Computer representation of the model covariance function resulting from travel-time tomography

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Summary

This paper represents a supplement to the paper by Klimeš (2002b) on the stochastic travel–time tomography. It contains brief generalization of the equations from interpolation of slowness to interpolation of general power of velocity, along with some notes on numerical implementation of the algorithm. The results of the inversion are represented by the velocity model and the model covariance function which describes the accuracy of the velocity model. Whereas the velocity model is composed of smooth functions of 3 coordinates, the model covariance function is a function of 6 coordinates with pronounced singularities. The computer representation of the model covariance function in terms of two matrices and the rays used for inversion is designed.

Keywords

Travel–time tomographic inversion, resolution, velocity model, medium covariance function, model covariance function, covariance matrices, heterogeneous media.

1. Introduction

It is well known that infinite number of different velocity models can fit the finite number of travel–time data. The results of the stochastic travel–time tomography (Franklin, 1970; Tarantola & Valette, 1982; Tarantola & Nercessian, 1984; Tarantola, 1987; 2005; Klimeš, 2002b) are the velocity model and the model covariance function describing the accuracy of this velocity model. The velocity model represents the statistical mean of the velocity models satisfying the travel–time data. The velocity model without the corresponding model covariance function is useless.

This paper represents a supplement to the paper by Klimeš (2002b) on the stochastic travel–time tomography. It contains brief generalization of the equations from interpolation of slowness to interpolation of general power of velocity, along with some notes on numerical implementation of the algorithm. Special attention is paid to the computer representation of the model covariance function which describes the accuracy of the velocity model resulting from the travel–time tomography.

For the stochastic travel–time tomography, we need to know approximately the statistical properties of the geological structure. These statistical properties are specified in terms of the a priori mean model and the medium covariance function (Klimeš, 2002b; 2002c; 2002d). The mean model and the medium covariance function describe the geological structure if no travel–time data are available.

The velocity model is usually specified in terms of a finite number of basis functions (e.g., tricubic B–splines) and a finite number of coefficients of these basis functions. We refer to these coefficients as the model parameters. In this paper, we concentrate on the

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computer representation of the model covariance function which describes the accuracy of the velocity model.

We index the model parameters by lower–case roman subscripts, and the travel–time data by upper–case roman subscripts. Synthetic travel times calculated in the velocity model resulting from travel–time tomography are indexed by primed upper–case roman subscripts. Lower–case greek subscripts are used in Section 5.2 to index perturbations of the velocity model. The Einstein summation over repetitive indices is used throughout the paper.

2. Mean model and the medium covariance function

Assume that wave–propagation velocity $v(\mathbf{x})$ in the geological structure is a realization of the random quantity. Let M be the power of propagation velocity which is linearized during travel–time tomography. We define the first statistical moment (mean value) of the M–th power of the velocity,

$$v^{(M)}(\mathbf{x}) = \langle [v(\mathbf{x})]^M \rangle \quad . \tag{1}$$

We shall refer to $v^{(M)}(\mathbf{x})$ as the *mean model.*

We also define the *medium covariance function*

$$C^{(M)}(\mathbf{x}_1, \mathbf{x}_2) = \langle \{[v(\mathbf{x}_1)]^M - v^{(M)}(\mathbf{x}_1)\} \, \{[v(\mathbf{x}_2)]^M - v^{(M)}(\mathbf{x}_2)\} \rangle \tag{2}$$

corresponding to the M–th power of propagation velocity. The mean model and the medium covariance function describe the properties of the geological structure in places where no travel times are available. The mean model and the medium covariance function must be available for the stochastic travel–time tomography.

Medium covariance function (2) strongly depends on the power M of propagation velocity. We thus define relative covariance function

$$c^{(M)}(\mathbf{x}_1, \mathbf{x}_2) = \frac{C^{(M)}(\mathbf{x}_1, \mathbf{x}_2)}{M^2 \, v^{(M)}(\mathbf{x}_1) \, v^{(M)}(\mathbf{x}_2)} \quad , \tag{3}$$

which should depend on power M only moderately, especially for small relative velocity deviations.

We shall consider the ellipsoidally anisotropic relative velocity–scaled ellipsoidally anisotropic power–law covariance function

$$c^{(M)}(\mathbf{x}_1, \mathbf{x}_2) = \sigma^2 [v^{(M)}(\mathbf{x}_1)]^{-\frac{K}{M}} [v^{(M)}(\mathbf{x}_2)]^{-\frac{K}{M}} [(\mathbf{x}_1 - \mathbf{x}_2)^{\mathrm{T}} \mathbf{L}^{-1} (\mathbf{x}_1 - \mathbf{x}_2)]^N \tag{4}$$

of a self–affine random medium (Klimeš, 2002c, eqs. 33, 68). Here N is the Hurst exponent,

$$-\tfrac{1}{2} < N < 0 \quad . \tag{5}$$

The positive–definite symmetric scaling matrix $\mathbf{L}$ accounts for the ellipsoidal anisotropy of the covariance function (Goff & Jordan, 1988). The units of reference deviation σ at the reference ellipsoid correspond to the units of K–th power of velocity. The medium covariance function of a self–affine random medium corresponding to assumptions (3) and (4) reads

$$C^{(M)}(\mathbf{x}_1, \mathbf{x}_2) = M^2 \sigma^2 [v^{(M)}(\mathbf{x}_1)]^{1-\frac{K}{M}} [v^{(M)}(\mathbf{x}_2)]^{1-\frac{K}{M}} [(\mathbf{x}_1 - \mathbf{x}_2)^{\mathrm{T}} \mathbf{L}^{-1} (\mathbf{x}_1 - \mathbf{x}_2)]^N \ . \tag{6}$$

We see that the covariance function is exactly stationary if $K = M$. Parameter K thus specifies the velocity power which has the stationary covariance function. Because the relative covariance function (4), which characterizes the relative velocity heterogeneities, can often be expected greater in positions where the velocity is lower, it seems that reasonable values of parameter K will be positive. Note that the ideal value of the velocity power for linearization of travel–time tomography is $M = -1$.

3. Velocity model and integrals required for inversion

3.1. Parametrization of the velocity model

We assume here that the M–th power of propagation velocity $\widetilde{v}(\mathbf{x})$ in the velocity model is a linear combination of a finite number of smooth basis functions $B_i(\mathbf{x})$ indexed by lower–case roman subscripts,

$$[\widetilde{v}(\mathbf{x})]^M = m_i\, B_i(\mathbf{x}) \quad , \tag{7}$$

where m_i are the model parameters. We denote by m_i^0 the model parameters of the mean model,

$$v^{(M)}(\mathbf{x}) = m_i^0\, B_i(\mathbf{x}) \quad . \tag{8}$$

In the current version of computer package `MODEL` (Bucha & Bulant, 2008), tricubic B-splines with optional tension are used as the basis functions.

3.2. Lebesgue and Sobolev scalar products of basis functions

We define the Lebesgue scalar products

$$B_{ij} = \int B_i(\mathbf{x}) B_j(\mathbf{x})\; \mathrm{d}\mathbf{x} \tag{9}$$

of the basis functions.

Various Sobolev scalar products $((B_i, B_j))^{\Phi}$, indexed here by Φ, are defined as the linear combinations of the Lebesgue scalar products of various derivatives of the basis functions. The Sobolev scalar products are used to limit the velocity model updates within individual iterations of the linearized inversion, and to quantify the roughness of the updated velocity model. Refer to Klimeš (2000a) for general description of the Sobolev scalar products, and to Klimeš (2002b, sec. 3.1) for description of the application of the Sobolev scalar products during the linearized inversion.

The Lebesgue scalar products and some Sobolev scalar products of the basis functions are required both for the least–square inversion and for the stochastic inversion of travel times. These scalar products are calculated by program `invsoft.for` of package `MODEL` (Bucha & Bulant, 2008).

3.3. Ray characteristic functions

Ray characteristic function $G_I^{(-1)}(\mathbf{x})$ characterizes the first–order sensitivity of travel times to slowness perturbations,

$$\Delta T_I = \int G_I^{(-1)}(\mathbf{x}) \Delta[v(\mathbf{x})]^{-1}\; \mathrm{d}\mathbf{x} \quad . \tag{10}$$

Ray characteristic function

$$G_I^{(M)}(\mathbf{x}) = -M^{-1} [\widetilde{v}(\mathbf{x})]^{-M-1} G_I^{(-1)}(\mathbf{x}) \tag{11}$$

then characterizes the first–order sensitivity of travel times to the perturbations of the M–th power of the velocity,

$$\Delta T_I = \int G_I^{(M)}(\mathbf{x}) \Delta[v(\mathbf{x})]^{M}\; \mathrm{d}\mathbf{x} \quad . \tag{12}$$

For travel–time data sparse with respect to the Fresnel volumes, the ray characteristic functions $G_I^{(-1)}(\mathbf{x})$ in the form of the 2-D Dirac distributions in the planes perpendicular to the rays are sufficient. In this case, the volume integrals with ray characteristic functions reduce to the path integrals along rays.

3.4. Derivatives of travel times with respect to model parameters

We define integrals

$$\boxed{G_{iK}^{(M)} = \int B_i(\mathbf{x}) G_K^{(M)}(\mathbf{x})\, \mathrm{d}\mathbf{x} \quad , \qquad (13)}$$

where $\mathrm{d}\mathbf{x}$ denotes the integration over the volume. Quantity $G_{iK}^{(M)}$ represents the derivatives of the K–th synthetic travel time with respect to the i–th model parameter. Integrals (13), frequently used in naive least–square travel–time inversion, are required in our stochastic travel–time inversion only if the velocity power $M \neq -1$. On the other hand, integrals analogous to (13) are required for calculating the covariance matrix of synthetic travel times.

For infinitely narrow rays, the ray characteristic functions $G_K^{(-1)}(\mathbf{x})$ take the form of the 2-D Dirac distributions in the planes perpendicular to the rays, and volume integrals (13) with (11) reduce to the path integrals along rays. These path integrals are calculated by program `invtt.for` of package `CRT` (Bucha & Bulant, 2008).

3.5. Covariance integrals for travel–time inversion

For the stochastic inversion of travel times, we shall need the following integrals: *geometric travel–time covariances*

$$\boxed{\Theta_{JK} = \iint G_J^{(M)}(\mathbf{x}_1)\, C^{(M)}(\mathbf{x}_1, \mathbf{x}_2)\, G_K^{(M)}(\mathbf{x}_2)\, \mathrm{d}\mathbf{x}_1\, \mathrm{d}\mathbf{x}_2 \qquad (14)}$$

and geometric covariances

$$\boxed{\Psi_{iK}^{(M)} = \iint B_i(\mathbf{x}_1)\, C^{(M)}(\mathbf{x}_1, \mathbf{x}_2)\, G_K^{(M)}(\mathbf{x}_2)\, \mathrm{d}\mathbf{x}_1\, \mathrm{d}\mathbf{x}_2 \quad , \qquad (15)}$$

between model parameters m_i and travel times T_K. Here $\mathrm{d}\mathbf{x}_1$ denotes the integration over the volume containing points $\mathbf{x}_1$, and $\mathrm{d}\mathbf{x}_2$ the integration over the volume containing points $\mathbf{x}_2$. Analogous integrals are also required for calculating the covariance matrix of synthetic travel times.

For the velocity–scaled power–law medium covariance function, we insert (6) and (11) into equations (14) and (15),

$$\begin{aligned} \Theta_{JK} = \sigma^2 \iint & [v^{(M)}(\mathbf{x}_1)]^{1-\frac{K}{M}} [\widetilde{v}(\mathbf{x}_1)]^{-M-1} [v^{(M)}(\mathbf{x}_2)]^{1-\frac{K}{M}} [\widetilde{v}(\mathbf{x}_2)]^{-M-1} \\ & \times [(\mathbf{x}_1 - \mathbf{x}_2)^{\mathrm{T}} \mathbf{L}^{-1} (\mathbf{x}_1 - \mathbf{x}_2)]^N G_J^{(-1)}(\mathbf{x}_1)\, G_K^{(-1)}(\mathbf{x}_2)\, \mathrm{d}\mathbf{x}_1\, \mathrm{d}\mathbf{x}_2 \quad , \qquad (16) \end{aligned}$$

$$\begin{aligned} \Psi_{iK}^{(M)} = -M\, \sigma^2 \iint & [v^{(M)}(\mathbf{x}_1)]^{1-\frac{K}{M}} [v^{(M)}(\mathbf{x}_2)]^{1-\frac{K}{M}} [\widetilde{v}(\mathbf{x}_2)]^{-M-1} \\ & \times [(\mathbf{x}_1 - \mathbf{x}_2)^{\mathrm{T}} \mathbf{L}^{-1} (\mathbf{x}_1 - \mathbf{x}_2)]^N B_i(\mathbf{x}_1)\, G_K^{(-1)}(\mathbf{x}_2)\, \mathrm{d}\mathbf{x}_1\, \mathrm{d}\mathbf{x}_2 \quad . \qquad (17) \end{aligned}$$

For $M = -1$, which represents the ideal value of the velocity power for linearization of travel–time tomography, integrals (16) and (17) depend on the current velocity model $\widetilde{v}(\mathbf{x})$ through the ray characteristic functions $G_K^{(-1)}(\mathbf{x})$ only.

For infinitely narrow rays, the ray characteristic functions $G_K^{(-1)}(\mathbf{x})$ take the form of the 2-D Dirac distributions in the planes perpendicular to the rays. For infinitely narrow rays, both volume integrals with weights $G_K^{(-1)}(\mathbf{x}_1)$ and $G_K^{(-1)}(\mathbf{x}_2)$ in (16), and volume integral with weight $G_K^{(-1)}(\mathbf{x}_2)$ over $\mathbf{x}_2$ in (17) thus reduce to the path integrals along rays.

The algorithm of calculating geometric travel–time covariances (16) for the velocity–scaled ellipsoidally anisotropic power–law medium covariance function (6) along infinitely narrow rays has been proposed by Klimeš (2008a).

The algorithm of calculating geometric covariances (17) between model parameters m_i and travel times T_K is still missing, which represents the main obstacle in numerical realization of the stochastic travel–time tomography.

4. Stochastic inversion of travel times

The model parameters depend nonlinearly on the travel–time data,

$$m_i = m_i(T_K) \quad . \tag{18}$$

In each iteration of the linearized stochastic inversion of travel times, we approximate (18) by linear Taylor expansion at $T_K = \overline{T}_K$,

$$m_i = R_{iK}(T_K - \overline{T}_K) + R_i^0 \quad , \tag{19}$$

where $\overline{T}_K$ are the travel times calculated in the best available velocity model, i.e., in the velocity model $\overline{m}_i B_i(\mathbf{x})$ resulting from the last iteration. Coefficients R_{iK} and R_i^0 in (19) are to be determined.

We approximate the travel times, measured in geological structure with unknown wave propagation velocity $v(\mathbf{x})$, by first–order perturbation expansion (12) in velocity model $\overline{m}_i B_i(\mathbf{x})$,

$$T_K - \overline{T}_K = \int G_I^{(M)}(\mathbf{x}) \{[v(\mathbf{x})]^M - \overline{m}_i B_i(\mathbf{x})\} \, \mathrm{d}\mathbf{x} + \delta T_I \quad . \tag{20}$$

Here δT_I are the picking errors with zero mean values,

$$\langle T_I \rangle = 0 \quad , \tag{21}$$

and *data covariance matrix*

$$\langle T_I \, T_J \rangle = T_{IJ} \quad . \tag{22}$$

Ray characteristic functions $G_I^{(M)}(\mathbf{x})$ are calculated in the velocity model $\overline{m}_i B_i(\mathbf{x})$ resulting from the last iteration. Inserting (19) with (20) into (7), we obtain updated velocity model

$$[\widetilde{v}(\mathbf{x})]^M = B_i(\mathbf{x}) \left[R_i^0 + R_{iK} \delta T_I + R_{iK} \int G_I^{(M)}(\mathbf{x}') \{[v(\mathbf{x}')]^M - \overline{m}_i B_i(\mathbf{x}')\} \, \mathrm{d}\mathbf{x}' \right] \, . \tag{23}$$

The *model covariance function* describing the accuracy of the updated velocity model is

$$\widetilde{C}^{(M)}(\mathbf{x}_1, \mathbf{x}_2) = \langle \{[\widetilde{v}(\mathbf{x}_1)]^M - [v(\mathbf{x}_1)]^M\} \, \{[\widetilde{v}(\mathbf{x}_2)]^M - [v(\mathbf{x}_2)]^M\} \rangle \quad . \tag{24}$$

We determine coefficients R_{iK} and R_i^0 in (19) so as to minimize objective function

$$y = \int \widetilde{C}^{(M)}(\mathbf{x}, \mathbf{x}) \, \mathrm{d}\mathbf{x} + \sum_{\Phi} \langle (([\widetilde{v}(\mathbf{x})]^M - m_i^{\Phi} B_i(\mathbf{x}), [\widetilde{v}(\mathbf{x})]^M - m_i^{\Phi} B_i(\mathbf{x})))^{\Phi} \rangle \quad , \tag{25}$$

where $((\bullet,\bullet))^{\Phi}$ are zero to several Sobolev scalar products, limiting the velocity model updates within individual iterations of the linearized inversion, or quantifying the roughness of the updated velocity model. Error of the linearization depends mostly on the updates of the first velocity derivatives (Klimeš, 2002e). It is thus useful to limit the velocity model updates within individual iterations of the linearized inversion by means of Sobolev scalar product $((\bullet,\bullet))^{\Phi}$ composed of the first–order derivatives with $m_i^{\Phi}=\overline{m}_i$. The complicated behaviour of rays depends on the second velocity derivatives (Klimeš, 2002a) and may thus be restricted using Sobolev scalar product $((\bullet,\bullet))^{\Phi}$ composed of the second–order derivatives with $m_i^{\Phi}=0$ (Klimeš, 2000a; Žáček, 2002; Bulant, 2002; 2004). Refer to Klimeš (2002b, sec. 3.1) for the description of the application of the Sobolev scalar products during the linearized inversion. Note that the Sobolev scalar products must carefully be adjusted for each iteration of the linearized travel–time inversion.

To calculate model covariance function (24), we apply decomposition

$$\{[\widetilde{v}(\mathbf{x})]^M-[v(\mathbf{x})]^M\}=\{[\widetilde{v}(\mathbf{x})]^M-v^{(M)}(\mathbf{x})\}-\{[v(\mathbf{x})]^M-v^{(M)}(\mathbf{x})\} \quad . \tag{26}$$

Considering (7), (8) and (13), we express the deviation of (23) from the mean model in form

$$\begin{aligned}&[\widetilde{v}(\mathbf{x})]^M-v^{(M)}(\mathbf{x})=B_i(\mathbf{x})\Big[R_i^0-m_i^0\\&+R_{iK}\Big((m_k^0-\overline{m}_k)G_{kK}^{(M)}+\delta T_K+\int G_K^{(M)}(\mathbf{x}')\{[v(\mathbf{x}')]^M-v^{(M)}(\mathbf{x}')\}\,\mathrm{d}\mathbf{x}'\Big)\Big] \quad .\end{aligned} \tag{27}$$

Analogously,

$$\begin{aligned}&[\widetilde{v}(\mathbf{x})]^M-m_i^{\Phi}B_i(\mathbf{x})=B_i(\mathbf{x})\Big[R_i^0-m_i^{\Phi}\\&+R_{iK}\Big((m_k^0-\overline{m}_k)G_{kK}^{(M)}+\delta T_K+\int G_K^{(M)}(\mathbf{x}')\{[v(\mathbf{x}')]^M-v^{(M)}(\mathbf{x}')\}\,\mathrm{d}\mathbf{x}'\Big)\Big] \quad .\end{aligned} \tag{28}$$

The mean value and covariance matrix of δT_K in (27) and (28) are given by equations (21) and (22). The mean values of the integrals in (27) and (28) are zero because of (1). The covariance matrix of the integrals in (27) and (28) is given by equation (14), see (2). Other terms on the right–hand sides of (27) and (28) are deterministic quantities. The covariances between the integrals in (27) and random function $\{[v(\mathbf{x}')]^M-v^{(M)}(\mathbf{x}')\}$ appearing in (26) are

$$\Psi_K^{(M)}(\mathbf{x}_1)=\int C^{(M)}(\mathbf{x}_1,\mathbf{x}_2)\,G_K^{(M)}(\mathbf{x}_2)\,\mathrm{d}\mathbf{x}_2 \quad . \tag{29}$$

Inserting (26) with (27) into (24), and considering the above mentioned mean values and covariances together with equation (2), the model covariance function reads

$$\begin{aligned}\widetilde{C}^{(M)}(\mathbf{x}_1,\mathbf{x}_2)=&\,B_i(\mathbf{x}_1)B_j(\mathbf{x}_2)\big[R_i^0-m_i^0+R_{iK}\,(m_k^0-\overline{m}_k)\,G_{kK}^{(M)}\big]\\&\times\big[R_j^0-m_j^0+R_{jL}\,(m_l^0-\overline{m}_l)\,G_{lL}^{(M)}\big]\\&+B_i(\mathbf{x}_1)B_j(\mathbf{x}_2)R_{iK}R_{jL}(T_{KL}+\Theta_{KL})\\&-B_i(\mathbf{x}_1)R_{iK}\Psi_K^{(M)}(\mathbf{x}_2)-B_i(\mathbf{x}_2)R_{iK}\Psi_K^{(M)}(\mathbf{x}_1)+C^{(M)}(\mathbf{x}_1,\mathbf{x}_2) \quad .\end{aligned} \tag{30}$$

Inserting (30) and (28) into (25), and considering the above mentioned mean values and covariances together with equation (15), the objective function reads

$$\begin{aligned} y = {} & B_{ij}\left[R_i^0 - m_i^0 + R_{iK}(m_k^0 - \overline{m}_k)G_{kK}^{(M)}\right]\left[R_j^0 - m_j^0 + R_{jL}(m_l^0 - \overline{m}_l)G_{lL}^{(M)}\right] \\ & + \sum_{\Phi}((B_i, B_j))^{\Phi}\left[R_i^0 - m_i^{\Phi} + R_{iK}(m_k^0 - \overline{m}_k)G_{kK}^{(M)}\right]\left[R_j^0 - m_j^{\Phi} + R_{jL}(m_l^0 - \overline{m}_l)G_{lL}^{(M)}\right] \\ & + \left[B_{ij} + \sum_{\Phi}((B_i, B_j))^{\Phi}\right] R_{iK}R_{jL}(T_{KL} + \Theta_{KL}) - 2R_{iK}\Psi_{iK}^{(M)} + \int C^{(M)}(\mathbf{x}, \mathbf{x})\, \mathrm{d}\mathbf{x} \ . \end{aligned} \quad (31)$$

Minimization of (31) with respect to R_j^0 yields

$$R_i^0 = N_{ij}^{-1}\left[B_{jk}m_k^0 + \sum_{\Phi}((B_j, B_k))^{\Phi} m_k^{\Phi}\right] + R_{iL}\left(\overline{m}_l - m_l^0\right) G_{lL}^{(M)} \quad , \quad (32)$$

where N_{ij}^{-1} are components of the matrix inverse to

$$N_{ij} = B_{ij} + \sum_{\Phi}((B_i, B_j))^{\Phi} \quad . \quad (33)$$

Minimization of (31) with respect to R_{jK} then yields

$$\boxed{R_{iK} = N_{ij}^{-1}\, \Psi_{jJ}^{(M)}\, S_{JK}^{-1} \quad , \quad (34)}$$

where S_{KL}^{-1} are components of the matrix inverse to the complete travel–time covariance matrix

$$S_{KL} = T_{KL} + \Theta_{KL} \quad . \quad (35)$$

4.1. Updated velocity model

Equation (19) with (32) and (34) yield parameters

$$m_i = N_{ij}^{-1}\left[B_{jk}m_k^0 + \sum_{\Phi}((B_j, B_k))^{\Phi} m_k^{\Phi} + \Psi_{jJ}^{(M)} S_{JK}^{-1}[T_K - \overline{T}_K + (\overline{m}_k - m_k^0)\, G_{kK}^{(M)}]\right] \quad (36)$$

of the updated velocity model obtained from the previous velocity model described by parameters $\overline{m}_i$. This equation can be expressed in form

$$\boxed{m_i = m_i^0 + N_{ij}^{-1}\left[\Phi_j + \Psi_{jJ}^{(M)} S_{JK}^{-1}[T_K - \overline{T}_K + (\overline{m}_k - m_k^0)\, G_{kK}^{(M)}]\right] \quad , \quad (37)}$$

where

$$\Phi_j = \sum_{\Phi}((B_j, B_k))^{\Phi}(m_k^{\Phi} - m_k^0) \quad . \quad (38)$$

Quantities $\overline{T}_K$, $G_{iK}^{(M)}$, S_{JK}, $\Psi_{iK}^{(M)}$ in these equations are calculated in the velocity model described by parameters $\overline{m}_i$.

5. Model covariance function

Equation (30) for the model covariance function reads

$$\boxed{\begin{aligned}\widetilde{C}^{(M)}(\mathbf{x}_1,\mathbf{x}_2) = B_i(\mathbf{x}_1)B_j(\mathbf{x}_2)K_{ij} - B_i(\mathbf{x}_1)R_{iK}\Psi_K^{(M)}(\mathbf{x}_2) \\ - B_i(\mathbf{x}_2)R_{iK}\Psi_K^{(M)}(\mathbf{x}_1) + C^{(M)}(\mathbf{x}_1,\mathbf{x}_2) \quad , \qquad (39)\end{aligned}}$$

where

$$\boxed{K_{ij} = D_i^0 D_j^0 + R_{iK}R_{jL}S_{KL} \qquad (40)}$$

with

$$D_i^0 = R_i^0 - m_i^0 + R_{iK}\,(m_k^0 - \overline{m}_k)\,G_{kK}^{(M)} \quad . \qquad (41)$$

For vectors D_i^0 appearing in (40), equation (32) with definition (38) yield

$$D_i^0 = N_{ij}^{-1}\,\Phi_j \quad . \qquad (42)$$

5.1. Covariance matrix of synthetic travel times

The covariance matrix describing the accuracy of synthetic travel times corresponding to rays $\Omega_{K'}$ traced in the velocity model resulting from the travel–time inversion is

$$\widetilde{\Theta}_{J'K'} = \iint G_{J'}^{(M)}(\mathbf{x}_1)\,\widetilde{C}^{(M)}(\mathbf{x}_1,\mathbf{x}_2)\,G_{K'}^{(M)}(\mathbf{x}_2)\;\mathrm{d}\mathbf{x}_1\,\mathrm{d}\mathbf{x}_2 \quad . \qquad (43)$$

Inserting (39) into (43), we obtain

$$\boxed{\widetilde{\Theta}_{J'K'} = \Theta_{J'K'} - G_{iJ'}^{(M)}R_{iL}\Theta_{LK'} - G_{iK'}^{(M)}R_{iL}\Theta_{LJ'} + G_{iJ'}^{(M)}K_{ij}G_{jK'}^{(M)} \quad . \qquad (44)}$$

Integrals $G_{iK'}^{(M)}$ along rays $\Omega_{K'}$ are analogous to integrals (13) and may be calculated in the same way. Geometrical covariances $\Theta_{J'K'}$ between rays corresponding to synthetic travel times and geometrical covariances $\Theta_{JK'}$ between rays corresponding to synthetic travel times and rays Ω_K used for inversion are analogous to integrals (14) and may be calculated in the same way.

5.2. Covariance matrix of model perturbations

The covariance matrix of some perturbations $g_\alpha(\mathbf{x})$ of velocity model $[\widetilde{v}(\mathbf{x})]^M$ is

$$\widetilde{\Theta}_{\alpha\beta} = \iint g_\alpha(\mathbf{x})\,\widetilde{C}^{(M)}(\mathbf{x}_1,\mathbf{x}_2)\,g_\beta(\mathbf{x})\;\mathrm{d}\mathbf{x}_1\,\mathrm{d}\mathbf{x}_2 \quad . \qquad (45)$$

This covariance matrix may be useful during wavefield inversion based on the Born approximation (Pratt et al., 1996; Pratt, 1999; Pratt & Shipp, 1999; Klimeš, 2008b).

Inserting (39) into (45), we obtain

$$\widetilde{\Theta}_{\alpha\beta} = \Theta_{\alpha\beta} - G_{i\alpha}^{(M)}R_{iL}\Theta_{L\beta} - G_{i\beta}^{(M)}R_{iL}\Theta_{L\alpha} + G_{i\alpha}^{(M)}K_{ij}G_{j\beta}^{(M)} \quad , \qquad (46)$$

where

$$G_{i\alpha}^{(M)} = \int B_i(\mathbf{x})\,g_\alpha(\mathbf{x})\;\mathrm{d}\mathbf{x} \quad , \qquad (47)$$

$$\Theta_{J\alpha} = \iint G_J^{(M)}(\mathbf{x}_1)\,C^{(M)}(\mathbf{x}_1,\mathbf{x}_2)\,g_\alpha(\mathbf{x})\;\mathrm{d}\mathbf{x}_1\,\mathrm{d}\mathbf{x}_2 \quad , \qquad (48)$$

and

$$\Theta_{\alpha\beta} = \iint g_\alpha(\mathbf{x}_1)\,C^{(M)}(\mathbf{x}_1,\mathbf{x}_2)\,g_\beta(\mathbf{x})\;\mathrm{d}\mathbf{x}_1\,\mathrm{d}\mathbf{x}_2 \quad . \qquad (49)$$

5.3. Computer representation of the model covariance function

For computer representation of model covariance function $\widetilde{C}^{(M)}(\mathbf{x}_1,\mathbf{x}_2)$, we need to store rays Ω_K used for inversion. Functions $\Psi_K^{(M)}(\mathbf{x})$ in (39) are then determined by given medium covariance function $C^{(M)}(\mathbf{x}_1,\mathbf{x}_2)$ and stored rays Ω_K through equation (29).

Vector Φ_i and matrices N_{ij}, $\Psi_{iK}^{(M)}$ and S_{KL} are known from the last iteration of the linearized travel–time inversion. These quantities are sufficient for determination of matrices K_{ij} and R_{iK} in (39) through equations (34), (40) and (42).

For computer representation of model covariance function $\widetilde{C}^{(M)}(\mathbf{x}_1,\mathbf{x}_2)$, we should store rays Ω_K used for inversion, and matrices K_{ij} and R_{iK}.

If we cannot perform matrix inversions in (34), we may store vector Φ_i and matrices N_{ij}, $\Psi_{iK}^{(M)}$ and S_{KL} instead of proceeding to matrices K_{ij} and R_{iK}. In this emergency case, we can calculate individual covariances $\widetilde{\Theta}_{J'K'}$ (or $\widetilde{\Theta}_{\alpha\beta}$) in the following way:
(a) Calculate $a_{iL'} = N_{ij}^{-1} G_{jL'}^{(M)}$ from $G_{jL'}^{(M)}$ (or $a_{i\alpha} = N_{ij}^{-1} G_{j\alpha}^{(M)}$ from $G_{j\alpha}^{(M)}$) by solving the linear system of equations.
(b) Calculate $b_{0L'} = \Phi_i a_{iL'}$ and $b_{KL'} = \Psi_{iK}^{(M)} a_{iL'}$ (or $b_{0\alpha} = \Phi_i a_{i\alpha}$ and $b_{K\alpha} = \Psi_{iK}^{(M)} a_{i\alpha}$).
(c) Calculate $c_{JL'} = S_{JK}^{-1} b_{KL'}$ (or $c_{J\alpha} = S_{JK}^{-1} b_{K\alpha}$) by solving the linear system of equations. We now have $G_{iL'}^{(M)} R_{iJ} = c_{JL'}$ (or $G_{i\alpha}^{(M)} R_{iJ} = c_{J\alpha}$), see (34).
(d) Calculate $d_{L'N'} = c_{JL'} \Theta_{JN'}$ (or $d_{\alpha\beta} = c_{J\alpha} \Theta_{J\beta}$). We now have $G_{iL'}^{(M)} R_{iJ} \Theta_{JN'} = d_{L'N'}$ in equation (44) (or $G_{i\alpha}^{(M)} R_{iJ} \Theta_{J\beta} = d_{\alpha\beta}$ in equation (46)).
(e) Calculate $e_{L'N'} = b_{0L'} b_{0N'} + b_{JL'} c_{JN'}$ (or $e_{\alpha\beta} = b_{0\alpha} b_{0\beta} + b_{J\alpha} c_{J\beta}$). We now have $G_{iL'}^{(M)} K_{ij} G_{jN'}^{(M)} = e_{L'N'}$ in equation (44) (or $G_{i\alpha}^{(M)} K_{ij} G_{j\beta}^{(M)} = e_{\alpha\beta}$ in equation (46)).

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