2 p_k p_k + 2\alpha^2[\epsilon_x p_1^2 + \epsilon_y p_2^2 + \epsilon_z p_3^2 + (\epsilon_{16}+\epsilon_{26})p_1p_2 + (\epsilon_{15}+\epsilon_{35})p_1p_3 + (\epsilon_{24}+\epsilon_{34})p_2p_3]$$
$$+2\beta^2[p_kp_k + \epsilon_{45}p_1p_2 + \epsilon_{46}p_1p_3 + \epsilon_{56}p_2p_3 + \gamma_y(p_2^2+p_3^2) + \gamma_x(p_1^2+p_3^2) + \gamma_z(p_1^2+p_2^2)]. \quad (A1)$$

The first derivatives of $Tr(\mathbf{\Gamma})$ with respect to p_i and x_i appearing in the first-order common S-wave ray tracing, see (11) and (13) have the following explicit form:

$$\frac{\partial Tr(\mathbf{\Gamma})}{\partial p_1} = 2\alpha^2[(1+2\epsilon_x)p_1 + (\epsilon_{16}+\epsilon_{26})p_2 + (\epsilon_{15}+\epsilon_{35})p_3] + 2\beta^2[2(1+\gamma_x+\gamma_z)p_1 + \epsilon_{45}p_2 + \epsilon_{46}p_3],$$

$$\frac{\partial Tr(\mathbf{\Gamma})}{\partial p_2} = 2\alpha^2[(1+2\epsilon_y)p_2 + (\epsilon_{16}+\epsilon_{26})p_1 + (\epsilon_{24}+\epsilon_{34})p_3] + 2\beta^2[2(1+\gamma_y+\gamma_z)p_2 + \epsilon_{45}p_1 + \epsilon_{56}p_3],$$

$$\frac{\partial Tr(\mathbf{\Gamma})}{\partial p_3} = 2\alpha^2[(1+2\epsilon_z)p_3 + (\epsilon_{15}+\epsilon_{35})p_1 + (\epsilon_{24}+\epsilon_{34})p_2] + 2\beta^2[2(1+\gamma_y+\gamma_x)p_3 + \epsilon_{46}p_1 + \epsilon_{56}p_2],$$

$$\frac{\partial Tr(\mathbf{\Gamma})}{\partial x_i} = 2\alpha^2[\epsilon_{x,i}p_1^2 + \epsilon_{y,i}p_2^2 + \epsilon_{z,i}p_3^2 + (\epsilon_{16,i}+\epsilon_{26,i})p_1p_2 + (\epsilon_{15,i}+\epsilon_{35,i})p_1p_3 + (\epsilon_{24,i}+\epsilon_{34,i})p_2p_3]$$
$$+2\beta^2[\epsilon_{45,i}p_1p_2 + \epsilon_{46,i}p_1p_3 + \epsilon_{56,i}p_2p_3 + \gamma_{x,i}(p_1^2+p_3^2) + \gamma_{y,i}(p_2^2+p_3^2) + \gamma_{z,i}(p_1^2+p_2^2)]. \quad (A2)$$

The second derivatives of $Tr(\mathbf{\Gamma})$ with respect to p_i and x_i appearing in the first-order common S-wave ray tracing, see (11) and (13) have the following explicit form:

$$\frac{\partial^2 \partial Tr(\mathbf{\Gamma})}{\partial p_1^2} = 2\alpha^2(1+2\epsilon_x) + 4\beta^2(1+\gamma_x+\gamma_z), \quad \frac{\partial^2 \partial Tr(\mathbf{\Gamma})}{\partial p_1 \partial p_2} = 2\alpha^2(\epsilon_{16}+\epsilon_{26}) + 2\beta^2\epsilon_{45},$$

$$\frac{\partial^2 \partial Tr(\mathbf{\Gamma})}{\partial p_1 \partial p_3} = 2\alpha^2(\epsilon_{15}+\epsilon_{35}) + 2\beta^2\epsilon_{46}, \quad \frac{\partial^2 \partial Tr(\mathbf{\Gamma})}{\partial p_2^2} = 2\alpha^2(1+2\epsilon_y) + 4\beta^2(1+\gamma_y+\gamma_z),$$

$$\frac{\partial^2 \partial Tr(\mathbf{\Gamma})}{\partial p_2 \partial p_3} = 2\alpha^2(\epsilon_{24}+\epsilon_{34}) + 2\beta^2\epsilon_{56}, \quad \frac{\partial^2 \partial Tr(\mathbf{\Gamma})}{\partial p_3^2} = 2\alpha^2(1+2\epsilon_z) + 4\beta^2(1+\gamma_x+\gamma_y). \quad (A3)$$

The WA parameters used in the above equations are defined in Appendix B.

Appendix B: WA parameters and elements of the matrix B

If we denote by α and β the P- and S-wave velocity of a reference isotropic medium, we can introduce 21 WA parameters in the way slightly different from that in Farra & Pšenčík (2003):

$$\epsilon_x = \frac{A_{11}-\alpha^2}{2\alpha^2}, \quad \epsilon_y = \frac{A_{22}-\alpha^2}{2\alpha^2}, \quad \epsilon_z = \frac{A_{33}-\alpha^2}{2\alpha^2},$$

$$\delta_x = \frac{A_{13}+2A_{55}-\alpha^2}{\alpha^2}, \quad \delta_y = \frac{A_{23}+2A_{44}-\alpha^2}{\alpha^2}, \quad \delta_z = \frac{A_{12}+2A_{66}-\alpha^2}{\alpha^2},$$

$$\chi_x = \frac{A_{14} + 2A_{56}}{\alpha^2}, \quad \chi_y = \frac{A_{25} + 2A_{46}}{\alpha^2}, \quad \chi_z = \frac{A_{36} + 2A_{45}}{\alpha^2},$$

$$\epsilon_{15} = \frac{A_{15}}{\alpha^2}, \quad \epsilon_{16} = \frac{A_{16}}{\alpha^2}, \quad \epsilon_{24} = \frac{A_{24}}{\alpha^2}, \quad \epsilon_{26} = \frac{A_{26}}{\alpha^2}, \quad \epsilon_{34} = \frac{A_{34}}{\alpha^2}, \quad \epsilon_{35} = \frac{A_{35}}{\alpha^2},$$

$$\epsilon_{46} = \frac{A_{46}}{\beta^2}, \quad \epsilon_{56} = \frac{A_{56}}{\beta^2}, \quad \epsilon_{45} = \frac{A_{45}}{\beta^2}, \quad \gamma_x = \frac{A_{55} - \beta^2}{2\beta^2}, \quad \gamma_y = \frac{A_{44} - \beta^2}{2\beta^2}, \quad \gamma_z = \frac{A_{66} - \beta^2}{2\beta^2}. \tag{B1}$$

In the text, the following variables are also used:

$$\eta_x = \delta_y - \epsilon_y - \epsilon_z, \quad \eta_y = \delta_x - \epsilon_x - \epsilon_z, \quad \eta_z = \delta_z - \epsilon_x - \epsilon_y. \tag{B2}$$

The elements of the matrix $\bar{\mathbf{B}}$, see equation (21), can be expressed explicitly in the following way:

$$\begin{aligned}
\bar{B}_{11} = \beta^2 + 2\alpha^2 D^{-2}\Big(& n_3^3[(\epsilon_{15} - \epsilon_{35})n_1^3 + (\epsilon_{24} - \epsilon_{34})n_2^3 + (\chi_x - \epsilon_{34})n_1^2 n_2 + (\chi_y - \epsilon_{35})n_1 n_2^2] \\
& + n_3^2[(\eta_z - \eta_x - \eta_y)n_1^2 n_2^2 - \eta_y n_1^4 - \eta_x n_2^4 \\
& + 2(\epsilon_{16} - \chi_z)n_1^3 n_2 + 2(\epsilon_{26} - \chi_z)n_1 n_2^3] \\
& + n_3[(\epsilon_{35} - \epsilon_{15})n_1^5 + (\epsilon_{34} - \epsilon_{24})n_2^5 + (\epsilon_{34} - \chi_x)n_1^4 n_2 + (\epsilon_{35} - \chi_y)n_1 n_2^4 \\
& + (2\epsilon_{35} - \chi_y - \epsilon_{15})n_1^3 n_2^2 + (2\epsilon_{34} - \chi_x - \epsilon_{24})n_1^2 n_2^3]\Big) \\
& + 2\beta^2 D^{-2}(\gamma_x n_1^2 + \gamma_y n_2^2 + \epsilon_{45} n_1 n_2),
\end{aligned}$$

$$\begin{aligned}
\bar{B}_{12} = \alpha^2 D^{-2}\Big(& n_3^2[(\chi_x - \epsilon_{34})n_1^3 - (\chi_y - \epsilon_{35})n_2^3 + (\epsilon_{35} + \chi_y - 2\epsilon_{15})n_1^2 n_2 + (2\epsilon_{24} - \chi_x - \epsilon_{34})n_1 n_2^2] \\
& + n_3[(\epsilon_{16} - \chi_z)n_1^4 - (\epsilon_{26} - \chi_z)n_2^4 + (\eta_y - \eta_x + \eta_z)n_1^3 n_2 + (\eta_y - \eta_x - \eta_z)n_1 n_2^3 + 3(\epsilon_{26} - \epsilon_{16})n_1^2 n_2^2] \\
& + [(\epsilon_{15} - \chi_y)n_1^4 n_2 + (\chi_x - \epsilon_{24})n_1 n_2^4 + (\epsilon_{15} - \chi_y)n_1^2 n_2^3 + (\chi_x - \epsilon_{24})n_1^3 n_2^2 \\
& + \epsilon_{46} n_2^3 - \epsilon_{56} n_1^3 + \epsilon_{46} n_1^2 n_2 - \epsilon_{56} n_1 n_2^2]\Big) + \beta^2 D^{-2} n_3[2(\gamma_y - \gamma_x)n_1 n_2 + \epsilon_{45} n_1^2 - \epsilon_{45} n_2^2],
\end{aligned}$$

$$\begin{aligned}
\bar{B}_{22} = & \beta^2 + 2\beta^2 D^{-2} n_3^2(\gamma_x n_2^2 + \gamma_y n_1^2 - \epsilon_{45} n_1 n_2) \\
& + 2\alpha^2 D^{-2}\Big(n_3[\epsilon_{46} n_1^3 + \epsilon_{56} n_2^3) + (\epsilon_{56} - \chi_x + \epsilon_{24})n_1^2 n_2 + (\epsilon_{46} - \chi_y + \epsilon_{15})n_1 n_2^2] \\
& - \eta_z n_1^2 n_2^2 + (\epsilon_{26} - \epsilon_{16})n_1^3 n_2 - (\epsilon_{26} - \epsilon_{16})n_1 n_2^3]\Big) + 2\beta^2 \gamma_z (n_1^2 + n_2^2),
\end{aligned}$$

$$\begin{aligned}
\bar{B}_{13} = & \alpha^2 D^{-1}\{n_3^4(\epsilon_{34} n_2 + \epsilon_{35} n_1) + n_3^3[\eta_y n_1^2 + \eta_x n_2^2 + 2\chi_z n_1 n_2] \\
& + n_3^2[(4\chi_x - 3\epsilon_{34})n_1^2 n_2 + (4\chi_y - 3\epsilon_{35})n_1 n_2^2 + (4\epsilon_{15} - 3\epsilon_{35})n_1^3 + (4\epsilon_{24} - 3\epsilon_{34})n_2^3] \\
& + n_3[(2\eta_z - \eta_x - \eta_y)n_1^2 n_2^2 + 2(2\epsilon_{16} - \chi_z)n_1^3 n_2 + 2(2\epsilon_{26} - \chi_z)n_1 n_2^3
\end{aligned}$$

$$-\eta_y n_1^4 - \eta_x n_2^4 + (\epsilon_x - \epsilon_z)n_1^2 + (\epsilon_y - \epsilon_z)n_2^2] - \chi_x n_1^2 n_2 - \chi_y n_1 n_2^2 - \epsilon_{15} n_1^3 - \epsilon_{24} n_2^3\},$$

$$\bar{B}_{23} = \alpha^2 D^{-1}\{n_3^3(\epsilon_{34}n_1 - \epsilon_{35}n_2) + n_3^2[(\eta_x - \eta_y)n_1 n_2 + \chi_z n_1^2 - \chi_z n_2^2]$$
$$+n_3[(2\chi_y - 3\epsilon_{15})n_1^2 n_2 - (2\chi_x - 3\epsilon_{24})n_1 n_2^2 + \chi_x n_1^3 - \chi_y n_2^3]$$
$$+\eta_z n_1^3 n_2 - \eta_z n_1 n_2^3 + 3(\epsilon_{26} - \epsilon_{16})n_1^2 n_2^2 + \epsilon_{16} n_1^4 - \epsilon_{26} n_2^4 + (\epsilon_y - \epsilon_x)n_1 n_2\},$$

$$\bar{B}_{33} = \alpha^2 + 2\alpha^2[2n_3^3(\epsilon_{34}n_2 + \epsilon_{35}n_1) + n_3^2(\eta_y n_1^2 + \eta_x n_2^2 + 2\chi_z n_1 n_2 + \epsilon_z)$$
$$+2n_3(\chi_x n_1^2 n_2 + \chi_y n_1 n_2^2 + \epsilon_{15} n_1^3 + \epsilon_{24} n_2^3) + \epsilon_x n_1^2 + \epsilon_y n_2^2 + \eta_z n_1^2 n_2^2 + 2\epsilon_{16} n_1^3 n_2 + 2\epsilon_{26} n_1 n_2^3]. \quad (B3)$$

The expressions in eq. (B3) correspond to the following choice of vectors $\mathbf{e}^i$:

$$\bar{\mathbf{e}}^1 \equiv D^{-1}(n_1 n_3, n_2 n_3, n_3^2 - 1), \quad \bar{\mathbf{e}}^2 \equiv D^{-1}(-n_2, n_1, 0), \quad \bar{\mathbf{e}}^3 = \mathbf{n} \equiv (n_1, n_2, n_3), \quad (B4)$$

where

$$D = (n_1^2 + n_2^2)^{1/2}, \qquad n_1^2 + n_2^2 + n_3^2 = 1. \quad (B5)$$

The symbols n_i denote components of the unit normal $\mathbf{n}$ to the wavefront.

Appendix C: Proof that the coefficients of iω in (30) are $O(1/\omega)$

We assume that deviations of eigenvalues $G^{[1]}$ and $G^{[2]}$ of S1 and S2 waves from their mean value $G^{[\mathcal{M}]}$ are $O(1/\omega)$, i.e., we assume that the differences $\Delta G^{[K]} = G^{[K]} - G^{[\mathcal{M}]}$ are $O(1/\omega)$. Following Bakker (2002), we can show that

$$G^{[K]} = \frac{1}{2}[\mathbf{f}^{[1]T}\mathbf{\Gamma}\mathbf{f}^{[1]} + \mathbf{f}^{[2]T}\mathbf{\Gamma}\mathbf{f}^{[2]} \pm \sqrt{(\mathbf{f}^{[1]T}\mathbf{\Gamma}\mathbf{f}^{[1]} - \mathbf{f}^{[2]T}\mathbf{\Gamma}\mathbf{f}^{[2]})^2 + 4(\mathbf{f}^{[1]T}\mathbf{\Gamma}\mathbf{f}^{[2]})^2}]. \quad (C1)$$

From (C1), we immediately get

$$G^{[\mathcal{M}]} = \frac{1}{2}[\mathbf{f}^{[1]T}\mathbf{\Gamma}\mathbf{f}^{[1]} + \mathbf{f}^{[2]T}\mathbf{\Gamma}\mathbf{f}^{[2]}] \quad (C2)$$

and

$$\Delta G^{[K]} = \pm\sqrt{(\mathbf{f}^{[1]T}\mathbf{\Gamma}\mathbf{f}^{[1]} - \mathbf{f}^{[2]T}\mathbf{\Gamma}\mathbf{f}^{[2]})^2 + 4(\mathbf{f}^{[1]T}\mathbf{\Gamma}\mathbf{f}^{[2]})^2}]. \quad (C3)$$

Important consequence of the above assumption that the differences $\Delta G^{[K]}$ are $O(1/\omega)$ is that $\mathbf{f}^{[1]T}\mathbf{\Gamma}\mathbf{f}^{[1]} - \mathbf{f}^{[2]T}\mathbf{\Gamma}\mathbf{f}^{[2]}$ and $\mathbf{f}^{[1]T}\mathbf{\Gamma}\mathbf{f}^{[2]}$ are $O(1/\omega)$. This clearly indicates that the coefficients of the first term (with iω), in (30) are of the order $O(1/\omega)$.