Calculation of the a priori geometrical covariances of travel times in a self-affine random medium

Luděk Klimeš

Department of Geophysics, Faculty of Mathematics and Physics, Charles University, Ke Karlovu 3, 121 16 Praha 2, Czech Republic, http://sw3d.cz/staff/klimes.htm

Summary

To perform the stochastic inversion of travel times, we need the geometrical covariances of travel times, represented by the integrals of the medium covariance function over corresponding pairs of rays. These covariances are also essential for estimating the deviations of synthetic travel times calculated in the velocity model from correct travel times. The medium covariance function describing the deviation of the medium from the mean model is expressed in terms of the velocity–scaled ellipsoidally anisotropic power–law correlation function of a self–affine random medium. The geometrical covariances of travel times are semianalytically expressed in terms of the hypergeometric functions which can efficiently be calculated.

Keywords

Stochastic inversion of travel times, velocity model, heterogeneous medium, self–affine random medium, medium covariance function, travel–time covariance.

In: Seismic Waves in Complex 3–D Structures, Report 18 (Department of Geophysics, Charles University, Prague, 2008), pp. 27–70

1. Introduction

For the stochastic inversion of travel times (Franklin, 1970; Tarantola, 1987; Klimeš, 2002a), we need to know the a priori mean velocity model of the region and the medium covariance function which describes its uncertainty.

To perform the stochastic inversion of travel times, we need to calculate two kinds of integrals of the medium covariance function:

1. The geometrical covariances of travel times, represented by the integrals of the medium covariance function over corresponding pairs of the ray characteristic functions.
2. The geometrical covariances between travel times and the basis functions of which the velocity model is composed. Each of these covariances is represented by the integral of the medium covariance function over the ray characteristic function and over the basis function.

This paper is devoted to the calculation of the geometrical covariances of travel times. These covariances are also essential for estimating the deviations of synthetic travel times calculated in the velocity model from correct travel times, and consequently for estimating the inaccuracy of anything calculated using the synthetic travel times, e.g., for estimating the inaccuracy of determined seismic hypocentres.

We assume that the ray characteristic functions are infinitesimally narrow, which represents the high–frequency limit. The integral over the ray characteristic function then reduces to the integral over the ray path. This approximation is suitable for sparse travel–time measurements during which the finite–frequency ray characteristic functions corresponding to different sources or different receivers do not considerably overlap.

We assume that the medium covariance function describing the deviation of the medium from the mean model is expressed in terms of the velocity–scaled ellipsoidally anisotropic power–law correlation function of a self–affine stationary random function (Klimeš, 2002b; 2002c).

In Section 2, we express the geometrical covariance between two rays and corresponding to the velocity–scaled ellipsoidally anisotropic power–law medium covariance function in terms of the geometrical covariances between straight line segments and corresponding to the isotropic power–law medium covariance function. The rest of the paper is devoted to the algorithm of calculating the geometrical covariance between two straight line segments using the hypergeometric functions. In Section 3, we express the geometrical covariance between two straight line segments in form of the double integral over the unit square. In Section 4, we convert this integral over the unit square into four integrals over the sides of the unit square. In Section 5, we transform the integrals over the sides of the unit square into the integrals along the straight lines containing the line segments. In Section 6, we analytically calculate the integrals and express them in terms of the hypergeometric functions which can efficiently be calculated. Since the resulting expression of Section 5, is not applicable to parallel or nearly parallel line segments, we propose in Section 7 the approximation of this expression for parallel or nearly parallel line segments and derive the error of the approximation. This error enables us to decide when the approximation for parallel or nearly parallel line segments may be used. Section 8 is devoted to some simple special cases of the derived equations.

We use the Einstein summation over those repetitive lower–case roman subscripts which correspond to spatial coordinates throughout the paper.

2. Geometrical covariance between two rays

2.1 Velocity–scaled power–law medium covariance function

We define the mean velocity model as

$$v^0(X_k) = \left\{ \langle [v(X_k)]^M \rangle \right\}^{\frac{1}{M}} \quad , \tag{1}$$

and the medium covariance function as

$$C^{(M)}(X_{i1}, X_{j2}) = \langle [v(X_{k1})]^M \; [v(X_{l2})]^M \rangle - \langle [v(X_{k1})]^M \rangle \, \langle [v(X_{l2})]^M \rangle \quad , \tag{2}$$

where $[v(X_i)]^M$ is the M–th power of the wave–propagation velocity in an isotropic medium at point X_i, and $\langle\rangle$ denotes the mean value. Parameter M represents the power of propagation velocity which is linearized during the travel–time tomography. In this paper, we consider the velocity–scaled ellipsoidally anisotropic power–law medium covariance function

$$C^{(M)}(X_{m1}, X_{n2}) = M^2 \sigma^2 [v^0(X_{k1})]^{M-K} \, [v^0(X_{l2})]^{M-K} \, [(X_{i2} - X_{i1}) L^{-1}_{ij} (X_{j2} - X_{j1})]^N \tag{3}$$

(Klimeš, 2008, eq. 6). Here N is the Hurst exponent,

$$-\tfrac{1}{2} < N < 0 \quad , \tag{4}$$

and L^{-1}_{ij} are the components of the matrix inverse to positive–semidefinite matrix L_{ij}. Parameter K specifies the velocity power which has the stationary covariance function. The units of reference deviation σ at the reference ellipsoid correspond to the units of K–th power of velocity.

It may be comfortable to specify matrix L_{ij} in terms of its three eigenvectors E_{i1}, E_{i2}, E_{i3}, and in terms the square roots L_1, L_2, L_3 of its eigenvalues,

$$L_{ij} = \sum_{a=1}^{3} E_{ia} E_{ja} L_a^2 \quad . \tag{5}$$

The components of the inverse matrix to L_{ij} then read

$$L^{-1}_{ij} = \sum_{a=1}^{3} E_{ia} E_{ja} L_a^{-2} \quad . \tag{6}$$

Note that coefficient σ may optionally be specified in terms of wavenumber–domain coefficient κ (Klimeš, 2002b, eq. 67)

$$\sigma^2 = \frac{\Gamma(-N)}{2^{3+2N} \pi^{\frac{3}{2}} \Gamma(\frac{3}{2} + N)} \, \kappa^2 \quad . \tag{7}$$

2.2 Geometrical covariance of travel times

For infinitely narrow rays, the geometrical covariance of the I–th and J–th travel times is represented by integral

$$\Theta_{IJ} = \sigma^2 \int_0^{S_1} \mathrm{d}S_1 \int_0^{S_2} \mathrm{d}S_2 \, (v_1^0 \, v_2^0)^{M-K} (\widetilde{v}_1 \, \widetilde{v}_2)^{-M-1} \, (r^2)^N \tag{8}$$

over the corresponding rays traced in the current velocity model (Klimeš, 2008, eq. 16).

The quadrature is performed with respect to arclengths S_1 and S_2 which parametrize the rays in the current velocity model. Quantity

$$r^2 = (X_{i1} - X_{i2}) L_{ij}^{-1} (X_{j1} - X_{j2}) \tag{9}$$

is the square of the distance between point $X_{i1} = X_{i1}(S_1)$ of the I–th ray and point $X_{i2} = X_{i2}(S_2)$ of the J–th ray with respect to covariant metric tensor L_{ij}^{-1}. Mean velocity v_1^0 is taken at point X_{i1} and v_2^0 at point X_{i2} in the mean velocity model. Estimated propagation velocity $\widetilde{v}_1$ is taken at point X_{i1} and $\widetilde{v}_2$ at point X_{i2} in the current velocity model. For $M = -1$, which represents the ideal value of the velocity power for linearization of travel–time tomography, geometrical covariance (8) depends on the current velocity model $\widetilde{v}$ through the ray paths only.

We assume that the rays are specified as polylines composed of straight line segments: the I–th ray is given by $M_1{+}1$ points $X_{i1}^{(0)}$, $X_{i1}^{(1)}$, ..., $X_{i1}^{(M_1)}$, the J–th ray is given by $M_2{+}1$ points $X_{i2}^{(0)}$, $X_{i2}^{(1)}$, ..., $X_{i2}^{(M_2)}$. The geometrical covariance is then composed of the geometrical covariances between individual line segments,

$$\boxed{\Theta_{IJ} = \sum_{m_1=1}^{M_1} \sum_{m_2=1}^{M_2} \Theta^{(m_1)(m_2)} \quad ,} \tag{10}$$

where

$$\Theta^{(m_1)(m_2)} = \sigma^2 \int_{S_1^{(m_1-1)}}^{S_1^{(m_1)}} \mathrm{d}S_1 \int_{S_2^{(m_2-1)}}^{S_2^{(m_2)}} \mathrm{d}S_2 \, (v_1^0 \, v_2^0)^{M-K} (\widetilde{v}_1 \, \widetilde{v}_2)^{-M-1} \, (r^2)^N \tag{11}$$

is the geometrical covariance between line segment $X_{i1}^{(m_1-1)} X_{i1}^{(m_1)}$ and line segment $X_{i2}^{(m_2-1)} X_{i2}^{(m_2)}$. The quadrature is performed along these line segments with respect to arclengths S_1 and S_2.

In order to make the medium covariance function isotropic, we now switch to new coordinates

$$x_a = X_i E_{ia} / L_a \quad \text{(no summation over } _a\text{)} \quad . \tag{12}$$

In these coordinates, equation (9) with (6) reads

$$r^2 = (x_{i1} - x_{i2})(x_{i1} - x_{i2}) \quad , \tag{13}$$

and geometrical covariance (11) between two line segments approximately reads

$$\boxed{\begin{aligned} \Theta^{(m_1)(m_2)} \approx \sigma^2 \, \frac{\Delta S_1^{(m_1)} \, \Delta S_2^{(m_2)}}{\Delta s_1^{(m_1)} \, \Delta s_2^{(m_2)}} \, \frac{1}{4} & \left[\frac{\left[v^0(x_{i1}^{(m_1-1)})\right]^{M-K}}{\left[\widetilde{v}(x_{i1}^{(m_1-1)})\right]^{M+1}} + \frac{\left[v^0(x_{i1}^{(m_1)})\right]^{M-K}}{\left[\widetilde{v}(x_{i1}^{(m_1)})\right]^{M+1}} \right] \\ & \times \left[\frac{\left[v^0(x_{i2}^{(m_2-1)})\right]^{M-K}}{\left[\widetilde{v}(x_{i2}^{(m_2-1)})\right]^{M+1}} + \frac{\left[v^0(x_{i2}^{(m_2)})\right]^{M-K}}{\left[\widetilde{v}(x_{i2}^{(m_2)})\right]^{M+1}} \right] \Theta \quad , \end{aligned}} \tag{14}$$

where

$$\Delta S_1^{(m_1)} = \sqrt{(X_{i1}^{(m_1)} - X_{i1}^{(m_1-1)})(X_{i1}^{(m_1)} - X_{i1}^{(m_1-1)})} \quad , \tag{15}$$

$$\Delta S_2^{(m_2)} = \sqrt{(X_{i2}^{(m_2)} - X_{i2}^{(m_2-1)})(X_{i2}^{(m_2)} - X_{i2}^{(m_2-1)})} \quad , \tag{16}$$

$$\Delta s_1^{(m_1)} = \sqrt{(x_{i1}^{(m_1)} - x_{i1}^{(m_1-1)})(x_{i1}^{(m_1)} - x_{i1}^{(m_1-1)})} \quad , \tag{17}$$

$$\Delta s_2^{(m_2)} = \sqrt{(x_{i2}^{(m_2)} - x_{i2}^{(m_2-1)})(x_{i2}^{(m_2)} - x_{i2}^{(m_2-1)})} \quad , \tag{18}$$

and

$$\boxed{\Theta = \int_{s_1^{(m_1-1)}}^{s_1^{(m_1)}} \mathrm{d}s_1 \int_{s_2^{(m_2-1)}}^{s_2^{(m_2)}} \mathrm{d}s_2 \, (r^2)^N} \tag{19}$$

with r^2 given by equation (13). The quadrature in (19) is performed over line segment $x_{i1}^{(m_1-1)} x_{i1}^{(m_1)}$ with respect to Euclidean arclength s_1, and over line segment $x_{i2}^{(m_2-1)} x_{i2}^{(m_2)}$ with respect to Euclidean arclength s_2. For $M = -1$, approximate equation (14) depends on the current velocity model $\widetilde{v}$ through the ray paths only. Since the mean model v^0 is very smooth, the accuracy of approximation (14) is very good for $M = -1$, which represents the ideal value of the velocity power for linearization of travel–time tomography. Approximate equation (14) becomes exact if $K = M = -1$, but it seems that reasonable values of parameter K will be positive.

The next sections of this paper are devoted to the algorithm of calculating integral (19) for two given line segments. We now specify the maximum numerical error $\Delta\Theta$ of integral (19).

The travel–time variance defined by (10) with (11) may roughly be approximated by (Klimeš, 2002c, eq. 30)

$$\Theta_{II}^{\text{approx}} = \frac{\sigma^2}{(1+N)(1+2N)} \, (s_I)^{1+2N} \int_0^{s_I} \mathrm{d}s \, \left(\frac{\mathrm{d}S}{\mathrm{d}s}\right)^2 \frac{(v^0)^{2(M-K)}}{(\widetilde{v})^{2(M+1)}} \quad . \tag{20}$$

The numerical approximation of (20) is

$$\Theta_{II}^{\text{approx}} = \frac{\sigma^2}{(1+N)(1+2N)} \left[\sum_{m_1=1}^{M_1} s^{(m_1)}\right]^{1+2N}$$
$$\times \frac{1}{4}\left[\sum_{m_1=1}^{M_1} \frac{\left[S^{(m_1)}\right]^2}{s^{(m_1)}} \left(\frac{\left[v^0(x_{i1}^{(m_1-1)})\right]^{2(M-K)}}{\left[\widetilde{v}(x_{i1}^{(m_1-1)})\right]^{2(M+1)}} + \frac{\left[v^0(x_{i1}^{(m_1)})\right]^{2(M-K)}}{\left[\widetilde{v}(x_{i1}^{(m_1)})\right]^{2(M+1)}}\right)\right] \quad . \tag{21}$$

We specify the maximum relative numerical error ρ of the travel–time variances, and set the maximum absolute numerical errors of the individual travel–time covariances to

$$\Delta\Theta_{IJ} = \sqrt{\Theta_{II}^{\text{approx}}} \sqrt{\Theta_{JJ}^{\text{approx}}} \, \rho \quad . \tag{22}$$

Considering (10) and (14), we can comply with maximum error (22) if we specify the maximum absolute numerical error of (19) as

$$\boxed{\begin{aligned}\Delta\Theta = \frac{\Delta s_1^{(m_1)} \Delta s_2^{(m_2)} \Delta\Theta_{IJ}}{\sigma^2} \, 4 &\left[\sum_{m_1'=1}^{M_1} \Delta S_1^{(m_1')} \left(\frac{\left[v^0(x_{i1}^{(m_1-1)})\right]^{M-K}}{\left[\widetilde{v}(x_{i1}^{(m_1-1)})\right]^{M+1}} + \frac{\left[v^0(x_{i1}^{(m_1)})\right]^{M-K}}{\left[\widetilde{v}(x_{i1}^{(m_1)})\right]^{M+1}}\right)\right]^{-1} \\ \times &\left[\sum_{m_2'=1}^{M_2} \Delta S_2^{(m_2')} \left(\frac{\left[v^0(x_{i2}^{(m_2-1)})\right]^{M-K}}{\left[\widetilde{v}(x_{i2}^{(m_2-1)})\right]^{M+1}} + \frac{\left[v^0(x_{i2}^{(m_2)})\right]^{M-K}}{\left[\widetilde{v}(x_{i2}^{(m_2)})\right]^{M+1}}\right)\right]^{-1} \, .\end{aligned}} \tag{23}$$

The rest of this paper is devoted to the numerical algorithm for calculating double integral (19). Hereinafter, endpoints $x_{i1}^{(m_1-1)}$, $x_{i1}^{(m_1)}$, $x_{i2}^{(m_2-1)}$, $x_{i2}^{(m_2)}$ of the line segments are denoted A_{i1}, B_{i1}, A_{i2}, B_{i2}, respectively.

3. Geometrical covariance between two line segments

We have given two line segments A_1B_1 and A_2B_2. We wish to calculate their geometrical covariance represented by integral (19), where distance r between point x_{i1} of segment A_1B_1 and point x_{i2} of segment A_2B_2 is given by equation (13). The quadrature is performed over segment A_1B_1 with respect to arclength s_1 and over segment A_2B_2 with respect to arclength s_2.

We parametrize segment A_1B_1 by coordinate $\xi_1 \in \langle 0,1\rangle$ and segment A_2B_2 by coordinate $\xi_2 \in \langle 0,1\rangle$,

$$x_{i1} = A_{i1} + (B_{i1} - A_{i1})\xi_1 \quad . \tag{24}$$

Then

$$r^2 = c_{00} + 2c_{0K}\xi_K + c_{KL}\xi_K\xi_L \quad , \tag{25}$$

where

$$c_{00} = (A_{i1} - A_{i2})(A_{i1} - A_{i2}) \quad , \tag{26}$$

$$c_{01} = (A_{i1} - A_{i2})(B_{i1} - A_{i1}) \quad , \tag{27}$$

$$c_{02} = (A_{i1} - A_{i2})(A_{i2} - B_{i2}) \quad , \tag{28}$$

$$c_{11} = (B_{i1} - A_{i1})(B_{i1} - A_{i1}) \quad , \tag{29}$$

$$c_{12} = (B_{i1} - A_{i1})(A_{i2} - B_{i2}) \quad , \tag{30}$$

and

$$c_{22} = (A_{i2} - B_{i2})(A_{i2} - B_{i2}) \quad . \tag{31}$$

We may consider a system of planes parallel with both given line segments. We then express distance r in terms of distance z between the parallel planes containing the given line segments, and in terms of distance ϱ measured in the 2-D projection onto one of these parallel planes,

$$r^2 = z^2 + \varrho^2 \quad . \tag{32}$$

Here

$$z^2 = c_{00} - c_{0K}c^{-1}_{KL}c_{0L} \quad , \tag{33}$$

where c^{-1}_{KL} denotes the components of the matrix inverse to matrix c_{KL}, and

$$\varrho^2 = c_{KL}(\xi_K - \xi^0_K)(\xi_L - \xi^0_L) \tag{34}$$

with

$$\xi^0_K = -c^{-1}_{KL}c_{0L} \quad . \tag{35}$$

Note that matrix c_{KL} is ill–conditioned and ξ^0_K are large if the line segments are nearly parallel, and matrix c_{KL} is singular and ξ^0_K are infinite if the line segments are parallel. For these two special cases refer to Section 7 "Nearly parallel line segments".

We thus need to calculate integral

$$\boxed{\Theta = \sqrt{c_{11}c_{22}} \int_0^1 \mathrm{d}\xi_1 \int_0^1 \mathrm{d}\xi_2 \,(z^2 + \varrho^2)^N \tag{36}}$$

over the unit square $\langle 0,1\rangle \times \langle 0,1\rangle$ parametrized by coordinates ξ_1 and ξ_2.

4. Converting the double integral to a single integral

We parametrize the sides of the square by parameters ζ_1 or ζ_2. The sides can then be parametrically expressed as $\xi_K = \zeta_K$, where two sides are parametrized by $\zeta_1 \in \langle 0,1\rangle$ for given $\zeta_2 = 0$ or $\zeta_2 = 1$, and two sides are parametrized by $\zeta_2 \in \langle 0,1\rangle$ for given $\zeta_1 = 0$ or $\zeta_1 = 1$. We now switch from coordinates ξ_1 and ξ_2 to coordinate ϱ and coordinate ζ_1 or ζ_2 along the boundary of the square.

We denote by ϱ_ζ the value of ϱ at the sides of the square,

$$\varrho_\zeta{}^2 = c_{KL}(\zeta_K - \xi^0_K)(\zeta_L - \xi^0_L) \quad . \tag{37}$$

The transformation from new coordinates ϱ, ζ_1 for ζ_2 given (or ϱ, ζ_2 for ζ_1 given) to old coordinates ξ_1 and ξ_2 is determined by equation

$$\xi_K = \xi^0_K + \tfrac{\varrho}{\varrho_\zeta}(\zeta_K - \xi^0_K) \quad . \tag{38}$$

We calculate the Jacobian of this transform from coordinates ϱ, ζ_1 for ζ_2 given. The derivatives of ϱ_ζ in (38) follow from (37),

$$\tfrac{\partial \varrho_\zeta}{\partial \varrho} = 0 \quad , \tag{39}$$

$$\tfrac{\partial \varrho_\zeta}{\partial \zeta_1} = \tfrac{1}{\varrho_\zeta} c_{1L}(\zeta_L - \xi^0_L) \quad . \tag{40}$$

We denote

$$\eta_K = \zeta_K - \xi^0_K \quad , \tag{41}$$

and calculate transformation matrix

$$\begin{pmatrix} \frac{\partial \xi_1}{\partial \varrho} & \frac{\partial \xi_2}{\partial \varrho} \\ \frac{\partial \xi_1}{\partial \zeta_1} & \frac{\partial \xi_2}{\partial \zeta_1} \end{pmatrix} = \begin{pmatrix} \frac{1}{\varrho_\zeta}\eta_1 & \frac{1}{\varrho_\zeta}\eta_2 \\ \frac{\varrho}{\varrho_\zeta} - \frac{\varrho}{\varrho_\zeta{}^3}\eta_1 c_{1L}\eta_L & -\frac{\varrho}{\varrho_\zeta{}^3}\eta_2 c_{1L}\eta_L \end{pmatrix} \quad . \tag{42}$$

Its determinant is

$$\det\begin{pmatrix} \frac{\partial \xi_1}{\partial \varrho} & \frac{\partial \xi_2}{\partial \varrho} \\ \frac{\partial \xi_1}{\partial \zeta_1} & \frac{\partial \xi_2}{\partial \zeta_1} \end{pmatrix} = -\frac{\varrho\, \eta_2}{\varrho_\zeta{}^2} \quad . \tag{43}$$

Analogously for the transform from coordinates ϱ, ζ_2 with ζ_1 given. Since η_K and ϱ_ζ in (43) are independent of ϱ, we may integrate $(z^2 + \varrho^2)^N$ in (36) with respect to ϱ,

$$\int_0^{\varrho_\zeta} \mathrm{d}\varrho\, \varrho\, (z^2 + \varrho^2)^N = \tfrac{1}{2(1+N)}[(z^2 + \varrho_\zeta{}^2)^{1+N} - z^{2(1+N)}] \quad . \tag{44}$$

Using (43) and (44), we may convert the integral over the unit square in (36) into the integral over the boundary of the unit square,

$$\boxed{\begin{aligned} \int_0^1 \mathrm{d}\xi_1 \int_0^1 \mathrm{d}\xi_2\, (z^2 + \varrho^2)^N = {} & \left[\eta_2 \int_0^1 \mathrm{d}\zeta_1 \frac{(z^2 + \varrho_\zeta{}^2)^{1+N} - z^{2(1+N)}}{2(1+N)\ \varrho_\zeta{}^2}\right]_{\zeta_2=0}^{\zeta_2=1} \\ & + \left[\eta_1 \int_0^1 \mathrm{d}\zeta_2 \frac{(z^2 + \varrho_\zeta{}^2)^{1+N} - z^{2(1+N)}}{2(1+N)\ \varrho_\zeta{}^2}\right]_{\zeta_1=0}^{\zeta_1=1} \quad . \qquad (45) \end{aligned}}$$

Note that, for ζ_2 given, $\varrho_\zeta{}^2$ in (45) depends on coordinate ζ_1 as

$$\varrho_\zeta{}^2 = c_{11}\eta_1^2 + 2c_{12}\eta_1\eta_2 + c_{22}\eta_2^2 = c_{11}(\eta_1 + \tfrac{c_{12}}{c_{11}}\eta_2)^2 + \tfrac{c_{11}c_{22} - c_{12}c_{12}}{c_{11}}\eta_2^2 \quad . \tag{46}$$

5. Quadrature with respect to arclength

We calculate distance between points A_{i1} and A_{i2},

$$d_{A1A2} = \sqrt{(A_{i1} - A_{i2})(A_{i1} - A_{i2})} \quad , \tag{47}$$

distance between points B_{i1} and A_{i2},

$$d_{B1A2} = \sqrt{(B_{i1} - A_{i2})(B_{i1} - A_{i2})} \quad , \tag{48}$$

distance between points A_{i1} and B_{i2},

$$d_{A1B2} = \sqrt{(A_{i1} - B_{i2})(A_{i1} - B_{i2})} \quad , \tag{49}$$

and distance between points B_{i1} and B_{i2},

$$d_{B1B2} = \sqrt{(B_{i1} - B_{i2})(B_{i1} - B_{i2})} \quad . \tag{50}$$

We then exchange points A_{i1} and B_{i1}, and points A_{i2} and B_{i2} so that d_{A1A2} is the smallest of the above four distances. This reordering minimizes coefficient (26) and may be useful for minimizing the rounding errors.

The distance between points A_{i1} and B_{i1} is

$$d_{A1B1} = \sqrt{c_{11}} \quad , \tag{51}$$

see (29), and the distance between points A_{i2} and B_{i2} is

$$d_{A2B2} = \sqrt{c_{22}} \quad , \tag{52}$$

see (31).

Although distance z may be calculated according to equation (33) in terms of the scalar products, it is more accurate to express z using vectorial product

$$n_i = \varepsilon_{ijk}(B_{j1} - A_{j1})(A_{k2} - B_{k2}) \quad . \tag{53}$$

If its norm

$$n = \sqrt{n_i n_i} = \sqrt{c_{11}c_{22} - c_{12}c_{12}} \tag{54}$$

is small with respect to $d_{A1B1}d_{A2B2}$, i.e., if

$$\sin\alpha = \frac{n}{d_{A1B1}d_{A2B2}} \quad , \tag{55}$$

is small, the line segments are nearly parallel and we apply equation (238). Here α is the positive angle between half–lines $A_{i1}B_{i1}$ and $A_{i2}B_{i2}$. We shall specify what "small" means later in Section 7 "Nearly parallel line segments", see (253). If $\sin\alpha$ calculated using (55) is not small,

$$z = |(A_{i1} - A_{i2})n_i/n| \quad . \tag{56}$$

If $\sin\alpha$ is small, we do not need to calculate z.

Denote by A'_{i1}, B'_{i1}, A'_{i2}, B'_{i2} the projections of points A_{i1}, B_{i1}, A_{i2}, B_{i2} onto a plane parallel with lines $A_{i1}B_{i1}$ and $A_{i2}B_{i2}$. Along line $A'_{i1}B'_{i1}$, the point of intersection with line $A'_{i2}B'_{i2}$ corresponds to $\zeta_1 = \xi_1^0$ or $\eta_1 = 0$, point A'_{i1} to $\zeta_1 = 0$ or $\eta_1 = \eta_{A1}$, and point B'_{i1} to $\zeta_1 = 1$ or $\eta_1 = \eta_{B1}$, where

$$\eta_{A1} = -\xi_1^0 \quad , \qquad \eta_{B1} = 1 - \xi_1^0 \quad . \tag{57}$$

Along line $A'_{i2}B'_{i2}$, the point of intersection with line $A'_{i1}B'_{i1}$ corresponds to $\zeta_2 = \xi_2^0$ or $\eta_2 = 0$, point A'_{i2} to $\zeta_2 = 0$ or $\eta_2 = \eta_{A2}$, and point B'_{i2} to $\zeta_2 = 1$ or $\eta_2 = \eta_{B2}$, where

$$\eta_{A2} = -\xi_2^0 \quad , \qquad \eta_{B2} = 1 - \xi_2^0 \quad . \tag{58}$$

Arclengths

$$s_{A1} = \eta_{A1}\sqrt{c_{11}} \tag{59}$$

and

$$s_{B1} = \eta_{B1}\sqrt{c_{11}} \tag{60}$$

thus represent the coordinates of points A'_{i1} and B'_{i1} along straight line $A'_{i1}B'_{i1}$, with the origin at the point of intersection of lines $A'_{i1}B'_{i1}$ and $A'_{i2}B'_{i2}$, increasing from A'_{i1} towards B'_{i1}. Analogously

$$s_{A2} = \eta_{A2}\sqrt{c_{22}} \quad , \tag{61}$$

and

$$s_{B2} = \eta_{B2}\sqrt{c_{22}} \quad . \tag{62}$$

It may be rewarding to slightly correct arclengths s_{A1}, s_{B1}, s_{A2} and s_{B2}, calculated using (59)–(62), so that

$$|s_{B1} - s_{A1}| = d_{A1B1} \quad , \tag{63}$$

and

$$|s_{B2} - s_{A2}| = d_{A2B2} \quad . \tag{64}$$

We do not need to calculate s_{A1}, s_{B1}, s_{A2} and s_{B2} if $\sin\alpha$ calculated using (55) is small.

Coordinate of point A_1 along straight line $A_{i1}B_{i1}$, with the origin at the orthogonal projection of point A_2 onto line $A_{i1}B_{i1}$, increasing from A_{i1} towards B_{i1} is

$$s_{A1A2} = (A_{i1} - A_{i2})(B_{i1} - A_{i1})/d_{A1B1} \quad . \tag{65}$$

Analogously

$$s_{B1A2} = (B_{i1} - A_{i2})(B_{i1} - A_{i1})/d_{A1B1} \quad , \tag{66}$$

$$s_{A1B2} = (A_{i1} - B_{i2})(B_{i1} - A_{i1})/d_{A1B1} \quad , \tag{67}$$

$$s_{B1B2} = (B_{i1} - B_{i2})(B_{i1} - A_{i1})/d_{A1B1} \quad , \tag{68}$$

$$s_{A2A1} = (A_{i2} - A_{i1})(B_{i2} - A_{i2})/d_{A2B2} \quad , \tag{69}$$

$$s_{B2A1} = (B_{i2} - A_{i1})(B_{i2} - A_{i2})/d_{A2B2} \quad , \tag{70}$$

$$s_{A2B1} = (A_{i2} - B_{i1})(B_{i2} - A_{i2})/d_{A2B2} \quad , \tag{71}$$

and

$$s_{B2B1} = (B_{i2} - B_{i1})(B_{i2} - A_{i2})/d_{A2B2} \quad . \tag{72}$$

Note that

$$s_{A1A2} = s_{A1} - s_{A2}\cos\alpha \quad , \tag{73a}$$

$$s_{B1A2} = s_{B1} - s_{A2}\cos\alpha \quad , \tag{73b}$$

$$s_{A1B2} = s_{A1} - s_{B2}\cos\alpha \quad , \tag{73c}$$

$$s_{B1B2} = s_{B1} - s_{B2}\cos\alpha \quad , \tag{73d}$$

$$s_{A2A1} = s_{A2} - s_{A1}\cos\alpha \quad , \tag{73e}$$

$$s_{B2A1} = s_{B2} - s_{A1}\cos\alpha \quad , \tag{73f}$$

$$s_{A2B1} = s_{A2} - s_{B1}\cos\alpha \quad , \tag{73g}$$

$$s_{B2B1} = s_{B2} - s_{B1}\cos\alpha \quad , \tag{73h}$$

and that

$$s_{A1A2} = (\eta_{A1} + \tfrac{c_{12}}{c_{11}}\eta_{A2})\sqrt{c_{11}} = c_{01}/\sqrt{c_{11}} \quad , \tag{74}$$

etc.

If $\sin\alpha$ calculated using (55) is not small, vector

$$n_{i1} = \varepsilon_{ijk} n_j (B_{k1} - A_{k1}) \tag{75}$$

is perpendicular to straight line $A'_{i1}B'_{i1}$, and vector

$$n_{i2} = \varepsilon_{ijk} n_j (B_{k2} - A_{k2}) \tag{76}$$

is perpendicular to straight line $A'_{i2}B'_{i2}$. Distance of point A'_{i1} from straight line $A'_{i2}B'_{i2}$ is

$$q_{A1} = |(A_{i1} - A_{i2})n_{i2}| / \sqrt{n_{j2}n_{j2}} \quad . \tag{77}$$

Distance of point B'_{i1} from straight line $A'_{i2}B'_{i2}$ is

$$q_{B1} = |(B_{i1} - A_{i2})n_{i2}| / \sqrt{n_{j2}n_{j2}} \tag{78}$$

if $d_{B1A2} \leq d_{B1B2}$, otherwise

$$q_{B1} = |(B_{i1} - B_{i2})n_{i2}| / \sqrt{n_{j2}n_{j2}} \quad . \tag{79}$$

Distance of point A'_{i2} from straight line $A'_{i1}B'_{i1}$ is

$$q_{A2} = |(A_{i1} - A_{i2})n_{i1}| / \sqrt{n_{j1}n_{j1}} \quad . \tag{80}$$

Distance of point B'_{i2} from straight line $A'_{i1}B'_{i1}$ is

$$q_{B2} = |(B_{i2} - A_{i1})n_{i1}| / \sqrt{n_{j1}n_{j1}} \tag{81}$$

if $d_{A1B2} \leq d_{B1B2}$, otherwise

$$q_{B2} = |(B_{i2} - B_{i1})n_{i1}| / \sqrt{n_{j1}n_{j1}} \quad . \tag{82}$$

Note that

$$q_{A1} = |s_{A1}| \sin\alpha \quad , \tag{83a}$$
$$q_{B1} = |s_{B1}| \sin\alpha \quad , \tag{83b}$$
$$q_{A2} = |s_{A2}| \sin\alpha \quad , \tag{83c}$$
$$q_{B2} = |s_{B2}| \sin\alpha \quad , \tag{83d}$$

and that

$$q_{A1} = \eta_{A1} \sqrt{\tfrac{c_{11}c_{22} - c_{12}c_{12}}{c_{22}}} \quad , \tag{84a}$$
$$q_{B1} = \eta_{B1} \sqrt{\tfrac{c_{11}c_{22} - c_{12}c_{12}}{c_{22}}} \quad , \tag{84b}$$
$$q_{A2} = \eta_{A2} \sqrt{\tfrac{c_{11}c_{22} - c_{12}c_{12}}{c_{11}}} \quad , \tag{84c}$$
$$q_{B2} = \eta_{B2} \sqrt{\tfrac{c_{11}c_{22} - c_{12}c_{12}}{c_{11}}} \quad . \tag{84d}$$

We do not need to calculate z, q_{A1}, q_{B1}, q_{A2} and q_{B2} if $\sin\alpha$ calculated using (55) is small. Instead, we directly calculate distance

$$\sqrt{z^2 + q_{A1}^2} \tag{85}$$

of point A_{i1} from straight line $A_{i2}B_{i2}$, distance

$$\sqrt{z^2 + q_{B1}^2} \tag{86}$$

of point B_{i1} from straight line $A_{i2}B_{i2}$, distance

$$\sqrt{z^2 + q_{A2}^2} \tag{87}$$

of point A_{i2} from straight line $A_{i1}B_{i1}$, and distance

$$\sqrt{z^2 + q_{B2}^2} \tag{88}$$

of point B_{i2} from straight line $A_{i1}B_{i1}$.

Note also that

$$d_{A1A2} = \sqrt{z^2 + q_{A2}^2 + s_{A1A2}^2} = \sqrt{z^2 + q_{A1}^2 + s_{A2A1}^2} \quad , \tag{89}$$

$$d_{B1A2} = \sqrt{z^2 + q_{A2}^2 + s_{B1A2}^2} = \sqrt{z^2 + q_{B1}^2 + s_{A2B1}^2} \quad , \tag{90}$$

$$d_{A1B2} = \sqrt{z^2 + q_{B2}^2 + s_{A1B2}^2} = \sqrt{z^2 + q_{A1}^2 + s_{B2A1}^2} \quad , \tag{91}$$

and

$$d_{B1B2} = \sqrt{z^2 + q_{B2}^2 + s_{B1B2}^2} = \sqrt{z^2 + q_{B1}^2 + s_{B2B1}^2} \quad . \tag{92}$$

Considering (59)–(62),

$$\mathrm{d}\zeta_1 = \mathrm{d}s_{A1A2}/\sqrt{c_{11}} = \mathrm{d}s_{B1A2}/\sqrt{c_{11}} = \mathrm{d}s_{A1B2}/\sqrt{c_{11}} = \mathrm{d}s_{B1B2}/\sqrt{c_{11}} \tag{93}$$

and

$$\mathrm{d}\zeta_2 = \mathrm{d}s_{A2A1}/\sqrt{c_{22}} = \mathrm{d}s_{B2A1}/\sqrt{c_{22}} = \mathrm{d}s_{A2B1}/\sqrt{c_{22}} = \mathrm{d}s_{B2B1}/\sqrt{c_{22}} \quad , \tag{94}$$

we may express covariance (36) with integrals (45) in form

$$\boxed{\begin{aligned}\Theta = \tfrac{1}{2(1+N)} \{ & s_{B2}[I(z,q_{B2},s_{B1B2}) - I(z,q_{B2},s_{A1B2})] \\ & -s_{A2}[I(z,q_{A2},s_{B1A2}) - I(z,q_{A2},s_{A1A2})] \\ & +s_{B1}[I(z,q_{B1},s_{B2B1}) - I(z,q_{B1},s_{A2B1})] \\ & -s_{A1}[I(z,q_{A1},s_{B2A1}) - I(z,q_{A1},s_{A2A1})] \} \quad , \end{aligned}} \tag{95}$$

where

$$\boxed{I(z,q,s) = \int_0^s \mathrm{d}s \, \frac{(z^2+q^2+s^2)^{1+N} - (z^2)^{1+N}}{q^2+s^2} \quad .} \tag{96}$$

Equation (95) cannot be used for numerical calculations if the line segments are nearly parallel and $\sin\alpha$ is small. In this case, s_{A1}, s_{B1}, s_{A2} and s_{B2} are large, see $(83a)$–$(83d)$, and we would have to subtract terms considerably larger than the result in equation (95). This case is studied in Section 7.

6. Expressing the integral in terms of hypergeometric functions

Program Mathematica yields integrals

$$\int \mathrm{d}s \, \frac{(z^2+q^2+s^2)^{-b}}{q^2+s^2} = \tfrac{s}{q^2}\,(z^2+q^2)^{-b}\,\mathrm{F}_1(\tfrac{1}{2},b,1,\tfrac{3}{2};-\tfrac{s^2}{z^2+q^2};-\tfrac{s^2}{q^2}) \tag{97}$$

and

$$\int \mathrm{d}s\,(q^2+s^2)^{-b} = s\,(q^2)^{-b}\,{}_2\mathrm{F}_1(\tfrac{1}{2},b;\tfrac{3}{2};-\tfrac{s^2}{q^2}) \quad , \tag{98}$$

which can be used to analytically integrate (96). Here ${}_2\mathrm{F}_1(a,b;c;x)$ is the hypergeometric function of a single variable, and $\mathrm{F}_1(a,b,b',c;x,y)$ is the hypergeometric function of two variables, see Gradshteyn & Ryzhik (2000, sec. 9.18).

We may apply (97) with $b=-(1+N)$ and (98) with $b=1$ directly to (96),

$$\begin{aligned} I(z,q,s) = {} & \tfrac{s}{q^2}\,(z^2+q^2)^{1+N}\,\mathrm{F}_1(\tfrac{1}{2},-(1+N),1,\tfrac{3}{2};-\tfrac{s^2}{z^2+q^2};-\tfrac{s^2}{q^2}) \\ & -\tfrac{s}{q^2}\,(z^2)^{1+N}\,{}_2\mathrm{F}_1(\tfrac{1}{2},1;\tfrac{3}{2};-\tfrac{s^2}{q^2}) \quad . \end{aligned} \tag{99}$$

Another possibility is to use decomposition $(z^2+q^2+s^2)^{1+N} = z^2(z^2+q^2+s^2)^N + (q^2+s^2)(z^2+q^2+s^2)^N$ in (96), and apply (97) with $b=-N$ and (98) with $b=-N$ and $b=1$,

$$\begin{aligned} I(z,q,s) = {} & s\,\tfrac{z^2}{q^2}\,(z^2+q^2)^N\,\mathrm{F}_1(\tfrac{1}{2},-N,1,\tfrac{3}{2};-\tfrac{s^2}{z^2+q^2};-\tfrac{s^2}{q^2}) \\ & + s\,(z^2+q^2)^N\,{}_2\mathrm{F}_1(\tfrac{1}{2},-N;\tfrac{3}{2};-\tfrac{s^2}{z^2+q^2}) - \tfrac{s}{q^2}\,(z^2)^{1+N}\,{}_2\mathrm{F}_1(\tfrac{1}{2},1;\tfrac{3}{2};-\tfrac{s^2}{q^2}) \ . \end{aligned} \tag{100}$$

Of these two possibilities, we shall choose option (100) for further derivation. Note that the hypergeometric function in the last terms in equations (99) and (100) can be expressed in terms of the arc tangent function (Abramowitz & Stegun, 1964, eq. 15.1.5),

$${}_2\mathrm{F}_1(\tfrac{1}{2},1;\tfrac{3}{2};-\tfrac{s^2}{q^2}) = \tfrac{q}{s}\,\arctan(\tfrac{s}{q}) \quad . \tag{101}$$

We now wish to convert the negative variables in equation (100) to positive variables from interval $\langle 0,1\rangle$. For this purpose, we use identities

$$\mathrm{F}_1(a,b,b',c;x,y) = (1-x)^{-b}(1-y)^{-b'}\,\mathrm{F}_1(c-a,b,b',c;\tfrac{x}{x-1},\tfrac{y}{y-1}) \tag{102}$$

by Gradshteyn & Ryzhik (2000, eq. 9.183.1), and

$${}_2\mathrm{F}_1(a,b;c;x) = (1-x)^{-b}\,{}_2\mathrm{F}_1(c-a,b;c;\tfrac{x}{x-1}) \tag{103}$$

by Abramowitz & Stegun (1964, eq. 15.3.5). Applying identities (102) and (103) to equation (100), we obtain equation

$$\boxed{\begin{aligned} I(z,q,s) = {} & s\,\tfrac{z^2}{q^2+s^2}\,(z^2+q^2+s^2)^N\,\mathrm{F}_1(1,-N,1,\tfrac{3}{2};\tfrac{s^2}{z^2+q^2+s^2};\tfrac{s^2}{q^2+s^2}) \\ & + s\,(z^2+q^2+s^2)^N\,{}_2\mathrm{F}_1(1,-N;\tfrac{3}{2};\tfrac{s^2}{z^2+q^2+s^2}) \\ & - \tfrac{s}{q^2+s^2}\,(z^2)^{1+N}\,{}_2\mathrm{F}_1(1,1;\tfrac{3}{2};\tfrac{s^2}{q^2+s^2}) \quad . \end{aligned}} \tag{104}$$

Note that the hypergeometric function in the last term in equation (104) can be expressed in terms of the arc sine function using identity (129), see (131).

6.1. Hypergeometric series

We define one hypergeometric function of a single variable and three different hypergeometric functions of two variables, which we shall use in our equations, in the form of hypergeometric (Gauss) series.

The hypergeometric function ${}_2\mathrm{F}_1(a,b;c;x)$ of single variable x can be expressed for $|x|<1$ in the form of the Gauss hypergeometric series

$$ {}_2\mathrm{F}_1(a,b;c;x) = \sum_{m=0}^{+\infty} \frac{(a)_m (b)_m}{(c)_m (1)_m} \, x^m \tag{105} $$

(Abramowitz & Stegun, 1964, eq. 15.1.1; Gradshteyn & Ryzhik, 2000, eq. 9.100). Here

$$ (a)_0 = 1 \quad , \tag{106} $$

and

$$ (a)_m = \prod_{k=0}^{m-1} (a+k) \tag{107} $$

for integer $m > 0$. We may also write

$$ (a)_m = \frac{\Gamma(a+m)}{\Gamma(a)} \quad . \tag{108} $$

Note that

$$ (1)_m = m! \quad . \tag{109} $$

We shall use the hypergeometric functions

$$ \mathrm{F}_1(a,b,b',c;x,y) = \sum_{m=0}^{+\infty} \sum_{n=0}^{+\infty} \frac{(a)_{m+n} (b)_m (b')_n}{(c)_{m+n} (1)_m (1)_n} \, x^m y^n \tag{110} $$

(Gradshteyn & Ryzhik, 2000, eq. 9.180.1) and

$$ \mathrm{F}_2(a,b,b',c,c';x,y) = \sum_{m=0}^{+\infty} \sum_{n=0}^{+\infty} \frac{(a)_{m+n} (b)_m (b')_n}{(c)_m (c')_n (1)_m (1)_n} \, x^m y^n \tag{111} $$

(Gradshteyn & Ryzhik, 2000, eq. 9.180.2) of two variables x and y. Series (110) converges for $|x|<1$ and $|y|<1$, series (111) converges for $|x|+|y|<1$.

Replacing $(1)_m(1)_n$ by $(1)_{m+n}$ in (111), we define additional hypergeometric function

$$ \widetilde{\mathrm{F}}_2(a,b,b',c,c';x,y) = \sum_{m=0}^{+\infty} \sum_{n=0}^{+\infty} \frac{(a)_{m+n} (b)_m (b')_n}{(1)_{m+n} (c)_m (c')_n} \, x^m y^n \tag{112} $$

of two variables. Series (112) should converge for $|x|<1$ and $|y|<1$.

Note that we shall use only special cases

$$ \mathrm{F}_1(a,b,1,c;x,y) \quad , \qquad \mathrm{F}_2(a,b,1,c,c';x,y) \quad , \qquad \widetilde{\mathrm{F}}_2(a,b,1,c,1;x,y) \tag{113} $$

of the above three hypergeometric functions in our final equations. Series (105), (110), (111) and (112) can be used for reasonably fast calculation of the hypergeometric functions for positive variables x and y sufficiently smaller than 1. If variable x or y is not sufficiently smaller than 1, we shall switch to hypergeometric functions of variable $1-x$ or $1-y$.

Within the resulting numerical algorithm, we shall need to calculate functions (113) with $a, b, c, c' > 0$ only, except for $\widetilde{\mathrm{F}}_2(a, b, 1, c, 1; x, y)$ with $b \geq -\frac{1}{2}$, see (157). Hypergeometric series (110)–(112) are coded for the discussed range of parameters and variables in subroutines of file `hg.for` of package `FORMS` (Bucha & Bulant, 2008).

6.2. Decreasing the greater variable of hypergeometric functions

If the greater variable y of hypergeometric functions in ... is not sufficiently smaller than 1, we shall switch to hypergeometric functions of variable $1-y$. Let us now derive the corresponding equations.

We rearrange series (110) in form

$$\mathrm{F}_1(a, b, b', c; x, y) = \sum_{m=0}^{+\infty} \frac{(a)_m (b)_m}{(c)_m (1)_m} x^m \sum_{n=0}^{+\infty} \frac{(a+m)_n (b')_n}{(c+m)_n (1)_n} y^n \quad . \tag{114}$$

Considering series (105), we may express (114) as

$$\mathrm{F}_1(a, b, b', c; x, y) = \sum_{m=0}^{+\infty} \frac{(a)_m (b)_m}{(c)_m (1)_m} x^m {}_2\mathrm{F}_1(a+m, b'; c+m; y) \quad . \tag{115}$$

We may now insert identity (Abramowitz & Stegun, 1964, eq. 15.3.6; Gradshteyn & Ryzhik, 2000, eq. 9.131.2)

$$\begin{aligned} {}_2\mathrm{F}_1(a, b, c; x) = {} & \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} {}_2\mathrm{F}_1(a, b; a+b-c+1; 1-x) \\ & + \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} (1-x)^{c-a-b} {}_2\mathrm{F}_1(c-a, c-b; c-a-b+1; 1-x) \end{aligned} \tag{116}$$

valid for $x < 1$, and arrive at

$$\begin{aligned} \mathrm{F}_1(a, b, b', c; x, y) = {} & \sum_{m=0}^{+\infty} \frac{(a)_m (b)_m}{(c)_m (1)_m} x^m \\ & \times \left[\frac{\Gamma(c+m)\Gamma(c-a-b')}{\Gamma(c-a)\Gamma(c+m-b')} {}_2\mathrm{F}_1(a+m, b'; a+b'-c+1; 1-y) \right. \\ & \left. + \frac{\Gamma(c+m)\Gamma(a+b'-c)}{\Gamma(a+m)\Gamma(b')} (1-y)^{c-a-b'} {}_2\mathrm{F}_1(c-a, c+m-b'; c-a-b'+1; 1-y) \right] . \end{aligned} \tag{117}$$

Considering that $\Gamma(a+m) = \Gamma(a)\,(a)_m$, we rewrite this equation in form

$$\begin{aligned} & \mathrm{F}_1(a, b, b', c; x, y) \\ & = \frac{\Gamma(c)\Gamma(c-a-b')}{\Gamma(c-a)\Gamma(c-b')} \sum_{m=0}^{+\infty} \frac{(a)_m (b)_m}{(c-b')_m (1)_m} x^m {}_2\mathrm{F}_1(a+m, b'; a+b'-c+1; 1-y) \\ & + \frac{\Gamma(c)\Gamma(a+b'-c)}{\Gamma(a)\Gamma(b')} (1-y)^{c-a-b'} \sum_{m=0}^{+\infty} \frac{(b)_m}{(1)_m} x^m {}_2\mathrm{F}_1(c-a, c+m-b'; c-a-b'+1; 1-y) . \end{aligned} \tag{118}$$

We now insert series (105),

$$\begin{aligned}
&\mathrm{F}_1(a,b,b',c;x,y)\\
&\quad=\frac{\Gamma(c)\Gamma(c-a-b')}{\Gamma(c-a)\Gamma(c-b')}\sum_{m=0}^{+\infty}\frac{(a)_m(b)_m}{(c-b')_m(1)_m}x^m\sum_{n=0}^{+\infty}\frac{(a+m)_n(b')_n}{(a+b'-c+1)_n(1)_n}(1-y)^n\\
&\quad+\frac{\Gamma(c)\Gamma(a+b'-c)}{\Gamma(a)\Gamma(b')}(1-y)^{c-a-b'}\sum_{m=0}^{+\infty}\frac{(b)_m}{(1)_m}x^m\sum_{n=0}^{+\infty}\frac{(c-a)_n(c+m-b')_n}{(c-a-b'+1)_n(1)_n}(1-y)^n\ .
\end{aligned}\tag{119}$$

Considering that $(a)_m(a+m)_n=(a)_{m+n}$, we rearrange the double summation towards the form of series (111),

$$\begin{aligned}
&\mathrm{F}_1(a,b,b',c;x,y)\\
&=\frac{\Gamma(c)\Gamma(c-a-b')}{\Gamma(c-a)\Gamma(c-b')}\sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty}\frac{(a)_{m+n}(b)_m(b')_n}{(c-b')_m(a+b'-c+1)_n(1)_m(1)_n}x^m(1-y)^n\\
&+\frac{\Gamma(c)\Gamma(a+b'-c)}{\Gamma(a)\Gamma(b')}(1-y)^{c-a-b'}\sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty}\frac{(c-b')_{m+n}(b)_m(c-a)_n}{(c-b')_m(c-a-b'+1)_n(1)_m(1)_n}x^m(1-y)^n\ .
\end{aligned}\tag{120}$$

We may now express function (110) of variables x and y in terms of function (111) of variables x and $1-y$,

$$\boxed{\begin{aligned}
&\mathrm{F}_1(a,b,b',c;x,y)\\
&\quad=\frac{\Gamma(c)\Gamma(c-a-b')}{\Gamma(c-a)\Gamma(c-b')}\mathrm{F}_2(a,b,b',c-b',a+b'-c+1;x,1-y)\\
&\quad+\frac{\Gamma(c)\Gamma(a+b'-c)}{\Gamma(a)\Gamma(b')}(1-y)^{c-a-b'}\mathrm{F}_2(c-b',b,c-a,c-b',c-a-b'+1;x,1-y)\ .
\end{aligned}}\tag{121}$$

We now apply this general equation (121) to our special case,

$$\begin{aligned}
\mathrm{F}_1(1,b,1,\tfrac{3}{2};x,y)&=-\mathrm{F}_2(1,b,1,\tfrac{1}{2},\tfrac{3}{2};x,1-y)\\
&+\Gamma(\tfrac{1}{2})\Gamma(\tfrac{3}{2})(1-y)^{-\frac{1}{2}}\mathrm{F}_2(\tfrac{1}{2},b,\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2};x,1-y)\quad .
\end{aligned}\tag{122}$$

We use identity

$$\mathrm{F}_2(a,b,b',c,b';x,y)=(1-y)^{-a}{}_2\mathrm{F}_1(a,b;c;\tfrac{x}{1-y})\tag{123}$$

by Gradshteyn & Ryzhik (2000, eq. 9.182.2), and simplify (122) into

$$\begin{aligned}
\mathrm{F}_1(1,b,1,\tfrac{3}{2};x,y)&=-\mathrm{F}_2(1,b,1,\tfrac{1}{2},\tfrac{3}{2};x,1-y)\\
&+\Gamma(\tfrac{1}{2})\Gamma(\tfrac{3}{2})(1-y)^{-\frac{1}{2}}y^{-\frac{1}{2}}{}_2\mathrm{F}_1(\tfrac{1}{2},b;\tfrac{1}{2};\tfrac{x}{y})\quad .
\end{aligned}\tag{124}$$

We now use identity

$${}_2\mathrm{F}_1(a,b;b;x)=(1-x)^{-a}\tag{125}$$

by Abramowitz & Stegun (1964, eq. 15.1.8) or Gradshteyn & Ryzhik (2000, eq. 9.121.1), insert

$$\Gamma(\tfrac{1}{2})=\sqrt{\pi}\quad,\qquad\Gamma(\tfrac{3}{2})=\tfrac{\sqrt{\pi}}{2}\quad,\tag{126}$$

and arrive to

$$\boxed{\mathrm{F}_1(1,b,1,\tfrac{3}{2};x,y) = -\mathrm{F}_2(1,b,1,\tfrac{1}{2},\tfrac{3}{2};x,1-y) + \tfrac{\pi}{2}\,(1-y)^{-\frac{1}{2}}y^{-\frac{1}{2}}\big(1-\tfrac{x}{y}\big)^{-b} \quad . \qquad (127)}$$

To transform the hypergeometric function of single variable y to the function of $1-y$, we apply identity (116) to our special case,

$$ {}_2\mathrm{F}_1(1,1;\tfrac{3}{2};y) = -{}_2\mathrm{F}_1(1,1;\tfrac{3}{2};1-y) + \Gamma(\tfrac{1}{2})\Gamma(\tfrac{3}{2})\,(1-y)^{-\frac{1}{2}}{}_2\mathrm{F}_1(\tfrac{1}{2},\tfrac{1}{2};\tfrac{1}{2};1-y) \quad . \qquad (128)$$

Using identity

$$ {}_2\mathrm{F}_1(1,1;\tfrac{3}{2};x) = x^{-\frac{1}{2}}(1-x)^{-\frac{1}{2}}\arcsin\big(\sqrt{x}\,\big) \qquad (129)$$

(Abramowitz & Stegun, 1964, eq. 15.1.6) applicable to $0 < x < 1$, identity (125) and values (126), we arrive at expression

$$\boxed{ {}_2\mathrm{F}_1(1,1;\tfrac{3}{2};y) = -(1-y)^{-\frac{1}{2}}y^{-\frac{1}{2}}\arcsin\big(\sqrt{1-y}\,\big) + \tfrac{\pi}{2}\,(1-y)^{-\frac{1}{2}}\,y^{-\frac{1}{2}} \quad . \qquad (130)}$$

For $s^2 \lesssim q^2$, equation (104) with (129) reads

$$\boxed{\begin{aligned} I(z,q,s) &= s\,(z^2+q^2+s^2)^N\,{}_2\mathrm{F}_1(1,-N;\tfrac{3}{2};\tfrac{s^2}{z^2+q^2+s^2}) \\ &\quad + s\,\tfrac{z^2}{q^2+s^2}\,(z^2+q^2+s^2)^N\,\mathrm{F}_1(1,-N,1,\tfrac{3}{2};\tfrac{s^2}{z^2+q^2+s^2};\tfrac{s^2}{q^2+s^2}) \\ &\quad - \tfrac{(z^2)^{1+N}}{|q|}\,\arcsin(\tfrac{s}{\sqrt{q^2+s^2}}) \quad . \qquad (131) \end{aligned}}$$

For $s^2 \gtrsim q^2$, equation (104) with (127) and (130) yields

$$\begin{aligned} I(z,q,s) &= s\,(z^2+q^2+s^2)^N\,{}_2\mathrm{F}_1(1,-N;\tfrac{3}{2};\tfrac{s^2}{z^2+q^2+s^2}) \\ &\quad - s\,\tfrac{z^2}{q^2+s^2}\,(z^2+q^2+s^2)^N\,\mathrm{F}_2(1,-N,1,\tfrac{1}{2},\tfrac{3}{2};\tfrac{s^2}{z^2+q^2+s^2};\tfrac{q^2}{q^2+s^2}) \\ &\quad + \tfrac{\pi}{2}\,\tfrac{s(z^2)^{1+N}}{|s||q|} \\ &\quad + \tfrac{s(z^2)^{1+N}}{|s||q|}\,\arcsin(\tfrac{q}{\sqrt{q^2+s^2}}) \\ &\quad - \tfrac{\pi}{2}\,\tfrac{s(z^2)^{1+N}}{|s||q|} \quad . \qquad (132) \end{aligned}$$

Two terms in (132) cancel, and we obtain equation suitable for calculation in domain $q^2 \lesssim s^2 \lesssim z^2+q^2$,

$$\boxed{\begin{aligned} I(z,q,s) &= s\,(z^2+q^2+s^2)^N\,{}_2\mathrm{F}_1(1,-N;\tfrac{3}{2};\tfrac{s^2}{z^2+q^2+s^2}) \\ &\quad - s\,\tfrac{z^2}{q^2+s^2}\,(z^2+q^2+s^2)^N\,\mathrm{F}_2(1,-N,1,\tfrac{1}{2},\tfrac{3}{2};\tfrac{s^2}{z^2+q^2+s^2};\tfrac{q^2}{q^2+s^2}) \\ &\quad + \tfrac{s(z^2)^{1+N}}{|s||q|}\,\arcsin(\tfrac{|q|}{\sqrt{q^2+s^2}}) \quad . \qquad (133) \end{aligned}}$$

Note that, for $s \neq 0$ and $z \neq 0$,

$$\tfrac{s^2}{z^2+q^2+s^2} + \tfrac{q^2}{q^2+s^2} < 1 \quad . \qquad (134)$$

6.3. Decreasing both variables of hypergeometric functions

If both variables of hypergeometric functions in (104) are not sufficiently smaller than 1, we shall first use (127) and (130) to decrease the grater one of the two variables, and then decrease the other variable x to $1-x$. Let us now derive the corresponding equations.

Since hypergeometric function (111) is symmetric with respect to exchange $b \leftrightarrow b'$, $c \leftrightarrow c'$, $x \leftrightarrow y$, we may start with deriving equations for the transform of its variable y to $1-y$, similarly as we did for function (110) in previous Section 6.2. We rearrange series (111) in form

$$\mathrm{F}_2(a,b,b',c,c';x,y) = \sum_{m=0}^{+\infty} \frac{(a)_m (b)_m}{(c)_m (1)_m} x^m \sum_{n=0}^{+\infty} \frac{(a+m)_n (b')_n}{(c')_n (1)_n} y^n \quad . \tag{135}$$

Considering series (105), we may express (135) as

$$\mathrm{F}_2(a,b,b',c,c';x,y) = \sum_{m=0}^{+\infty} \frac{(a)_m (b)_m}{(c)_m (1)_m} x^m {}_2\mathrm{F}_1(a+m,b';c';y) \quad . \tag{136}$$

We may now insert identity (116),

$$\begin{aligned}
\mathrm{F}_2(a,b,b',c,c';x,y) &= \sum_{m=0}^{+\infty} \frac{(a)_m (b)_m}{(c)_m (1)_m} x^m \\
&\times \left[\frac{\Gamma(c')\Gamma(c'-a-b'-m)}{\Gamma(c'-a-m)\Gamma(c'-b')} {}_2\mathrm{F}_1(a+m,b';a+b'-c'+m+1;1-y) \right. \\
&\left. + \frac{\Gamma(c')\Gamma(a+b'-c'+m)}{\Gamma(a+m)\Gamma(b')} (1-y)^{c'-a-b'-m} {}_2\mathrm{F}_1(c'-a-m,c'-b';c'-a-b'-m+1;1-y) \right] .
\end{aligned} \tag{137}$$

Considering that $\Gamma(a+m) = \Gamma(a)\,(a)_m$, we rewrite this equation in form

$$\begin{aligned}
&\mathrm{F}_2(a,b,b',c,c';x,y) \\
&\quad = \frac{\Gamma(c')\Gamma(c'-a-b')}{\Gamma(c'-a)\Gamma(c'-b')} \sum_{m=0}^{+\infty} \frac{(a)_m (b)_m (c'-a-m)_m}{(c)_m (c'-a-b'-m)_m (1)_m} x^m \\
&\qquad\qquad \times {}_2\mathrm{F}_1(a+m,b';a+b'-c'+m+1;1-y) \\
&\quad + \frac{\Gamma(c')\Gamma(a+b'-c')}{\Gamma(a)\Gamma(b')} \sum_{m=0}^{+\infty} \frac{(b)_m (a+b'-c')_m}{(c)_m (1)_m} x^m (1-y)^{c'-a-b'-m} \\
&\qquad\qquad \times {}_2\mathrm{F}_1(c'-a-m,c'-b';c'-a-b'-m+1;1-y) \quad ,
\end{aligned} \tag{138}$$

and considering that $(a-m)_m = (-1)^m\,(-a+1)_m$, in form

$$\begin{aligned}
&\mathrm{F}_2(a,b,b',c,c';x,y) \\
&\quad = \frac{\Gamma(c')\Gamma(c'-a-b')}{\Gamma(c'-a)\Gamma(c'-b')} \sum_{m=0}^{+\infty} \frac{(a)_m (b)_m (a-c'+1)_m}{(c)_m (a+b'-c'+1)_m (1)_m} x^m \\
&\qquad\qquad \times {}_2\mathrm{F}_1(a+m,b';a+b'-c'+m+1;1-y)
\end{aligned}$$

$$+\frac{\Gamma(c')\Gamma(a+b'-c')}{\Gamma(a)\Gamma(b')}\sum_{m=0}^{+\infty}\frac{(b)_m(a+b'-c')_m}{(c)_m(1)_m}x^m(1-y)^{c'-a-b'-m}$$
$$\times\,{}_2\mathrm{F}_1(c'-a-m,c'-b';c'-a-b'-m+1;1-y)\quad . \tag{139}$$

We now insert series (105),

$$\begin{aligned}
&\mathrm{F}_2(a,b,b',c,c';x,y)\\
&=\frac{\Gamma(c')\Gamma(c'-a-b')}{\Gamma(c'-a)\Gamma(c'-b')}\sum_{m=0}^{+\infty}\frac{(a)_m(b)_m(a-c'+1)_m}{(c)_m(a+b'-c'+1)_m(1)_m}x^m\\
&\qquad\times\sum_{n=0}^{+\infty}\frac{(a+m)_n(b')_n}{(a+b'-c'+m+1)_n(1)_n}(1-y)^n\\
&+\frac{\Gamma(c')\Gamma(a+b'-c')}{\Gamma(a)\Gamma(b')}\sum_{m=0}^{+\infty}\frac{(b)_m(a+b'-c')_m}{(c)_m(1)_m}x^m(1-y)^{c'-a-b'-m}\\
&\qquad\times\sum_{n=0}^{+\infty}\frac{(c'-a-m)_n(c'-b')_n}{(c'-a-b'-m+1)_n(1)_n}(1-y)^n\quad .
\end{aligned}\tag{140}$$

Considering that $(a)_m(a+m)_n=(a)_{m+n}$, we rearrange the double summation,

$$\begin{aligned}
&\mathrm{F}_2(a,b,b',c,c';x,y)\\
&=\frac{\Gamma(c')\Gamma(c'-a-b')}{\Gamma(c'-a)\Gamma(c'-b')}\sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty}\frac{(a)_{m+n}(b)_m(a-c'+1)_m(b')_n}{(a+b'-c'+1)_{m+n}(c)_m(1)_m(1)_n}x^m(1-y)^n\\
&+\frac{\Gamma(c')\Gamma(a+b'-c')}{\Gamma(a)\Gamma(b')}(1-y)^{c'-a-b'}\\
&\times\sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty}\frac{(b)_m(a+b'-c')_m(c'-b')_n(c'-a-m)_n}{(c)_m(c'-a-b'-m+1)_n(1)_m(1)_n}x^m(1-y)^{-m}(1-y)^n\quad .
\end{aligned}\tag{141}$$

We shall restrict our further derivation to the special case $c'=a+1-c$ of function $\mathrm{F}_2(a,b,b',c,c';x,y)$ appearing in the first term on the right–hand side of general equation (121),

$$\begin{aligned}
&\mathrm{F}_2(a,b,b',c,a+1-c;x,y)\\
&=\frac{\Gamma(1-c+a)\Gamma(1-c-b')}{\Gamma(1-c)\Gamma(1-c-b'+a)}\sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty}\frac{(a)_{m+n}(b)_m(b')_n}{(c+b')_{m+n}(1)_m(1)_n}x^m(1-y)^n\\
&+\frac{\Gamma(a-c+1)\Gamma(b'+c-1)}{\Gamma(a)\Gamma(b')}(1-y)^{1-c-b'}\\
&\times\sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty}\frac{(b)_m(c+b'-1)_m(a+1-c-b')_n(1-c-m)_n}{(c)_m(2-c-b'-m)_n(1)_m(1)_n}x^m(1-y)^{-m}(1-y)^n\quad .
\end{aligned}\tag{142}$$

The first double sum in (142) represents hypergeometric function (110), but the second double sum is more intricate. We decompose the second double sum into three summations, for $m \leq n$, $m \geq n$ and $m = n$,

$$
\begin{aligned}
&\mathrm{F}_2(a,b,b',c,a+1-c;x,y)\\
&\quad = \frac{\Gamma(1-c+a)\Gamma(1-c-b')}{\Gamma(1-c)\Gamma(1-c-b'+a)}\ \mathrm{F}_1(a,b,b',c+b';x,1-y)\\
&\quad + \frac{\Gamma(a-c+1)\Gamma(b'+c-1)}{\Gamma(a)\Gamma(b')}(1-y)^{1-c-b'}\\
&\quad \times \left[\sum_{m=0}^{+\infty}\sum_{k=0}^{+\infty} \frac{(b)_m(c+b'-1)_m(a+1-c-b')_{m+k}(1-c-m)_{m+k}}{(c)_m(2-c-b'-m)_{m+k}(1)_m(1)_{m+k}} x^m(1-y)^{-m}(1-y)^{m+k}\right.\\
&\quad + \sum_{n=0}^{+\infty}\sum_{k=0}^{+\infty} \frac{(b)_{n+k}(c+b'-1)_{n+k}(a+1-c-b')_n(1-c-n-k)_n}{(c)_{n+k}(2-c-b'-n-k)_n(1)_{n+k}(1)_n} x^{n+k}(1-y)^{-n-k}(1-y)^{n}\\
&\quad \left. - \sum_{m=0}^{+\infty} \frac{(b)_m(a+1-c-b')_m}{(1)_m(1)_m} x^m \right] \quad . \qquad (143)
\end{aligned}
$$

In the double sums, we use identity $(a-m)_m = (-1)^m\,(-a+1)_m$ to convert the terms which can, together with terms $(...)_{m+k}$ or $(...)_{m+k}$, reduce the corresponding fraction. The last sum represents hypergeometric function (105).

$$
\begin{aligned}
&\mathrm{F}_2(a,b,b',c,a+1-c;x,y)\\
&\quad = \frac{\Gamma(1-c+a)\Gamma(1-c-b')}{\Gamma(1-c)\Gamma(1-c-b'+a)}\ \mathrm{F}_1(a,b,b',c+b';x,1-y)\\
&\quad + \frac{\Gamma(a-c+1)\Gamma(b'+c-1)}{\Gamma(a)\Gamma(b')}(1-y)^{1-c-b'}\\
&\quad \times \left[\sum_{m=0}^{+\infty}\sum_{k=0}^{+\infty} \frac{(b)_m(2-c-b'-m)_m(a+1-c-b')_{m+k}(1-c-m)_{m+k}}{(1-c-m)_m(2-c-b'-m)_{m+k}(1)_m(1)_{m+k}} x^m(1-y)^{k}\right.\\
&\quad + \sum_{n=0}^{+\infty}\sum_{k=0}^{+\infty} \frac{(b)_{n+k}(c+b'-1)_{n+k}(a+1-c-b')_n(c+k)_n}{(c)_{n+k}(c+b'-1+k)_n(1)_{n+k}(1)_n} x^{n}\left(\tfrac{x}{1-y}\right)^{k}\\
&\quad \left. - {}_2\mathrm{F}_1(b,a+1-c-b';1;x) \right] \quad . \qquad (144)
\end{aligned}
$$

We use identity $(a)_m(a+m)_n = (a)_{m+n}$ to reduce the fractions,

$$
\begin{aligned}
&\mathrm{F}_2(a,b,b',c,a+1-c;x,y)\\
&\quad = \frac{\Gamma(1-c+a)\Gamma(1-c-b')}{\Gamma(1-c)\Gamma(1-c-b'+a)}\ \mathrm{F}_1(a,b,b',c+b';x,1-y)\\
&\quad + \frac{\Gamma(a-c+1)\Gamma(b'+c-1)}{\Gamma(a)\Gamma(b')}(1-y)^{1-c-b'}\\
&\quad \times \left[\sum_{m=0}^{+\infty}\sum_{k=0}^{+\infty} \frac{(a+1-c-b')_{m+k}(b)_m(1-c)_k}{(1)_{m+k}(1)_m(2-c-b')_k} x^m(1-y)^{k}\right.
\end{aligned}
$$

$$
+\sum_{n=0}^{+\infty}\sum_{k=0}^{+\infty}\frac{(b)_{n+k}(a+1-c-b')_n(c-1+b')_k}{(1)_{n+k}(1)_n(c)_k}x^n(\tfrac{x}{1-y})^k
$$

$$
\left.-\,{}_2\mathrm{F}_1(b,a+1-c-b';1;x)\right] \quad . \tag{145}
$$

We may now replace the double sums by functions (112),

$$
\begin{aligned}
&\mathrm{F}_2(a,b,b',c,a+1-c;x,y)\\
&\quad=\frac{\Gamma(1-c+a)\Gamma(1-c-b')}{\Gamma(1-c)\Gamma(1-c-b'+a)}\,\mathrm{F}_1(a,b,b',c+b';x,1-y)\\
&\quad+\frac{\Gamma(a-c+1)\Gamma(b'+c-1)}{\Gamma(a)\Gamma(b')}(1-y)^{1-c-b'}\\
&\quad\times\Big[\widetilde{\mathrm{F}}_2(a+1-c-b',b,1-c,1,2-c-b';x,1-y)\\
&\quad\quad+\widetilde{\mathrm{F}}_2(b,a+1-c-b',c-1+b',1,c;x,\tfrac{x}{1-y})\\
&\quad\quad-\,{}_2\mathrm{F}_1(b,a+1-c-b';1;x)\Big] \quad .
\end{aligned} \tag{146}
$$

Considering the symmetry of hypergeometric functions (110), (111) and (112) with respect to variables x and y, we may convert (146) into

$$
\boxed{
\begin{aligned}
&\mathrm{F}_2(a,b,b',c,a+1-c;x,y)\\
&\quad=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\,\mathrm{F}_1(a,b,b',a+b-c+1;1-x,y)\\
&\quad+\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}(1-x)^{c-a-b}\\
&\quad\times\Big[\widetilde{\mathrm{F}}_2(c-b,c-a,b',c-a-b+1,1;1-x,y)\\
&\quad\quad+\widetilde{\mathrm{F}}_2(b',a+b-c,c-b,a-c+1,1;\tfrac{y}{1-x},y)\\
&\quad\quad-\,{}_2\mathrm{F}_1(b',c-b;1;y)\Big] \quad . \qquad (147)
\end{aligned}}
$$

We now apply this general equation (147) to our special case (122),

$$
\begin{aligned}
\mathrm{F}_2(1,b,1,\tfrac{1}{2},\tfrac{3}{2};x,y)&=\frac{\Gamma(\frac{1}{2})\Gamma(-\frac{1}{2}-b)}{\Gamma(-\frac{1}{2})\Gamma(\frac{1}{2}-b)}\,\mathrm{F}_1(1,b,1,\tfrac{3}{2}+b;1-x,y)\\
&+\frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}+b)}{\Gamma(b)}(1-x)^{-\frac{1}{2}-b}\\
&\times\Big[\widetilde{\mathrm{F}}_2(\tfrac{1}{2}-b,-\tfrac{1}{2},1,\tfrac{1}{2}-b,1;1-x,y)\\
&\quad+\widetilde{\mathrm{F}}_2(1,\tfrac{1}{2}+b,\tfrac{1}{2}-b,\tfrac{3}{2},1;\tfrac{y}{1-x},y)\\
&\quad-\,{}_2\mathrm{F}_1(1,\tfrac{1}{2}-b;1;y)\Big] \quad .
\end{aligned} \tag{148}
$$

Considering identity

$$
\widetilde{\mathrm{F}}_2(1,b,b',c,c';x,y)={}_2\mathrm{F}_1(1,b;c;x)\;{}_2\mathrm{F}_1(1,b';c';y) \quad , \tag{149}
$$

which directly follows from definitions (105) and (112), we obtain

$$\begin{aligned}\mathrm{F}_2(1,b,1,\tfrac{1}{2},\tfrac{3}{2};x,y) = {}& \tfrac{1}{1+2b}\,\mathrm{F}_1(1,b,1,\tfrac{3}{2}+b;1-x,y)\\ &+\frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}+b)}{\Gamma(b)}(1-x)^{-\frac{1}{2}-b}\\ &\times\Big\{\widetilde{\mathrm{F}}_2(\tfrac{1}{2}-b,-\tfrac{1}{2},1,\tfrac{1}{2}-b,1;1-x,y)\\ &+\Big[{}_2\mathrm{F}_1(1,\tfrac{1}{2}+b,\tfrac{3}{2};\tfrac{y}{1-x})-1\Big]\,{}_2\mathrm{F}_1(1,\tfrac{1}{2}-b;1;y)\Big\}\quad . \qquad (150)\end{aligned}$$

Inserting (125), we arrive at

$$\boxed{\begin{aligned}\mathrm{F}_2(1,b,1,\tfrac{1}{2},\tfrac{3}{2};x,y) = {}& \tfrac{1}{1+2b}\,\mathrm{F}_1(1,b,1,\tfrac{3}{2}+b;1-x,y)\\ &+\frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}+b)}{\Gamma(b)}(1-x)^{-\frac{1}{2}-b}\\ &\times\Big\{\widetilde{\mathrm{F}}_2(\tfrac{1}{2}-b,-\tfrac{1}{2},1,\tfrac{1}{2}-b,1;1-x,y)\\ &+(1-y)^{-\frac{1}{2}+b}\big[{}_2\mathrm{F}_1(1,\tfrac{1}{2}+b;\tfrac{3}{2};\tfrac{y}{1-x})-1\big]\Big\}\quad . \qquad (151)\end{aligned}}$$

To transform the hypergeometric function of a single variable in equation (133), we apply identity (116) to our special case,

$$\begin{aligned}{}_2\mathrm{F}_1(1,b,\tfrac{3}{2};x) = {}& \frac{\Gamma(\frac{3}{2})\Gamma(\frac{1}{2}-b)}{\Gamma(\frac{1}{2})\Gamma(\frac{3}{2}-b)}\,{}_2\mathrm{F}_1(1,b;\tfrac{1}{2}+b;1-x)\\ &+\frac{\Gamma(\frac{3}{2})\Gamma(-\frac{1}{2}+b)}{\Gamma(b)}(1-x)^{\frac{1}{2}-b}\,{}_2\mathrm{F}_1(\tfrac{1}{2},\tfrac{3}{2}-b;\tfrac{3}{2}-b;1-x)\quad . \qquad (152)\end{aligned}$$

Inserting (125), we arrive at

$$\boxed{\begin{aligned}{}_2\mathrm{F}_1(1,b,\tfrac{3}{2};x) = {}& \tfrac{1}{1-2b}\,{}_2\mathrm{F}_1(1,b;\tfrac{1}{2}+b;1-x)\\ &+\frac{\Gamma(\frac{3}{2})\Gamma(-\frac{1}{2}+b)}{\Gamma(b)}\,x^{-\frac{1}{2}}(1-x)^{\frac{1}{2}-b}\quad . \qquad (153)\end{aligned}}$$

We transform the hypergeometric function of a single variable in equation (133) using equation (153),

$$\begin{aligned}{}_2\mathrm{F}_1(1,-N;\tfrac{3}{2};\tfrac{s^2}{z^2+q^2+s^2}) = {}& \tfrac{1}{1+2N}\,{}_2\mathrm{F}_1(1,-N;\tfrac{1}{2}-N;\tfrac{z^2+q^2}{z^2+q^2+s^2})\\ &+\frac{\Gamma(\frac{3}{2})\Gamma(-\frac{1}{2}-N)}{\Gamma(-N)}(\tfrac{s^2}{z^2+q^2+s^2})^{-\frac{1}{2}}(\tfrac{z^2+q^2}{z^2+q^2+s^2})^{\frac{1}{2}+N}\quad , \qquad (154)\end{aligned}$$

and the hypergeometric function of two variables in equation (133) using equation (151),

$$\begin{aligned}&\mathrm{F}_2(1,-N,1,\tfrac{1}{2},\tfrac{3}{2};\tfrac{s^2}{z^2+q^2+s^2},\tfrac{q^2}{q^2+s^2})\\ &\quad= \tfrac{1}{1-2N}\,\mathrm{F}_1(1,-N,1,\tfrac{3}{2}-N;\tfrac{z^2+q^2}{z^2+q^2+s^2},\tfrac{q^2}{q^2+s^2})\\ &\quad+\frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}-N)}{\Gamma(-N)}(\tfrac{z^2+q^2}{z^2+q^2+s^2})^{-\frac{1}{2}+N}\end{aligned}$$

$$\times \Big[\widetilde{\mathrm{F}}_2(\tfrac{1}{2}+N,-\tfrac{1}{2},1,\tfrac{1}{2}+N,1;\tfrac{z^2+q^2}{z^2+q^2+s^2},\tfrac{q^2}{q^2+s^2})$$
$$+ (\tfrac{s^2}{q^2+s^2})^{-\frac{1}{2}-N}\,[{}_2\mathrm{F}_1(1,\tfrac{1}{2}-N;\tfrac{3}{2};\tfrac{(z^2+q^2+s^2)q^2}{(z^2+q^2+s^2)q^2+z^2s^2})-1]\Big] \quad . \tag{155}$$

For $s^2 \gtrsim z^2+q^2$, we insert (154) and (155) into (133),

$$\begin{aligned} I(z,q,s) = {} & \tfrac{1}{1+2N}\, s\,(z^2+q^2+s^2)^N\,{}_2\mathrm{F}_1(1,-N;\tfrac{1}{2}-N;\tfrac{z^2+q^2}{z^2+q^2+s^2}) \\ & + \frac{\Gamma(\frac{3}{2})\Gamma(-\frac{1}{2}-N)}{\Gamma(-N)}\, s\,(z^2+q^2+s^2)^N\,(\tfrac{s^2}{z^2+q^2+s^2})^{-\frac{1}{2}}(\tfrac{z^2+q^2}{z^2+q^2+s^2})^{\frac{1}{2}+N} \\ & - \tfrac{1}{1-2N}\, s\,\tfrac{z^2}{q^2+s^2}\,(z^2+q^2+s^2)^N\,\mathrm{F}_1(1,-N,1,\tfrac{3}{2}-N;\tfrac{z^2+q^2}{z^2+q^2+s^2},\tfrac{q^2}{q^2+s^2}) \\ & - \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}-N)}{\Gamma(-N)}\, s\,\tfrac{z^2}{q^2+s^2}\,(z^2+q^2+s^2)^N\,(\tfrac{z^2+q^2}{z^2+q^2+s^2})^{-\frac{1}{2}+N} \\ & \times \widetilde{\mathrm{F}}_2(\tfrac{1}{2}+N,-\tfrac{1}{2},1,\tfrac{1}{2}+N,1;\tfrac{z^2+q^2}{z^2+q^2+s^2},\tfrac{q^2}{q^2+s^2}) \\ & - \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}-N)}{\Gamma(-N)}\, s\,\tfrac{z^2}{q^2+s^2}\,(z^2+q^2+s^2)^N\,(\tfrac{z^2+q^2}{z^2+q^2+s^2})^{-\frac{1}{2}+N}(\tfrac{s^2}{q^2+s^2})^{-\frac{1}{2}-N} \\ & \times [{}_2\mathrm{F}_1(1,\tfrac{1}{2}-N;\tfrac{3}{2};\tfrac{(z^2+q^2+s^2)q^2}{(z^2+q^2+s^2)q^2+z^2s^2})-1] \\ & + \tfrac{s\,(z^2)^{1+N}}{|s|\,|q|}\,\arcsin(\tfrac{q}{\sqrt{q^2+s^2}}) \quad . \end{aligned} \tag{156}$$

We rearrange some products in (156) and convert the gamma function from a negative argument to a positive argument. For $s^2 \gtrsim z^2+q^2$, we then obtain equation

$$\boxed{\begin{aligned} I(z,q,s) = {} & \tfrac{1}{1+2N}\, s\,(z^2+q^2+s^2)^N\,{}_2\mathrm{F}_1(1,-N;\tfrac{1}{2}-N;\tfrac{z^2+q^2}{z^2+q^2+s^2}) \\ & - \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}-N)}{(1+2N)\,\Gamma(-N)}\,\tfrac{s}{|s|}\,(z^2+q^2)^{\frac{1}{2}+N} \\ & - \tfrac{1}{1-2N}\, s\,\tfrac{z^2}{q^2+s^2}\,(z^2+q^2+s^2)^N\,\mathrm{F}_1(1,-N,1,\tfrac{3}{2}-N;\tfrac{z^2+q^2}{z^2+q^2+s^2},\tfrac{q^2}{q^2+s^2}) \\ & - \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}-N)}{\Gamma(-N)}\, s\,\tfrac{\sqrt{z^2+q^2+s^2}}{q^2+s^2}\,\tfrac{z^2}{(z^2+q^2)^{\frac{1}{2}-N}} \\ & \times \widetilde{\mathrm{F}}_2(\tfrac{1}{2}+N,-\tfrac{1}{2},1,\tfrac{1}{2}+N,1;\tfrac{z^2+q^2}{z^2+q^2+s^2},\tfrac{q^2}{q^2+s^2}) \\ & + \tfrac{s\,(z^2)^{1+N}}{|s|\,|q|}\,\arcsin(\tfrac{q}{\sqrt{q^2+s^2}}) - J(q,s) \quad , \end{aligned}} \tag{157}$$

where

$$\begin{aligned} J(z,q,s) = {} & \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}-N)}{\Gamma(-N)}\,\tfrac{s}{|s|}\,(\tfrac{z^2+q^2+s^2}{q^2+s^2})^{\frac{1}{2}}\,(\tfrac{(q^2+s^2)}{s^2})^N\,\tfrac{z^2}{(z^2+q^2)^{\frac{1}{2}-N}} \\ & \times [{}_2\mathrm{F}_1(1,\tfrac{1}{2}-N;\tfrac{3}{2};\tfrac{(z^2+q^2+s^2)q^2}{(z^2+q^2+s^2)q^2+z^2s^2})-1] \quad . \end{aligned} \tag{158}$$

For $(z^2+q^2+s^2)\,q^2 \lesssim z^2s^2$, we use identity

$${}_2\mathrm{F}_1(1,b;c;x)-1 = \tfrac{b}{c}\,x\,{}_2\mathrm{F}_1(1,b+1;c+1;x) \tag{159}$$

following from (105), and insert it into (158),

$$\boxed{\begin{aligned} J(z,q,s) = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}-N)}{\Gamma(-N)} & \, \tfrac{1-2N}{3} \, \tfrac{s}{|s|} \, (\tfrac{z^2+q^2+s^2}{q^2+s^2})^{\frac{3}{2}} \, (\tfrac{q^2+s^2}{s^2})^N \, \tfrac{z^2\, q^2}{(z^2+q^2)^{\frac{3}{2}-N}} \\ & \times {}_2\mathrm{F}_1(1, \tfrac{3}{2}-N; \tfrac{5}{2}; \tfrac{(z^2+q^2+s^2)q^2}{(z^2+q^2+s^2)q^2+z^2s^2}) \quad . \end{aligned}} \tag{160}$$

Note that hypergeometric function ${}_2\mathrm{F}_1(1, \frac{3}{2}-N; \frac{5}{2}; x)$ in (160) can be converted using the Gauss' relation for contiguous hypergeometric functions (Abramowitz & Stegun, 1964, eq. 15.2.19) to ${}_2\mathrm{F}_1(1, \frac{1}{2}-N; \frac{5}{2}; x)$.

For $(z^2+q^2+s^2)\, q^2 \gtrsim z^2 s^2$, we transform the hypergeometric function in (158) using identity (153), which in our special case reads

$$\begin{aligned} {}_2\mathrm{F}_1(1, \tfrac{1}{2}-N; \tfrac{3}{2}; & \tfrac{(z^2+q^2+s^2)q^2}{(z^2+q^2+s^2)q^2+z^2s^2}) \\ & = \tfrac{1}{2N} \, {}_2\mathrm{F}_1(1, \tfrac{1}{2}-N; 1-N; \tfrac{z^2s^2}{(z^2+q^2+s^2)q^2+z^2s^2}) \\ & + \frac{\Gamma(\frac{3}{2})\Gamma(-N)}{\Gamma(\frac{1}{2}-N)} (\tfrac{(z^2+q^2+s^2)q^2}{(z^2+q^2+s^2)q^2+z^2s^2})^{-\frac{1}{2}} (\tfrac{z^2s^2}{(z^2+q^2+s^2)q^2+z^2s^2})^N \quad . \end{aligned} \tag{161}$$

We insert (161) into (158),

$$\begin{aligned} J(z,q,s) = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}-N)}{\Gamma(-N)} & \, \tfrac{s}{|s|} \, (\tfrac{z^2+q^2+s^2}{q^2+s^2})^{\frac{1}{2}} \, (\tfrac{(q^2+s^2)}{s^2})^N \, \tfrac{z^2}{(z^2+q^2)^{\frac{1}{2}-N}} \\ \times & \left[\frac{\Gamma(\frac{3}{2})\Gamma(-N)}{\Gamma(\frac{1}{2}-N)} (\tfrac{(z^2+q^2+s^2)q^2}{(q^2+s^2)(z^2+q^2)})^{-\frac{1}{2}} (\tfrac{z^2s^2}{(q^2+s^2)(z^2+q^2)})^N \right. \\ & \left. - \tfrac{1}{-2N} \, {}_2\mathrm{F}_1(1, \tfrac{1}{2}-N; 1-N; \tfrac{z^2s^2}{(z^2+q^2+s^2)q^2+z^2s^2}) - 1 \right] \quad . \end{aligned} \tag{162}$$

We carry out some multiplications in (162). For $(z^2+q^2+s^2)\, q^2 \gtrsim z^2 s^2$, we then obtain equation

$$\boxed{\begin{aligned} J(z,q,s) = & \, \tfrac{\pi}{2} \, \tfrac{s}{|s|} \, \tfrac{(z^2)^{1+N}}{(q^2)^{\frac{1}{2}}} \\ & - \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}-N)}{\Gamma(-N)} \, \tfrac{s}{|s|} \, (\tfrac{z^2+q^2+s^2}{q^2+s^2})^{\frac{1}{2}} \, (\tfrac{(q^2+s^2)}{s^2})^N \, \tfrac{z^2}{(z^2+q^2)^{\frac{1}{2}-N}} \\ & \times \left[1 + \tfrac{1}{-2N} \, {}_2\mathrm{F}_1(1, \tfrac{1}{2}-N; 1-N; \tfrac{z^2s^2}{(z^2+q^2+s^2)q^2+z^2s^2}) \right] \quad . \end{aligned}} \tag{163}$$

7. Nearly parallel line segments

For nearly parallel lines $A'_{i1}B'_{i1}$ and $A'_{i2}B'_{i2}$, we define

$$\overline{s}_{A1A2} = \tfrac{1}{2}(s_{A1A2} \mp s_{A2A1}) = \tfrac{1}{2}(s_{A1} \mp s_{A2})(1+|\cos\alpha|) \quad , \tag{164a}$$
$$\overline{s}_{B1A2} = \tfrac{1}{2}(s_{B1A2} \mp s_{A2B1}) = \tfrac{1}{2}(s_{B1} \mp s_{A2})(1+|\cos\alpha|) \quad , \tag{164b}$$
$$\overline{s}_{A1B2} = \tfrac{1}{2}(s_{A1B2} \mp s_{B2A1}) = \tfrac{1}{2}(s_{A1} \mp s_{B2})(1+|\cos\alpha|) \quad , \tag{164c}$$
$$\overline{s}_{B1B2} = \tfrac{1}{2}(s_{B1B2} \mp s_{B2B1}) = \tfrac{1}{2}(s_{B1} \mp s_{B2})(1+|\cos\alpha|) \quad , \tag{164d}$$

where $\mp = -\mathrm{sign}(\cos\alpha)$, and

$$\delta s_{A1A2} = \tfrac{1}{2}(s_{A1A2} \pm s_{A2A1}) = \tfrac{1}{2}(s_{A1} \pm s_{A2})(1-|\cos\alpha|) \quad , \tag{165a}$$
$$\delta s_{B1A2} = \tfrac{1}{2}(s_{B1A2} \pm s_{A2B1}) = \tfrac{1}{2}(s_{B1} \pm s_{A2})(1-|\cos\alpha|) \quad , \tag{165b}$$
$$\delta s_{A1B2} = \tfrac{1}{2}(s_{A1B2} \pm s_{B2A1}) = \tfrac{1}{2}(s_{A1} \pm s_{B2})(1-|\cos\alpha|) \quad , \tag{165c}$$
$$\delta s_{B1B2} = \tfrac{1}{2}(s_{B1B2} \pm s_{B2B1}) = \tfrac{1}{2}(s_{B1} \pm s_{B2})(1-|\cos\alpha|) \quad , \tag{165d}$$

where $\pm = \mathrm{sign}(\cos\alpha)$. Equations (164$a$)–(164$d$) and (165$a$)–(165$d$) then yield

$$s_{A1A2} = \overline{s}_{A1A2} + \delta s_{A1A2} \quad , \tag{166a}$$
$$s_{B1A2} = \overline{s}_{B1A2} + \delta s_{B1A2} \quad , \tag{166b}$$
$$s_{A1B2} = \overline{s}_{A1B2} + \delta s_{A1B2} \quad , \tag{166c}$$
$$s_{B1B2} = \overline{s}_{B1B2} + \delta s_{B1B2} \quad , \tag{166d}$$
$$s_{A2A1} = \mp\overline{s}_{A1A2} \pm \delta s_{A1A2} \quad , \tag{167a}$$
$$s_{A2B1} = \mp\overline{s}_{B1A2} \pm \delta s_{B1A2} \quad , \tag{167b}$$
$$s_{B2A1} = \mp\overline{s}_{A1B2} \pm \delta s_{A1B2} \quad , \tag{167c}$$
$$s_{B2B1} = \mp\overline{s}_{B1B2} \pm \delta s_{B1B2} \quad . \tag{167d}$$

Since $I(z,q,s) = I(z,-q,s)$, we do not mind possible minus signs of q_{A1}, q_{B1}, q_{A2} or q_{B2} related to $\sin\alpha$ through (83a)–(83d), and redefine them as

$$q_{A1} = s_{A1}\sin\alpha \quad , \tag{168a}$$
$$q_{B1} = s_{B1}\sin\alpha \quad , \tag{168b}$$
$$q_{A2} = \pm s_{A2}\sin\alpha \quad , \tag{169a}$$
$$q_{B2} = \pm s_{B2}\sin\alpha \quad , \tag{169b}$$

in order to simplify further derivations.

We select point Q'_{i1} on line $A'_{i1}B'_{i1}$ and point Q'_{i2} on line $A'_{i2}B'_{i2}$ so that their connecting line makes equal angles of size $|\frac{\pi}{2}-\alpha|$ with both lines $A'_{i1}B'_{i1}$ and $A'_{i2}B'_{i2}$. We denote coordinate s of point Q'_{i1} along line $A'_{i1}B'_{i1}$ by $\overline{s}$. Coordinate s of point Q'_{i2} along line $A'_{i2}B'_{i2}$ is then $\pm\overline{s}$. We select $\overline{s}$ close to values s_{A1}, s_{B1}, $\pm s_{A2}$ and $\pm s_{B2}$. We assume points Q'_{i1} and Q'_{i2} independent of angle α. Distance

$$\overline{q} = \overline{s}\sin\alpha \tag{170}$$

of point Q'_{i1} from line $A'_{i2}B'_{i2}$ and of point Q'_{i2} from line $A'_{i1}B'_{i1}$ is then independent of angle α, and q_{A1}, q_{B1}, q_{A2}, q_{B2} converge to $\overline{q}$ when the line segments approach to parallel. We define

$$\delta q_{A1} = (s_{A1}-\overline{s})\sin\alpha \quad , \tag{171a}$$
$$\delta q_{B1} = (s_{B1}-\overline{s})\sin\alpha \quad , \tag{171b}$$

$$\delta q_{A2} = (\pm s_{A2} - \overline{s}) \sin\alpha \quad , \tag{172a}$$
$$\delta q_{B2} = (\pm s_{B2} - \overline{s}) \sin\alpha \quad . \tag{172b}$$

Then

$$q_{A1} = \overline{q} + \delta q_{A1} \quad , \tag{173a}$$
$$q_{B1} = \overline{q} + \delta q_{B1} \quad , \tag{173b}$$
$$q_{A2} = \overline{q} + \delta q_{A2} \quad , \tag{173c}$$
$$q_{B2} = \overline{q} + \delta q_{B2} \quad . \tag{173d}$$

Differences (165a)–(165d) and (171a)–(172b) are of the first order with respect to $\sin\alpha$.

Since s_{A1}, s_{B1}, s_{A2} and s_{B2} in (95) are of the order of $(\sin\alpha)^{-1}$, we need the third–order expansion of $I(z,q,s)$ with respect to $\sin\alpha$ for the second–order expansion of (95) with respect to $\sin\alpha$.

The third–order Taylor expansion of (95) with respect to the above defined δq and δs reads

$$\Theta \approx \Theta_0 + \Theta_q + \Theta_s + \tfrac{1}{2}\Theta_{qq} + \Theta_{qs} + \tfrac{1}{2}\Theta_{ss} + \tfrac{1}{6}\Theta_{qqq} + \tfrac{1}{2}\Theta_{qqs} + \tfrac{1}{2}\Theta_{qss} + \tfrac{1}{6}\Theta_{sss} \ . \tag{174}$$

We denote by $I_q(z,q,s)$, $I_s(z,q,s)$, $I_{qq}(z,q,s)$, $I_{qs}(z,q,s)$, $I_{ss}(z,q,s)$, $I_{qqq}(z,q,s)$, $I_{qqs}(z,q,s)$, $I_{qss}(z,q,s)$ and $I_{sss}(z,q,s)$ the first–order, second–order and third–order partial derivatives of $I(z,q,s)$ with respect to q and s. The individual terms in Taylor expansion (174) then read

$$\begin{aligned}\Theta_0 = \tfrac{\pm 1}{2(1+N)} [& (s_{A1} \mp s_{B2})\, I(z,\overline{q},\overline{s}_{A1B2}) + (s_{B1} \mp s_{A2})\, I(z,\overline{q},\overline{s}_{B1A2}) \\ & -(s_{A1} \mp s_{A2})\, I(z,\overline{q},\overline{s}_{A1A2}) - (s_{B1} \mp s_{B2})\, I(z,\overline{q},\overline{s}_{B1B2})] \quad ,\end{aligned} \tag{175}$$

$$\begin{aligned}\Theta_q = \tfrac{\pm 1}{2(1+N)} [& (s_{A1}\delta q_{A1} \mp s_{B2}\delta q_{B2}) I_q(z,\overline{q},\overline{s}_{A1B2}) \\ & +(s_{B1}\delta q_{B1} \mp s_{A2}\delta q_{A2}) I_q(z,\overline{q},\overline{s}_{B1A2}) \\ & -(s_{A1}\delta q_{A1} \mp s_{A2}\delta q_{A2}) I_q(z,\overline{q},\overline{s}_{A1A2}) \\ & -(s_{B1}\delta q_{B1} \mp s_{B2}\delta q_{B2}) I_q(z,\overline{q},\overline{s}_{B1B2})] \, ,\end{aligned} \tag{176}$$

$$\begin{aligned}\Theta_s = \tfrac{\mp 1}{2(1+N)} [& (s_{A1} \pm s_{B2})\delta s_{A1B2} I_s(z,\overline{q},\overline{s}_{A1B2}) \\ & +(s_{B1} \pm s_{A2})\delta s_{B1A2} I_s(z,\overline{q},\overline{s}_{B1A2}) \\ & -(s_{A1} \pm s_{A2})\delta s_{A1A2} I_s(z,\overline{q},\overline{s}_{A1A2}) \\ & -(s_{B1} \pm s_{B2})\delta s_{B1B2} I_s(z,\overline{q},\overline{s}_{B1B2})] \, ,\end{aligned} \tag{177}$$

$$\begin{aligned}\Theta_{qq} = \tfrac{\pm 1}{2(1+N)} \{ & [s_{A1}\delta(q_{A1})^2 \mp s_{B2}\delta(q_{B2})^2]\, I_{qq}(z,\overline{q},\overline{s}_{A1B2}) \\ & +[s_{B1}\delta(q_{B1})^2 \mp s_{A2}\delta(q_{A2})^2]\, I_{qq}(z,\overline{q},\overline{s}_{B1A2}) \\ & -[s_{A1}\delta(q_{A1})^2 \mp s_{A2}\delta(q_{A2})^2]\, I_{qq}(z,\overline{q},\overline{s}_{A1A2}) \\ & -[s_{B1}\delta(q_{B1})^2 \mp s_{B2}\delta(q_{B2})^2]\, I_{qq}(z,\overline{q},\overline{s}_{B1B2})\} \quad ,\end{aligned} \tag{178}$$

$$\begin{aligned}\Theta_{qs} = \tfrac{\mp 1}{2(1+N)} [& (s_{A1}\delta q_{A1} \pm s_{B2}\delta q_{B2})\, \delta s_{A1B2}\, I_{qs}(z,\overline{q},\overline{s}_{A1B2}) \\ & +(s_{B1}\delta q_{B1} \pm s_{A2}\delta q_{A2})\, \delta s_{B1A2}\, I_{qs}(z,\overline{q},\overline{s}_{B1A2}) \\ & -(s_{A1}\delta q_{A1} \pm s_{A2}\delta q_{A2})\, \delta s_{A1A2}\, I_{qs}(z,\overline{q},\overline{s}_{A1A2}) \\ & -(s_{B1}\delta q_{B1} \pm s_{B2}\delta q_{B2})\, \delta s_{B1B2}\, I_{qs}(z,\overline{q},\overline{s}_{B1B2})] \quad ,\end{aligned} \tag{179}$$

$$\Theta_{ss} = \frac{\pm 1}{2(1+N)}[(s_{A1} \mp s_{B2})(\delta s_{A1B2})^2 I_{ss}(z,\overline{q},\overline{s}_{A1B2})
+(s_{B1} \mp s_{A2})(\delta s_{B1A2})^2 I_{ss}(z,\overline{q},\overline{s}_{B1A2})
-(s_{A1} \mp s_{A2})(\delta s_{A1A2})^2 I_{ss}(z,\overline{q},\overline{s}_{A1A2})
-(s_{B1} \mp s_{B2})(\delta s_{B1B2})^2 I_{ss}(z,\overline{q},\overline{s}_{B1B2})] \quad , \tag{180}$$

$$\Theta_{qqq} = \frac{\pm 1}{2(1+N)}\{[s_{A1}(\delta q_{A1})^3 \mp s_{B2}(\delta q_{B2})^3]\, I_{qqq}(z,\overline{q},\overline{s}_{A1B2})
+[s_{B1}(\delta q_{B1})^3 \mp s_{A2}(\delta q_{A2})^3]\, I_{qqq}(z,\overline{q},\overline{s}_{B1A2})
-[s_{A1}(\delta q_{A1})^3 \mp s_{A2}(\delta q_{A2})^3]\, I_{qqq}(z,\overline{q},\overline{s}_{A1A2})
-[s_{B1}(\delta q_{B1})^3 \mp s_{B2}(\delta q_{B2})^3]\, I_{qqq}(z,\overline{q},\overline{s}_{B1B2})\} \quad , \tag{181}$$

$$\Theta_{qqs} = \frac{\mp 1}{2(1+N)}\{[s_{A1}(\delta q_{A1})^2 \pm s_{B2}(\delta q_{B2})^2]\, \delta s_{A1B2}\, I_{qqs}(z,\overline{q},\overline{s}_{A1B2})
+[s_{B1}(\delta q_{B1})^2 \pm s_{A2}(\delta q_{A2})^2]\, \delta s_{B1A2}\, I_{qqs}(z,\overline{q},\overline{s}_{B1A2})
-[s_{A1}(\delta q_{A1})^2 \pm s_{A2}(\delta q_{A2})^2]\, \delta s_{A1A2}\, I_{qqs}(z,\overline{q},\overline{s}_{A1A2})
-[s_{B1}(\delta q_{B1})^2 \pm s_{B2}(\delta q_{B2})^2]\, \delta s_{B1B2}\, I_{qqs}(z,\overline{q},\overline{s}_{B1B2})\} \quad , \tag{182}$$

$$\Theta_{qss} = \frac{\pm 1}{2(1+N)}[(s_{A1}\delta q_{A1} \mp s_{B2}\delta q_{B2})(\delta s_{A1B2})^2 I_{qss}(z,\overline{q},\overline{s}_{A1B2})
+(s_{B1}\delta q_{B1} \mp s_{A2}\delta q_{A2})(\delta s_{B1A2})^2 I_{qss}(z,\overline{q},\overline{s}_{B1A2})
-(s_{A1}\delta q_{A1} \mp s_{A2}\delta q_{A2})(\delta s_{A1A2})^2 I_{qss}(z,\overline{q},\overline{s}_{A1A2})
-(s_{B1}\delta q_{B1} \mp s_{B2}\delta q_{B2})(\delta s_{B1B2})^2 I_{qss}(z,\overline{q},\overline{s}_{B1B2})] \quad , \tag{183}$$

$$\Theta_{sss} = \frac{\mp 1}{2(1+N)}[(s_{A1} \pm s_{B2})(\delta s_{A1B2})^3 I_{sss}(z,\overline{q},\overline{s}_{A1B2})
+(s_{B1} \pm s_{A2})(\delta s_{B1A2})^3 I_{sss}(z,\overline{q},\overline{s}_{B1A2})
-(s_{A1} \pm s_{A2})(\delta s_{A1A2})^3 I_{sss}(z,\overline{q},\overline{s}_{A1A2})
-(s_{B1} \pm s_{B2})(\delta s_{B1B2})^3 I_{sss}(z,\overline{q},\overline{s}_{B1B2})] \quad . \tag{184}$$

We now express the coefficients at $I(z,\overline{q},\overline{s}_{A1B2})$ and at its derivatives in equations (175)–(184) in alternative forms:

$$s_{A1} \mp s_{B2} = 2\,\overline{s}_{A1B2}/(1+|\cos\alpha|) \quad , \tag{185}$$

$$\begin{aligned} s_{A1}\delta q_{A1} \mp s_{B2}\delta q_{B2} &= [s_{A1}(s_{A1}-\overline{s}) \mp s_{B2}(\pm s_{B2}-\overline{s})]\sin\alpha \\ &= (s_{A1} \mp s_{B2})(s_{A1} \pm s_{B2}-\overline{s})\sin\alpha \\ &= (s_{A1} \mp s_{B2})(q_{A1}+q_{B2}-\overline{q}) \\ &= 2\,\overline{s}_{A1B2}(q_{A1}+q_{B2}-\overline{q})/(1+|\cos\alpha|) \\ &= 2\,\overline{s}_{A1B2}(\delta q_{A1}+\delta q_{B2}+\overline{q})/(1+|\cos\alpha|) \quad , \end{aligned} \tag{186}$$

$$\begin{aligned} (s_{A1} \pm s_{B2})\,\delta s_{A1B2} &= \tfrac{1}{2}(s_{A1} \pm s_{B2})^2(1-|\cos\alpha|) \\ &= \tfrac{1}{2}(s_{A1} \pm s_{B2})^2(\sin\alpha)^2/(1+|\cos\alpha|) \\ &= \tfrac{1}{2}(q_{A1}+q_{B2})^2/(1+|\cos\alpha|) \quad , \end{aligned} \tag{187}$$

$$\begin{aligned} &s_{A1}(\delta q_{A1})^2 \mp s_{B2}(\delta q_{B2})^2 \\ &\qquad = [s_{A1}(s_{A1}-\overline{s})^2 \mp s_{B2}(\pm s_{B2}-\overline{s})^2](\sin\alpha)^2 \end{aligned}$$

$$
\begin{aligned}
&= (s_{A1} \mp s_{B2})({s_{A1}}^2 \pm s_{A1}s_{B2} + {s_{B2}}^2 - 2s_{A1}\overline{s} \mp 2s_{B2}\overline{s} + \overline{s}^2)(\sin\alpha)^2 \\
&= (s_{A1} \mp s_{B2})[(q_{A1}+q_{B2}-\overline{q})^2 - q_{A1}q_{B2}] \\
&= 2\,\overline{s}_{A1B2}[(q_{A1}+q_{B2}-\overline{q})^2 - q_{A1}q_{B2}]/(1+|\cos\alpha|) \\
&= 2\,\overline{s}_{A1B2}[(\overline{q}+\delta q_{A1}+\delta q_{B2})^2 - (\overline{q}+\delta q_{A1})(\overline{q}+\delta q_{B2})]/(1+|\cos\alpha|) \\
&= 2\,\overline{s}_{A1B2}[\overline{q}(\delta q_{A1}+\delta q_{B2}) + (\delta q_{A1})^2 + \delta q_{A1}\delta q_{B2} + (\delta q_{B2})^2]/(1+|\cos\alpha|)\,, \qquad (188)
\end{aligned}
$$

$$
\begin{aligned}
&(s_{A1}\delta q_{A1} \pm s_{B2}\delta q_{B2})\,\delta s_{A1B2} \\
&\qquad = (s_{A1}\delta q_{A1} \pm s_{B2}\delta q_{B2})\tfrac{1}{2}(s_{A1} \pm s_{B2})(1-|\cos\alpha|) \\
&\qquad = (s_{A1}\delta q_{A1} \pm s_{B2}\delta q_{B2})\tfrac{1}{2}(s_{A1} \pm s_{B2})(\sin\alpha)^2/(1+|\cos\alpha|) \\
&\qquad = \tfrac{1}{2}(q_{A1}\delta q_{A1} + q_{B2}\delta q_{B2})(q_{A1}+q_{B2})/(1+|\cos\alpha|) \\
&\qquad = \tfrac{1}{2}[(\overline{q}+\delta q_{A1})\delta q_{A1} + (\overline{q}+\delta q_{B2})\delta q_{B2}](2\overline{q}+\delta q_{A1}+\delta q_{B2})/(1+|\cos\alpha|)\;, \qquad (189)
\end{aligned}
$$

$$
\begin{aligned}
(s_{A1} \mp s_{B2})(\delta s_{A1B2})^2 &= \tfrac{1}{4}(s_{A1} \mp s_{B2})(s_{A1} \pm s_{B2})^2(1-|\cos\alpha|)^2 \\
&= \tfrac{1}{4}(s_{A1} \mp s_{B2})(s_{A1} \pm s_{B2})^2(\sin\alpha)^4/(1+|\cos\alpha|)^2 \\
&= \tfrac{1}{4}(s_{A1} \mp s_{B2})(q_{A1}+q_{B2})^2(\sin\alpha)^2/(1+|\cos\alpha|)^2 \\
&= \tfrac{1}{2}\,\overline{s}_{A1B2}(q_{A1}+q_{B2})^2(\sin\alpha)^2/(1+|\cos\alpha|)^3 \quad , \qquad (190)
\end{aligned}
$$

$$
\begin{aligned}
&s_{A1}(\delta q_{A1})^3 \mp s_{B2}(\delta q_{B2})^3 \\
&\quad = [s_{A1}(s_{A1}-\overline{s})^3 \mp s_{B2}(\pm s_{B2}-\overline{s})^3](\sin\alpha)^3 \\
&\quad = (s_{A1} \mp s_{B2})[s_{A1}^3 \pm s_{A1}^2 s_{B2} + s_{A1}s_{B2}^2 \pm s_{B2}^3 \\
&\qquad\qquad -3(s_{A1}^2 \pm s_{A1}s_{B2} + s_{B2}^2)\overline{s} + 3(s_{A1} \pm s_{B2})\overline{s}^2 - \overline{s}^3](\sin\alpha)^3 \\
&\quad = 2\,\overline{s}_{A1B2}[(q_{A1}+q_{B2}-\overline{q})^3 - (2q_{A1}+2q_{B2}-3\overline{q})q_{A1}q_{B2}]/(1+|\cos\alpha|) \\
&\quad = 2\,\overline{s}_{A1B2}[(\overline{q}+\delta q_{A1}+\delta q_{B2})^3 - (\overline{q}+2\delta q_{A1}+2\delta q_{B2})(\overline{q}+\delta q_{A1})(\overline{q}+\delta q_{B2})]/(1+|\cos\alpha|)\;, \\
&\qquad\qquad (191)
\end{aligned}
$$

$$
\begin{aligned}
&[s_{A1}(\delta q_{A1})^2 \pm s_{B2}(\delta q_{B2})^2]\,\delta s_{A1B2} \\
&\qquad = [s_{A1}(\delta q_{A1})^2 \pm s_{B2}(\delta q_{B2})^2]\tfrac{1}{2}(s_{A1} \pm s_{B2})(1-|\cos\alpha|) \\
&\qquad = [s_{A1}(\delta q_{A1})^2 \pm s_{B2}(\delta q_{B2})^2]\tfrac{1}{2}(s_{A1} \pm s_{B2})(\sin\alpha)^2/(1+|\cos\alpha|) \\
&\qquad = \tfrac{1}{2}[q_{A1}(\delta q_{A1})^2 + q_{B2}(\delta q_{B2})^2](q_{A1}+q_{B2})/(1+|\cos\alpha|) \quad , \qquad (192)
\end{aligned}
$$

$$
\begin{aligned}
&(s_{A1}\delta q_{A1} \mp s_{B2}\delta q_{B2})(\delta s_{A1B2})^2 \\
&\qquad = \tfrac{1}{2}\overline{s}_{A1B2}(\overline{q}+\delta q_{A1}+\delta q_{B2})(s_{A1} \pm s_{B2})^2(1-|\cos\alpha|)^2/(1+|\cos\alpha|) \\
&\qquad = \tfrac{1}{2}\overline{s}_{A1B2}(\overline{q}+\delta q_{A1}+\delta q_{B2})(s_{A1} \pm s_{B2})^2(\sin\alpha|)^4/(1+|\cos\alpha|)^3 \\
&\qquad = \tfrac{1}{2}\overline{s}_{A1B2}(\overline{q}+\delta q_{A1}+\delta q_{B2})(q_{A1}+q_{B2})^2(\sin\alpha|)^2/(1+|\cos\alpha|)^3 \quad , \qquad (193)
\end{aligned}
$$

$$
\begin{aligned}
(s_{A1} \pm s_{B2})(\delta s_{A1B2})^3 &= \tfrac{1}{8}(s_{A1} \pm s_{B2})^4(1-|\cos\alpha|)^3 \\
&= \tfrac{1}{8}(s_{A1} \pm s_{B2})^4(\sin\alpha)^6/(1+|\cos\alpha|)^3 \\
&= \tfrac{1}{8}(q_{A1}+q_{B2})^4(\sin\alpha)^2/(1+|\cos\alpha|)^3 \quad . \qquad (194)
\end{aligned}
$$

7.1. Parallel line segments

Expressions (188)–(194) and higher–order terms vanish when the line segments approach to parallel. The geometrical covariance then converges to the limit

$$\overline{\overline{\Theta}} = \overline{\overline{\Theta}}_0 + \overline{\overline{\Theta}}_q + \overline{\overline{\Theta}}_s \tag{195}$$

of the first–order Taylor expansion. Inserting the limits $\overline{s}_{A1B2}$, $\overline{s}_{A1B2}\overline{q}$ and $\overline{q}^2$ of expressions (185)–(187) into equations (175)–(177), we obtain the individual terms in (195),

$$\begin{aligned}\overline{\overline{\Theta}}_0 = \tfrac{\pm 1}{2(1+N)}[&\overline{s}_{A1B2}I(z,\overline{q},\overline{s}_{A1B2}) + \overline{s}_{B1A2}I(z,\overline{q},\overline{s}_{B1A2}) \\ &-\overline{s}_{A1A2}I(z,\overline{q},\overline{s}_{A1A2}) - \overline{s}_{B1B2}I(z,\overline{q},\overline{s}_{B1B2})] \quad ,\end{aligned} \tag{196}$$

$$\begin{aligned}\overline{\overline{\Theta}}_q = \tfrac{\pm 1}{2(1+N)}\,\overline{q}\,[&\overline{s}_{A1B2}I_q(z,\overline{q},\overline{s}_{A1B2}) + \overline{s}_{B1A2}I_q(z,\overline{q},\overline{s}_{B1A2}) \\ &-\overline{s}_{A1A2}I_q(z,\overline{q},\overline{s}_{A1A2}) - \overline{s}_{B1B2}I_q(z,\overline{q},\overline{s}_{B1B2})] \,,\end{aligned} \tag{197}$$

$$\overline{\overline{\Theta}}_s = \tfrac{\mp 1}{2(1+N)}\,\overline{q}^2\,[I_s(z,\overline{q},\overline{s}_{A1B2}) + I_s(z,\overline{q},\overline{s}_{B1A2}) - I_s(z,\overline{q},\overline{s}_{A1A2}) - I_s(z,\overline{q},\overline{s}_{B1B2})] \,. \tag{198}$$

Geometrical covariance (195) of parallel line segments in distance $\sqrt{z^2+\overline{q}^2}$ can also be obtained directly from (95),

$$\begin{aligned}\overline{\overline{\Theta}} = \tfrac{\pm 1}{2(1+N)}[&\overline{s}_{A1B2}I(\sqrt{z^2+\overline{q}^2},0,\overline{s}_{A1B2}) + \overline{s}_{B1A2}I(\sqrt{z^2+\overline{q}^2},0,\overline{s}_{B1A2}) \\ &-\overline{s}_{A1A2}I(\sqrt{z^2+\overline{q}^2},0,\overline{s}_{A1A2}) - \overline{s}_{B1B2}I(\sqrt{z^2+\overline{q}^2},0,\overline{s}_{B1B2})] \quad .\end{aligned} \tag{199}$$

Note that (195) is equivalent to (199) if and only if

$$I(\sqrt{z^2+q^2},0,s) = I(z,q,s) + q\,I_q(z,q,s) - q^2 s^{-1}[I_s(z,q,s) - I_s(z,q,0)] \quad . \tag{200}$$

To check the correctness of our derivations, we shall prove identity (200). Identity (200) is satisfied at $s = 0$, and is thus satisfied elsewhere if and only if its derivative with respect to s is correct,

$$I_s(\sqrt{z^2+q^2},0,s) = I_s(z,q,s) + qI_{qs}(z,q,s) - q^2 I_{ss}(z,q,s)/s + q^2[I_s(z,q,s) - I_s(z,q,0)]/s^2. \tag{201}$$

We multiply (201) by s^2,

$$s^2 I_s(\sqrt{z^2+q^2},0,s) = (q^2+s^2)I_s(z,q,s) + qs^2 I_{qs}(z,q,s) - q^2 s I_{ss}(z,q,s) - q^2 I_s(z,q,0) \,. \tag{202}$$

Since the integrand in (96) is a function of z and $q^2 + s^2$, we have

$$s\,I_{qs}(z,q,s) = q\,I_{ss}(z,q,s) \quad . \tag{203}$$

Equation (202) is thus equivalent to

$$s^2 I_s(\sqrt{z^2+q^2},0,s) = (q^2+s^2)I_s(z,q,s) - q^2 I_s(z,q,0) \quad . \tag{204}$$

We insert the integrand from (96) for $I_s(z,q,s)$ into (204) and obtain equation

$$(z^2+q^2+s^2)^{1+N} - (z^2+q^2)^{1+N} = (z^2+q^2+s^2)^{1+N} - (z^2)^{1+N} - (z^2+q^2)^{1+N} + (z^2)^{1+N} \tag{205}$$

which is obviously satisfied.

7.2. Error caused by substituting parallel line segments for nearly parallel line segments

Differences of expressions (185)–(187) from their limits $\overline{s}_{A1B2}$, $\overline{s}_{A1B2}\overline{q}$ and $\overline{q}^2$ read

$$s_{A1} \mp s_{B2} - \overline{s}_{A1B2} = \overline{s}_{A1B2}\,(\sin\alpha)^2/(1+|\cos\alpha|)^2 \quad , \tag{206}$$

$$\begin{aligned} &s_{A1}\delta q_{A1} \mp s_{B2}\delta q_{B2} - \overline{s}_{A1B2}\overline{q} \\ &\quad = 2\,\overline{s}_{A1B2}(q_{A1}+q_{B2}-2\overline{q})/(1+|\cos\alpha|) + \overline{s}_{A1B2}\overline{q}(\sin\alpha)^2/(1+|\cos\alpha|)^2 \,, \end{aligned} \tag{207}$$

$$\begin{aligned} &(s_{A1} \pm s_{B2})\,\delta s_{A1B2} - \overline{q}^2 \\ &= \tfrac{1}{2}(q_{A1}+q_{B2} - 2\overline{q})(q_{A1}+q_{B2} + 2\overline{q})/(1+|\cos\alpha|) + \overline{q}^2(\sin\alpha)^2/(1+|\cos\alpha|)^2 \\ &= \tfrac{1}{2}(\delta q_{A1}+\delta q_{B2})(4\overline{q}+\delta q_{A1}+\delta q_{B2})/(1+|\cos\alpha|) + \overline{q}^2(\sin\alpha)^2/(1+|\cos\alpha|)^2 \quad . \end{aligned} \tag{208}$$

We approximate expressions (206)–(208) and (188)–(194) by second–order approximations with respect to $\sin\alpha$,

$$s_{A1} \mp s_{B2} - \overline{s}_{A1B2} \approx \tfrac{1}{4}(\sin\alpha)^2\overline{s}_{A1B2} \quad , \tag{209}$$

$$s_{A1}\delta q_{A1} \mp s_{B2}\delta q_{B2} - \overline{s}_{A1B2}\overline{q} \approx (\delta q_{A1}+\delta q_{B2})\overline{s}_{A1B2} + \tfrac{1}{4}\overline{q}(\sin\alpha)^2\overline{s}_{A1B2} \quad , \tag{210}$$

$$(s_{A1} \pm s_{B2})\,\delta s_{A1B2} - \overline{q}^2 \approx \overline{q}(\delta q_{A1}+\delta q_{B2}) + \tfrac{1}{4}(\delta q_{A1}+\delta q_{B2})^2 + \tfrac{1}{4}\overline{q}^2(\sin\alpha)^2 \quad , \tag{211}$$

$$s_{A1}(\delta q_{A1})^2 \mp s_{B2}(\delta q_{B2})^2 \approx \overline{q}(\delta q_{A1}+\delta q_{B2})\overline{s}_{A1B2} + [(\delta q_{A1})^2+\delta q_{A1}\delta q_{B2}+(\delta q_{B2})^2]\overline{s}_{A1B2} \,, \tag{212}$$

$$(s_{A1}\delta q_{A1} \pm s_{B2}\delta q_{B2})\,\delta s_{A1B2} \approx \tfrac{1}{2}\overline{q}^2(\delta q_{A1}+\delta q_{B2}) + \tfrac{1}{4}\overline{q}[3(\delta q_{A1})^2+2\delta q_{A1}\delta q_{B2}+3(\delta q_{B2})^2] \,, \tag{213}$$

$$(s_{A1} \mp s_{B2})(\delta s_{A1B2})^2 \approx \tfrac{1}{4}\,\overline{s}_{A1B2}\,\overline{q}^2\,(\sin\alpha)^2 \quad , \tag{214}$$

$$s_{A1}(\delta q_{A1})^3 \mp s_{B2}(\delta q_{B2})^3 \approx \overline{s}_{A1B2}\,\overline{q}\,[(\delta q_{A1})^2+\delta q_{A1}\delta q_{B2}+(\delta q_{B2})^2] \quad , \tag{215}$$

$$[s_{A1}(\delta q_{A1})^2 \pm s_{B2}(\delta q_{B2})^2]\,\delta s_{A1B2} \approx \tfrac{1}{2}\overline{q}^2[(\delta q_{A1})^2+(\delta q_{B2})^2] \quad , \tag{216}$$

$$(s_{A1}\delta q_{A1} \mp s_{B2}\delta q_{B2})(\delta s_{A1B2})^2 \approx \tfrac{1}{4}\overline{s}_{A1B2}\overline{q}^3(\sin\alpha)^2 \quad , \tag{217}$$

$$(s_{A1} \pm s_{B2})(\delta s_{A1B2})^3 \approx \tfrac{1}{4}\overline{q}^4(\sin\alpha)^2 \quad . \tag{218}$$

The first terms on the right–hand sides of equations (210)–(213) are of the first order with respect to $\sin\alpha$. All other terms on the right–hand sides of equations (209)–(218) are of the second order with respect to $\sin\alpha$.

We approximate the difference of the geometrical covariance from its approximation (195) by the second–order expansion with respect to $\sin\alpha$,

$$\Theta - \overline{\overline{\Theta}} \approx \delta\Theta^{(1)} + \delta\Theta^{(2)} \quad . \tag{219}$$

Here $\delta\Theta^{(1)}$ represents by the first–order expansion with respect to $\sin\alpha$, and $\delta\Theta^{(2)}$ represents by the second–order terms with respect to $\sin\alpha$.

7.2.1. First–order error

The first–order approximation of the difference of the geometrical covariance from its approximation (195) can be expressed in form

$$\delta\Theta^{(1)} = \delta\Theta^{(1)}_q + \delta\Theta^{(1)}_s + \tfrac{1}{2}\delta\Theta^{(1)}_{qq} + \delta\Theta^{(1)}_{qs} \quad , \tag{220}$$

where

$$\begin{aligned}\delta\Theta^{(1)}_q = \tfrac{\pm 1}{2(1+N)}[&(\delta q_{A1}+\delta q_{B2})\,\overline{s}_{A1B2} I_q(z,\overline{q},\overline{s}_{A1B2})\\ &+(\delta q_{B1}+\delta q_{A2})\,\overline{s}_{B1A2} I_q(z,\overline{q},\overline{s}_{B1A2})\\ &-(\delta q_{A1}+\delta q_{A2})\,\overline{s}_{A1A2} I_q(z,\overline{q},\overline{s}_{A1A2})\\ &-(\delta q_{B1}+\delta q_{B2})\,\overline{s}_{B1B2} I_q(z,\overline{q},\overline{s}_{B1B2})] \quad ,\end{aligned} \tag{221}$$

$$\begin{aligned}\delta\Theta^{(1)}_s = \tfrac{\mp 1}{2(1+N)}\,\overline{q}\,[&(\delta q_{A1}+\delta q_{B2})\, I_s(z,\overline{q},\overline{s}_{A1B2})\\ &+(\delta q_{B1}+\delta q_{A2})\, I_s(z,\overline{q},\overline{s}_{B1A2})\\ &-(\delta q_{A1}+\delta q_{A2})\, I_s(z,\overline{q},\overline{s}_{A1A2})\\ &-(\delta q_{B1}+\delta q_{B2})\, I_s(z,\overline{q},\overline{s}_{B1B2})] \quad ,\end{aligned} \tag{222}$$

$$\begin{aligned}\delta\Theta^{(1)}_{qq} = \tfrac{\pm 1}{2(1+N)}\,\overline{q}\,[&(\delta q_{A1}+\delta q_{B2})\,\overline{s}_{A1B2} I_{qq}(z,\overline{q},\overline{s}_{A1B2})\\ &+(\delta q_{B1}+\delta q_{A2})\,\overline{s}_{B1A2} I_{qq}(z,\overline{q},\overline{s}_{B1A2})\\ &-(\delta q_{A1}+\delta q_{A2})\,\overline{s}_{A1A2} I_{qq}(z,\overline{q},\overline{s}_{A1A2})\\ &-(\delta q_{B1}+\delta q_{B2})\,\overline{s}_{B1B2} I_{qq}(z,\overline{q},\overline{s}_{B1B2})] \quad ,\end{aligned} \tag{223}$$

$$\begin{aligned}\delta\Theta^{(1)}_{qs} = \tfrac{\mp 1}{2(1+N)}\,\tfrac{\overline{q}^2}{2}\,[&(\delta q_{A1}+\delta q_{B2})\, I_{qs}(z,\overline{q},\overline{s}_{A1B2})\\ &+(\delta q_{B1}+\delta q_{A2})\, I_{qs}(z,\overline{q},\overline{s}_{B1A2})\\ &-(\delta q_{A1}+\delta q_{A2})\, I_{qs}(z,\overline{q},\overline{s}_{A1A2})\\ &-(\delta q_{B1}+\delta q_{B2})\, I_{qs}(z,\overline{q},\overline{s}_{B1B2})] \quad .\end{aligned} \tag{224}$$

Differentiating identity (200) with respect to q, we obtain

$$\begin{aligned}s\tfrac{\partial}{\partial q} I(\sqrt{z^2+q^2},0,s) &= 2\,s\,I_q(z,q,s) - 2\,q\,[I_s(z,q,s)-I_s(z,q,0)]\\ &\quad + q\,s\,I_{qq}(z,q,s) - q^2\,[I_{qs}(z,q,s)-I_{qs}(z,q,0)] \quad .\end{aligned} \tag{225}$$

Using this identity and considering that we may substitute $I_s(z,q,s)-I_s(z,q,0)$ for $I_s(z,q,s)$ in (222) and $I_{qs}(z,q,s)-I_{qs}(z,q,0)$ for $I_{qs}(z,q,s)$ in (224), we express first–order error (220) in form

$$\begin{aligned}\delta\Theta^{(1)} = \tfrac{\pm 1}{2(1+N)}\tfrac{1}{2}[&(\delta q_{A1}+\delta q_{B2})\,\overline{s}_{A1B2}\tfrac{\partial}{\partial \overline{q}} I(\sqrt{z^2+\overline{q}^2},0,\overline{s}_{A1B2})\\ &+(\delta q_{B1}+\delta q_{A2})\,\overline{s}_{B1A2}\tfrac{\partial}{\partial \overline{q}} I(\sqrt{z^2+\overline{q}^2},0,\overline{s}_{B1A2})\\ &-(\delta q_{A1}+\delta q_{A2})\,\overline{s}_{A1A2}\tfrac{\partial}{\partial \overline{q}} I(\sqrt{z^2+\overline{q}^2},0,\overline{s}_{A1A2})\\ &-(\delta q_{B1}+\delta q_{B2})\,\overline{s}_{B1B2}\tfrac{\partial}{\partial \overline{q}} I(\sqrt{z^2+\overline{q}^2},0,\overline{s}_{B1B2})] \quad .\end{aligned} \tag{226}$$

7.2.2. Second–order error

The second–order approximation of the difference of the geometrical covariance from its approximation (195) can be expressed in form

$$\delta\Theta^{(2)} = \delta\Theta_0^{(2)} + \delta\Theta_q^{(2)} + \delta\Theta_s^{(2)} + \tfrac{1}{2}\delta\Theta_{qq}^{(2)} + \delta\Theta_{qs}^{(2)} + \tfrac{1}{2}\delta\Theta_{ss}^{(2)} + \tfrac{1}{6}\delta\Theta_{qqq}^{(2)} + \tfrac{1}{2}\delta\Theta_{qqs}^{(2)} + \tfrac{1}{2}\delta\Theta_{qss}^{(2)} + \tfrac{1}{6}\delta\Theta_{sss}^{(2)} \quad , \tag{227}$$

where

$$\begin{aligned}\delta\Theta_0^{(2)} = \tfrac{\pm 1}{2(1+N)}\,\tfrac{1}{4}\,(\sin\alpha)^2\,[&\overline{s}_{A1B2} I(z,\overline{q},\overline{s}_{A1B2})\\ &+\overline{s}_{B1A2} I(z,\overline{q},\overline{s}_{B1A2})\\ &-\overline{s}_{A1A2} I(z,\overline{q},\overline{s}_{A1A2})\\ &-\overline{s}_{B1B2} I(z,\overline{q},\overline{s}_{B1B2})] \quad ,\end{aligned} \tag{228}$$

$$\begin{aligned}\delta\Theta_q^{(2)} = \tfrac{\pm 1}{2(1+N)}\,\tfrac{1}{4}\,\overline{q}\,(\sin\alpha)^2\,[&\overline{s}_{A1B2} I_q(z,\overline{q},\overline{s}_{A1B2})\\ &+\overline{s}_{B1A2} I_q(z,\overline{q},\overline{s}_{B1A2})\\ &-\overline{s}_{A1A2} I_q(z,\overline{q},\overline{s}_{A1A2})\\ &-\overline{s}_{B1B2} I_q(z,\overline{q},\overline{s}_{B1B2})] \quad ,\end{aligned} \tag{229}$$

$$\begin{aligned}\delta\Theta_s^{(2)} = \tfrac{\mp 1}{2(1+N)}\,\tfrac{1}{4}\,\{&[(\delta q_{A1}+\delta q_{B2})^2+(\overline{q}\,\sin\alpha)^2] I_s(z,\overline{q},\overline{s}_{A1B2})\\ &+[(\delta q_{B1}+\delta q_{A2})^2+(\overline{q}\,\sin\alpha)^2] I_s(z,\overline{q},\overline{s}_{B1A2})\\ &-[(\delta q_{A1}+\delta q_{A2})^2+(\overline{q}\,\sin\alpha)^2] I_s(z,\overline{q},\overline{s}_{A1A2})\\ &-[(\delta q_{B1}+\delta q_{B2})^2+(\overline{q}\,\sin\alpha)^2] I_s(z,\overline{q},\overline{s}_{B1B2})\} \quad ,\end{aligned} \tag{230}$$

$$\begin{aligned}\delta\Theta_{qq}^{(2)} = \tfrac{\pm 1}{2(1+N)}\{&[(\delta q_{A1})^2+\delta q_{A1}\delta q_{B2}+(\delta q_{B2})^2]\,\overline{s}_{A1B2} I_{qq}(z,\overline{q},\overline{s}_{A1B2})\\ &+[(\delta q_{B1})^2+\delta q_{B1}\delta q_{A2}+(\delta q_{A2})^2]\,\overline{s}_{B1A2} I_{qq}(z,\overline{q},\overline{s}_{B1A2})\\ &-[(\delta q_{A1})^2+\delta q_{A1}\delta q_{A2}+(\delta q_{A2})^2]\,\overline{s}_{A1A2} I_{qq}(z,\overline{q},\overline{s}_{A1A2})\\ &-[(\delta q_{B1})^2+\delta q_{B1}\delta q_{B2}+(\delta q_{B2})^2]\,\overline{s}_{B1B2} I_{qq}(z,\overline{q},\overline{s}_{B1B2})\} \quad ,\end{aligned} \tag{231}$$

$$\begin{aligned}\delta\Theta_{qs}^{(2)} = \tfrac{\mp 1}{2(1+N)}\,\tfrac{1}{4}\,\overline{q}\,\{&[3(\delta q_{A1})^2+2\delta q_{A1}\delta q_{B2}+3(\delta q_{B2})^2]\,I_{qs}(z,\overline{q},\overline{s}_{A1B2})\\ &+[3(\delta q_{B1})^2+2\delta q_{B1}\delta q_{A2}+3(\delta q_{A2})^2]\,I_{qs}(z,\overline{q},\overline{s}_{B1A2})\\ &-[3(\delta q_{A1})^2+2\delta q_{A1}\delta q_{A2}+3(\delta q_{A2})^2]\,I_{qs}(z,\overline{q},\overline{s}_{A1A2})\\ &-[3(\delta q_{B1})^2+2\delta q_{B1}\delta q_{B2}+3(\delta q_{B2})^2]\,I_{qs}(z,\overline{q},\overline{s}_{B1B2})\} \quad ,\end{aligned} \tag{232}$$

$$\begin{aligned}\delta\Theta_{ss}^{(2)} = \tfrac{\pm 1}{2(1+N)}\,\tfrac{1}{4}\,\overline{q}^2\,(\sin\alpha)^2\,[&\overline{s}_{A1B2} I_{ss}(z,\overline{q},\overline{s}_{A1B2})\\ &+\overline{s}_{B1A2} I_{ss}(z,\overline{q},\overline{s}_{B1A2})\\ &-\overline{s}_{A1A2} I_{ss}(z,\overline{q},\overline{s}_{A1A2})\\ &-\overline{s}_{B1B2} I_{ss}(z,\overline{q},\overline{s}_{B1B2})] \quad ,\end{aligned} \tag{233}$$

$$\begin{aligned}\delta\Theta_{qqq}^{(2)} = \tfrac{\pm 1}{2(1+N)}\,\overline{q}\,\{&[(\delta q_{A1})^2+\delta q_{A1}\delta q_{B2}+(\delta q_{B2})^2]\,\overline{s}_{A1B2} I_{qqq}(z,\overline{q},\overline{s}_{A1B2})\\ &+[(\delta q_{B1})^2+\delta q_{B1}\delta q_{A2}+(\delta q_{A2})^2]\,\overline{s}_{B1A2} I_{qqq}(z,\overline{q},\overline{s}_{B1A2})\\ &-[(\delta q_{A1})^2+\delta q_{A1}\delta q_{A2}+(\delta q_{A2})^2]\,\overline{s}_{A1A2} I_{qqq}(z,\overline{q},\overline{s}_{A1A2})\\ &-[(\delta q_{B1})^2+\delta q_{B1}\delta q_{B2}+(\delta q_{B2})^2]\,\overline{s}_{B1B2} I_{qqq}(z,\overline{q},\overline{s}_{B1B2})\} \quad ,\end{aligned} \tag{234}$$

$$
\begin{aligned}
\delta\Theta^{(2)}_{qqs} = \tfrac{\mp 1}{2(1+N)}\,\tfrac{1}{2}\,\overline{q}^2\,\{ & [(\delta q_{A1})^2+(\delta q_{B2})^2]\, I_{qqs}(z,\overline{q},\overline{s}_{A1B2}) \\
& +[(\delta q_{B1})^2+(\delta q_{A2})^2]\, I_{qqs}(z,\overline{q},\overline{s}_{B1A2}) \\
& -[(\delta q_{A1})^2+(\delta q_{A2})^2]\, I_{qqs}(z,\overline{q},\overline{s}_{A1A2}) \\
& -[(\delta q_{B1})^2+(\delta q_{B2})^2]\, I_{qqs}(z,\overline{q},\overline{s}_{B1B2})\} \quad ,
\end{aligned} \tag{235}
$$

$$
\begin{aligned}
\delta\Theta^{(2)}_{qss} = \tfrac{\pm 1}{2(1+N)}\,\tfrac{1}{4}\,\overline{q}^3\,(\sin\alpha)^2\,[& \overline{s}_{A1B2} I_{qss}(z,\overline{q},\overline{s}_{A1B2}) \\
& +\overline{s}_{B1A2} I_{qss}(z,\overline{q},\overline{s}_{B1A2}) \\
& -\overline{s}_{A1A2} I_{qss}(z,\overline{q},\overline{s}_{A1A2}) \\
& -\overline{s}_{B1B2} I_{qss}(z,\overline{q},\overline{s}_{B1B2})] \quad ,
\end{aligned} \tag{236}
$$

$$
\begin{aligned}
\delta\Theta^{(2)}_{sss} = \tfrac{\mp 1}{2(1+N)}\,\tfrac{1}{4}\,\overline{q}^4\,(\sin\alpha)^2\,[& I_{sss}(z,\overline{q},\overline{s}_{A1B2}) \\
& +I_{sss}(z,\overline{q},\overline{s}_{B1A2}) \\
& -I_{sss}(z,\overline{q},\overline{s}_{A1A2}) \\
& -I_{sss}(z,\overline{q},\overline{s}_{B1B2})] \quad .
\end{aligned} \tag{237}
$$

7.3. Approximation for nearly parallel line segments

First–order error (226) of approximation (199) can be eliminated by using approximation

$$
\boxed{
\begin{aligned}
\overline{\Theta} = \tfrac{\pm 1}{2(1+N)}\tfrac{1}{2}\{ & \overline{s}_{A1B2}[I(\sqrt{z^2+q_{A1}^2},0,\overline{s}_{A1B2}) + I(\sqrt{z^2+q_{B2}^2},0,\overline{s}_{A1B2})] \\
& +\overline{s}_{B1A2}[I(\sqrt{z^2+q_{B1}^2},0,\overline{s}_{B1A2}) + I(\sqrt{z^2+q_{A2}^2},0,\overline{s}_{B1A2})] \\
& -\overline{s}_{A1A2}[I(\sqrt{z^2+q_{A1}^2},0,\overline{s}_{A1A2}) + I(\sqrt{z^2+q_{A2}^2},0,\overline{s}_{A1A2})] \\
& -\overline{s}_{B1B2}[I(\sqrt{z^2+q_{B1}^2},0,\overline{s}_{B1B2}) + I(\sqrt{z^2+q_{B2}^2},0,\overline{s}_{B1B2})]\}
\end{aligned}} \tag{238}
$$

for nearly parallel line segments.

7.3.1. Error of the approximation for nearly parallel line segments

Applying the second–order Taylor expansion with respect to q at $\overline{q}$, we obtain

$$
\begin{aligned}
\overline{\Theta} - \overline{\overline{\Theta}} = \tfrac{\pm 1}{2(1+N)}\tfrac{1}{2} & [\delta q_{A1}+\delta q_{B2}]\,\overline{s}_{A1B2}\tfrac{\partial^2}{\partial\overline{q}^2} I(\sqrt{z^2+\overline{q}^2},0,\overline{s}_{A1B2}) \\
& +[\delta q_{B1}+\delta q_{A2}]\,\overline{s}_{B1A2}\tfrac{\partial^2}{\partial\overline{q}^2} I(\sqrt{z^2+\overline{q}^2},0,\overline{s}_{B1A2}) \\
& -[\delta q_{A1}+\delta q_{A2}]\,\overline{s}_{A1A2}\tfrac{\partial^2}{\partial\overline{q}^2} I(\sqrt{z^2+\overline{q}^2},0,\overline{s}_{A1A2}) \\
& -[\delta q_{B1}+\delta q_{B2}]\,\overline{s}_{B1B2}\tfrac{\partial^2}{\partial\overline{q}^2} I(\sqrt{z^2+\overline{q}^2},0,\overline{s}_{B1B2})\} \\
+\tfrac{\pm 1}{2(1+N)}\tfrac{1}{4} & [(\delta q_{A1})^2+(\delta q_{B2})^2]\,\overline{s}_{A1B2}\tfrac{\partial^2}{\partial\overline{q}^2} I(\sqrt{z^2+\overline{q}^2},0,\overline{s}_{A1B2}) \\
& +[(\delta q_{B1})^2+(\delta q_{A2})^2]\,\overline{s}_{B1A2}\tfrac{\partial^2}{\partial\overline{q}^2} I(\sqrt{z^2+\overline{q}^2},0,\overline{s}_{B1A2}) \\
& -[(\delta q_{A1})^2+(\delta q_{A2})^2]\,\overline{s}_{A1A2}\tfrac{\partial^2}{\partial\overline{q}^2} I(\sqrt{z^2+\overline{q}^2},0,\overline{s}_{A1A2}) \\
& -[(\delta q_{B1})^2+(\delta q_{B2})^2]\,\overline{s}_{B1B2}\tfrac{\partial^2}{\partial\overline{q}^2} I(\sqrt{z^2+\overline{q}^2},0,\overline{s}_{B1B2})\} \quad .
\end{aligned} \tag{239}
$$

Subtracting (239) from (219), the linear terms in (239) eliminate the first–order error (226), and we obtain

$$
\Theta - \overline{\Theta} \approx \delta\overline{\Theta}^{(2)} \quad , \tag{240}
$$

where

$$\delta\overline{\Theta}^{(2)} = \delta\Theta^{(2)} - \tfrac{\pm 1}{2(1+N)}\tfrac{1}{4}[(\delta q_{A1})^2+(\delta q_{B2})^2]\,\overline{s}_{A1B2}\tfrac{\partial^2}{\partial\overline{q}^2}I(\sqrt{z^2+\overline{q}^2},0,\overline{s}_{A1B2})$$
$$+[(\delta q_{B1})^2+(\delta q_{A2})^2]\,\overline{s}_{B1A2}\tfrac{\partial^2}{\partial\overline{q}^2}I(\sqrt{z^2+\overline{q}^2},0,\overline{s}_{B1A2})$$
$$-[(\delta q_{A1})^2+(\delta q_{A2})^2]\,\overline{s}_{A1A2}\tfrac{\partial^2}{\partial\overline{q}^2}I(\sqrt{z^2+\overline{q}^2},0,\overline{s}_{A1A2})$$
$$-[(\delta q_{B1})^2+(\delta q_{B2})^2]\,\overline{s}_{B1B2}\tfrac{\partial^2}{\partial\overline{q}^2}I(\sqrt{z^2+\overline{q}^2},0,\overline{s}_{B1B2})\} \ . \qquad (241)$$

Differentiating identity (225) with respect to q, we obtain

$$s\tfrac{\partial^2}{\partial q^2}I(\sqrt{z^2+q^2},0,s) = -\,2\,[I_s(z,q,s)-I_s(z,q,0)]$$
$$+\,3\,s\,I_{qq}(z,q,s) - 4\,q\,[I_{qs}(z,q,s)-I_{qs}(z,q,0)]$$
$$+\,q\,s\,I_{qqq}(z,q,s) - q^2\,[I_{qqs}(z,q,s)-I_{qqs}(z,q,0)] \quad . \qquad (242)$$

We insert (242) for $s\frac{\partial^2}{\partial q^2}I(\sqrt{z^2+q^2},0,s)$ into (241). The terms containing $I_s(z,q,0)$, $I_{qs}(z,q,0)$ and $I_{qqs}(z,q,0)$ mutually cancel in (241). We then consider (241) with (227) and update individual terms (228)–(237). Terms (228), (229), (233), (236) and (237) remain unchanged,

$$\overline{\delta}\Theta^{(2)} = \delta\Theta_0^{(2)} + \delta\Theta_q^{(2)} + \delta\overline{\Theta}_s^{(2)} + \tfrac{1}{2}\delta\overline{\Theta}_{qq}^{(2)} + \delta\overline{\Theta}_{qs}^{(2)} + \tfrac{1}{2}\delta\Theta_{ss}^{(2)}$$
$$+\,\tfrac{1}{6}\delta\overline{\Theta}_{qqq}^{(2)} + \tfrac{1}{2}\delta\overline{\Theta}_{qqs}^{(2)} + \tfrac{1}{2}\delta\Theta_{qss}^{(2)} + \tfrac{1}{6}\delta\Theta_{sss}^{(2)} \quad . \qquad (243)$$

Terms (230)–(232), (234) and (235) change to

$$\delta\overline{\Theta}_s^{(2)} = \tfrac{\mp 1}{2(1+N)}\,\tfrac{1}{4}\,\{[(\overline{q}\,\sin\alpha)^2-(\delta q_{A1}-\delta q_{B2})^2]\,I_s(z,\overline{q},\overline{s}_{A1B2})$$
$$+[(\overline{q}\,\sin\alpha)^2-(\delta q_{B1}-\delta q_{A2})^2]\,I_s(z,\overline{q},\overline{s}_{B1A2})$$
$$-[(\overline{q}\,\sin\alpha)^2-(\delta q_{A1}-\delta q_{A2})^2]\,I_s(z,\overline{q},\overline{s}_{A1A2})$$
$$-[(\overline{q}\,\sin\alpha)^2-(\delta q_{B1}-\delta q_{B2})^2]\,I_s(z,\overline{q},\overline{s}_{B1B2})\} \quad , \qquad (244)$$

$$\delta\overline{\Theta}_{qq}^{(2)} = \tfrac{\pm 1}{2(1+N)}\,\tfrac{-1}{2}\,[(\delta q_{A1}-\delta q_{B2})^2\,\overline{s}_{A1B2}I_{qq}(z,\overline{q},\overline{s}_{A1B2})$$
$$+(\delta q_{B1}-\delta q_{A2})^2\,\overline{s}_{B1A2}I_{qq}(z,\overline{q},\overline{s}_{B1A2})$$
$$-(\delta q_{A1}-\delta q_{A2})^2\,\overline{s}_{A1A2}I_{qq}(z,\overline{q},\overline{s}_{A1A2})$$
$$-(\delta q_{B1}-\delta q_{B2})^2\,\overline{s}_{B1B2}I_{qq}(z,\overline{q},\overline{s}_{B1B2})] \quad , \qquad (245)$$

$$\delta\overline{\Theta}_{qs}^{(2)} = \tfrac{\mp 1}{2(1+N)}\,\tfrac{-1}{4}\,\overline{q}\,[(\delta q_{A1}-\delta q_{B2})^2 I_{qs}(z,\overline{q},\overline{s}_{A1B2})$$
$$+(\delta q_{B1}-\delta q_{A2})^2 I_{qs}(z,\overline{q},\overline{s}_{B1A2})$$
$$-(\delta q_{A1}-\delta q_{A2})^2 I_{qs}(z,\overline{q},\overline{s}_{A1A2})$$
$$-(\delta q_{B1}-\delta q_{B2})^2 I_{qs}(z,\overline{q},\overline{s}_{B1B2})] \quad , \qquad (246)$$

$$\delta\overline{\Theta}_{qqq}^{(2)} = \tfrac{\pm 1}{2(1+N)}\,\tfrac{-1}{2}\,\overline{q}\,[(\delta q_{A1}-\delta q_{B2})^2\,\overline{s}_{A1B2}I_{qqq}(z,\overline{q},\overline{s}_{A1B2})$$
$$+(\delta q_{B1}-\delta q_{A2})^2\,\overline{s}_{B1A2}I_{qqq}(z,\overline{q},\overline{s}_{B1A2})$$
$$-(\delta q_{A1}-\delta q_{A2})^2\,\overline{s}_{A1A2}I_{qqq}(z,\overline{q},\overline{s}_{A1A2})$$
$$-(\delta q_{B1}-\delta q_{B2})^2\,\overline{s}_{B1B2}I_{qqq}(z,\overline{q},\overline{s}_{B1B2})] \quad , \qquad (247)$$

$$\delta\overline{\Theta}^{(2)}_{qqs} = 0 \quad . \tag{248}$$

We insert $(171a)$–$(172b)$ into (244)–(247),

$$\begin{aligned}\delta\overline{\Theta}^{(2)}_{s} = \tfrac{\mp 1}{2(1+N)}\,\tfrac{1}{4}\,(\sin\alpha)^2\,\{&[\overline{q}^2-(\overline{s}_{A1B2})^2]\,I_s(z,\overline{q},\overline{s}_{A1B2})\\ &+[\overline{q}^2-(\overline{s}_{B1A2})^2]\,I_s(z,\overline{q},\overline{s}_{B1A2})\\ &-[\overline{q}^2-(\overline{s}_{A1A2})^2]\,I_s(z,\overline{q},\overline{s}_{A1A2})\\ &-[\overline{q}^2-(\overline{s}_{B1B2})^2]\,I_s(z,\overline{q},\overline{s}_{B1B2})\} \quad ,\end{aligned} \tag{249}$$

$$\begin{aligned}\delta\overline{\Theta}^{(2)}_{qq} = \tfrac{\pm 1}{2(1+N)}\,\tfrac{-1}{2}\,(\sin\alpha)^2\,[&(\overline{s}_{A1B2})^3 I_{qq}(z,\overline{q},\overline{s}_{A1B2})\\ &+(\overline{s}_{B1A2})^3 I_{qq}(z,\overline{q},\overline{s}_{B1A2})\\ &-(\overline{s}_{A1A2})^3 I_{qq}(z,\overline{q},\overline{s}_{A1A2})\\ &-(\overline{s}_{B1B2})^3 I_{qq}(z,\overline{q},\overline{s}_{B1B2})] \quad ,\end{aligned} \tag{250}$$

$$\begin{aligned}\delta\overline{\Theta}^{(2)}_{qs} = \tfrac{\mp 1}{2(1+N)}\,\tfrac{-1}{4}\,\overline{q}\,(\sin\alpha)^2\,[&(\overline{s}_{A1B2})^2 I_{qs}(z,\overline{q},\overline{s}_{A1B2})\\ &+(\overline{s}_{B1A2})^2 I_{qs}(z,\overline{q},\overline{s}_{B1A2})\\ &-(\overline{s}_{A1A2})^2 I_{qs}(z,\overline{q},\overline{s}_{A1A2})\\ &-(\overline{s}_{B1B2})^2 I_{qs}(z,\overline{q},\overline{s}_{B1B2})] \quad ,\end{aligned} \tag{251}$$

$$\begin{aligned}\delta\overline{\Theta}^{(2)}_{qqq} = \tfrac{\pm 1}{2(1+N)}\,\tfrac{-1}{2}\,\overline{q}\,(\sin\alpha)^2\,[&(\overline{s}_{A1B2})^3 I_{qqq}(z,\overline{q},\overline{s}_{A1B2})\\ &+(\overline{s}_{B1A2})^3 I_{qqq}(z,\overline{q},\overline{s}_{B1A2})\\ &-(\overline{s}_{A1A2})^3 I_{qqq}(z,\overline{q},\overline{s}_{A1A2})\\ &-(\overline{s}_{B1B2})^3 I_{qqq}(z,\overline{q},\overline{s}_{B1B2})] \quad .\end{aligned} \tag{252}$$

We insert (228), (229), (249)–(251), (233), (252), (248), (236) and (237) into (243). The second–order approximation of the difference (240) of the geometrical covariance from its approximation (238) can then be expressed in form

$$\boxed{\begin{aligned}\Theta - \overline{\Theta} = \tfrac{\pm 1}{2(1+N)}\,\tfrac{1}{4}\,(\sin\alpha)^2\,[&\overline{D}(z,\overline{q},\overline{s}_{A1B2})\\ &+\overline{D}(z,\overline{q},\overline{s}_{B1A2})\\ &-\overline{D}(z,\overline{q},\overline{s}_{A1A2})\\ &-\overline{D}(z,\overline{q},\overline{s}_{B1B2})] \quad , \qquad (253)\end{aligned}}$$

where

$$\begin{aligned}\overline{D}(z,q,s) = {}& s\,I(z,q,s) + q\,s\,I_q(z,q,s) + (s^2-q^2)\,I_s(z,q,s) + q^2 I_s(z,q,0)\\ &- s^3 I_{qq}(z,q,s) + q\,s^2 I_{qs}(z,q,s) + \tfrac{1}{2}\,q^2 s\,I_{ss}(z,q,s)\\ &- \tfrac{1}{3}\,q\,s^3 I_{qqq}(z,q,s) + \tfrac{1}{2}\,q^3 s\,I_{qss}(z,q,s) - \tfrac{1}{6}\,q^4[I_{sss}(z,q,s) - I_{sss}(z,q,0)]\,.\end{aligned} \tag{254}$$

We have added terms $q^2 I_s(z,q,0)$ and $\frac{1}{6}\,q^4 I_{sss}(z,q,0)$ into (254) in order to have $\overline{D}(z,q,0)=0$. These terms cancel in (253).

We now consider identity (203) and its derivatives

$$s\,I_{qqs}(z,q,s) = q\,I_{qss}(z,q,s) + I_{ss}(z,q,s) \tag{255}$$

and

$$I_{qs}(z,q,s) + s\, I_{qss}(z,q,s) = q\, I_{sss}(z,q,s) \tag{256}$$

to remove I_{ss}, I_{qss} and I_{sss} from (254). We first insert (256) for I_{sss} into (254),

$$\begin{aligned}\overline{D}(z,q,s) &= s\, I(z,q,s) + q\, s\, I_q(z,q,s) + (s^2 - q^2)\, I_s(z,q,s) + q^2 I_s(z,q,0)\\ &\quad - s^3 I_{qq}(z,q,s) + q\, s^2 I_{qs}(z,q,s) + \tfrac{1}{2}\, q^2 s\, I_{ss}(z,q,s)\\ &\quad - \tfrac{1}{3}\, q\, s^3 I_{qqq}(z,q,s) + \tfrac{1}{3}\, q^3 s\, I_{qss}(z,q,s) - \tfrac{1}{6}\, q^3\, [I_{qs}(z,q,s) - I_{qs}(z,q,0)]. \end{aligned} \tag{257}$$

We insert (255) for I_{qss} into (257),

$$\begin{aligned}\overline{D}(z,q,s) &= s\, I(z,q,s) + q\, s\, I_q(z,q,s) + (s^2 - q^2)\, I_s(z,q,s) + q^2 I_s(z,q,0)\\ &\quad - s^3 I_{qq}(z,q,s) + q\, s^2 I_{qs}(z,q,s) + \tfrac{1}{6}\, q^2 s\, I_{ss}(z,q,s)\\ &\quad - \tfrac{1}{3}\, q\, s^3 I_{qqq}(z,q,s) + \tfrac{1}{3}\, q^2 s^2 I_{qqs}(z,q,s) - \tfrac{1}{6}\, q^3\, [I_{qs}(z,q,s) - I_{qs}(z,q,0)]. \end{aligned} \tag{258}$$

We insert (203) for I_{ss} into (258),

$$\begin{aligned}\overline{D}(z,q,s) &= s\, I(z,q,s) + q\, s\, I_q(z,q,s) + (s^2 - q^2)\, I_s(z,q,s) + q^2 I_s(z,q,0)\\ &\quad - s^3 I_{qq}(z,q,s) + \tfrac{7}{6}\, q\, s^2 I_{qs}(z,q,s) - \tfrac{1}{6}\, q^3\, [I_{qs}(z,q,s) - I_{qs}(z,q,0)]\\ &\quad - \tfrac{1}{3}\, q\, s^3 I_{qqq}(z,q,s) + \tfrac{1}{3}\, q^2 s^2 I_{qqs}(z,q,s) \quad . \end{aligned} \tag{259}$$

We now consider identity (200) and its first–order derivatives

$$\begin{aligned}\tfrac{\partial}{\partial q} I(\sqrt{z^2+q^2}, 0, s) &= 2\, I_q(z,q,s) - 2\, q\, s^{-1}[I_s(z,q,s) - I_s(z,q,0)]\\ &\quad + q\, I_{qq}(z,q,s) - q^2 s^{-1}[I_{qs}(z,q,s) - I_{qs}(z,q,0)] \quad , \end{aligned} \tag{260}$$

which is equivalent to (225), and

$$\begin{aligned} I_s(\sqrt{z^2+q^2}, 0, s) &= I_s(z,q,s) + q^2 s^{-2}[I_s(z,q,s) - I_s(z,q,0)]\\ &\quad + q\, I_{qs}(z,q,s) - q^2 s^{-1} I_{ss}(z,q,s) \quad , \end{aligned} \tag{261}$$

which can be simplified using (203),

$$I_s(\sqrt{z^2+q^2}, 0, s) = I_s(z,q,s) + q^2 s^{-2}[I_s(z,q,s) - I_s(z,q,0)] \quad . \tag{262}$$

The value at $s = 0$ can be obtained using the L'Hospital rule,

$$I_s(\sqrt{z^2+q^2}, 0, 0) = I_s(z,q,0) + \tfrac{1}{2}\, q^2 I_{sss}(z,q,0) \quad . \tag{263}$$

Inserting (256), equation (263) reads

$$I_s(\sqrt{z^2+q^2}, 0, 0) = I_s(z,q,0) + \tfrac{1}{2}\, q\, I_{qs}(z,q,0) \quad . \tag{264}$$

Differentiating (260) with respect to q, we obtain

$$\begin{aligned}\tfrac{\partial^2}{\partial q^2} I(\sqrt{z^2+q^2}, 0, s) &= -\, 2\, s^{-1}[I_s(z,q,s) - I_s(z,q,0)]\\ &\quad + 3\, I_{qq}(z,q,s) - 4\, q\, s^{-1}[I_{qs}(z,q,s) - I_{qs}(z,q,0)]\\ &\quad + q\, I_{qqq}(z,q,s) - q^2 s^{-1}[I_{qqs}(z,q,s) - I_{qqs}(z,q,0)] \quad , \end{aligned} \tag{265}$$

which is equivalent to (242). Differentiating (262) with respect to q, we obtain

$$\begin{aligned}\tfrac{\partial}{\partial q} I_s(\sqrt{z^2+q^2}, 0, s) &= 2\, q\, s^{-2}[I_s(z,q,s) - I_s(z,q,0)] - 2\, q\, s^{-1} I_{ss}(z,q,s)\\ &\quad + 2\, I_{qs}(z,q,s) + q^2 s^{-2}[I_{qs}(z,q,s) - I_{qs}(z,q,0)]\\ &\quad + q\, I_{qqs}(z,q,s) - q^2 s^{-1} I_{qss}(z,q,s) \quad . \end{aligned} \tag{266}$$

We insert (203) for $I_{ss}(z,q,s)$ and (255) for $I_{qss}(z,q,s)$,

$$\begin{aligned}\tfrac{\partial}{\partial q} I_s(\sqrt{z^2+q^2},0,s) &= 2\,q\,s^{-2}[I_s(z,q,s) - I_s(z,q,0)] \\ &\quad + q^2 s^{-2}[I_{qs}(z,q,s) - I_{qs}(z,q,0)] + q\,s^{-1} I_{ss}(z,q,s) \quad , \end{aligned} \tag{267}$$

and again (203) for I_{ss},

$$\begin{aligned}\tfrac{\partial}{\partial q} I_s(\sqrt{z^2+q^2},0,s) &= 2\,q\,s^{-2}[I_s(z,q,s) - I_s(z,q,0)] \\ &\quad + q^2 s^{-2}[I_{qs}(z,q,s) - I_{qs}(z,q,0)] + I_{qs}(z,q,s) \quad . \end{aligned} \tag{268}$$

Differentiating (264) with respect to q, we obtain

$$\tfrac{\partial}{\partial q} I_s(\sqrt{z^2+q^2},0,0) = \tfrac{3}{2}\, I_{qs}(z,q,0) + \tfrac{1}{2}\, q\, I_{qqs}(z,q,0) \quad . \tag{269}$$

Differentiating (261) with respect to s, we obtain

$$\begin{aligned} I_{ss}(\sqrt{z^2+q^2},0,s) &= -\,2\,q^2 s^{-3}[I_s(z,q,s) - I_s(z,q,0)] + I_{ss}(z,q,s) + 2\,q^2 s^{-2} I_{ss}(z,q,s) \\ &\quad + q\, I_{qss}(z,q,s) - q^2 s^{-1} I_{sss}(z,q,s) \quad . \end{aligned} \tag{270}$$

Differentiating (262) with respect to s, we obtain

$$I_{ss}(\sqrt{z^2+q^2},0,s) = -2\,q^2 s^{-3}[I_s(z,q,s) - I_s(z,q,0)] + I_{ss}(z,q,s) + q^2 s^{-2} I_{ss}(z,q,s) \quad . \tag{271}$$

We insert (203) for $I_{ss}(z,q,s)$,

$$I_{ss}(\sqrt{z^2+q^2},0,s) = -2\,q^2 s^{-3}[I_s(z,q,s) - I_s(z,q,0)] + (q^2+s^2)\,q^{-1} s^{-1} I_{qs}(z,q,s) \quad . \tag{272}$$

We now express function (254) in form

$$\boxed{\begin{aligned} \overline{D}(z,q,s) &= s\, I(\sqrt{z^2+q^2},0,s) \\ &\quad + (\tfrac{q^2}{3} + \tfrac{s^2}{3})\, I_s(\sqrt{z^2+q^2},0,s) - (\tfrac{q^2}{3} - \tfrac{2\,s^2}{3})\, I_s(\sqrt{z^2+q^2},0,0) \\ &\quad - \tfrac{s^3}{3}\, \tfrac{\partial^2}{\partial q^2} I(\sqrt{z^2+q^2},0,s) \\ &\quad - \tfrac{q\,s^2}{3}\, \tfrac{\partial}{\partial q} I_s(\sqrt{z^2+q^2},0,s) + \tfrac{2\,q\,s^2}{3}\, \tfrac{\partial}{\partial q} I_s(\sqrt{z^2+q^2},0,0) \\ &\quad + \tfrac{q^2 s}{6}\, I_{ss}(\sqrt{z^2+q^2},0,s) \quad , \qquad (273) \end{aligned}}$$

Equation (273) can be obtained from equation (254) using identities (200), (261), (263), (265), (266), (269) and (270), or from equation (259) using identities (200), (262), (264), (265), (268), (269) and (272).

For special case

$$I(\sqrt{z^2+q^2},0,s) = \int_0^s \mathrm{d}s\, \frac{(z^2+q^2+s^2)^{1+N} - (z^2+q^2)^{1+N}}{s^2} \tag{274}$$

of equation (96), equation (133) yields

$$\begin{aligned} I(\sqrt{z^2+q^2},0,s) &= -s\, \tfrac{z^2+q^2}{s^2}\, (z^2+q^2+s^2)^N\, {}_2\mathrm{F}_1(1,-N;\tfrac{1}{2};\tfrac{s^2}{z^2+q^2+s^2}) \\ &\quad + s\,(z^2+q^2+s^2)^N\, {}_2\mathrm{F}_1(1,-N;\tfrac{3}{2};\tfrac{s^2}{z^2+q^2+s^2}) \\ &\quad + \tfrac{(z^2+q^2)^{1+N}}{s} \quad . \end{aligned} \tag{275}$$

The first hypergeometric function can be expressed in terms of the second one using identity

$$ {}_2\mathrm{F}_1(a,b;c+1;x) = \tfrac{c}{c-b}\,\tfrac{1}{x}\,{}_2\mathrm{F}_1(a-1,b;c;x) - \tfrac{c}{c-b}\,\tfrac{1-x}{x}\,{}_2\mathrm{F}_1(a,b;c;x) \tag{276} $$

(Abramowitz & Stegun, 1964, eq. 15.2.25), which for $a=1$ reads

$$ \tfrac{1-x}{x}\,{}_2\mathrm{F}_1(1,b;c;x) = \tfrac{1}{x} - \tfrac{c-b}{c}\,{}_2\mathrm{F}_1(1,b;c+1;x) \quad . \tag{277} $$

Inserting (277) into (275), we obtain

$$ \boxed{\begin{aligned} I(\sqrt{z^2+q^2},0,s) =& 2\,(1+N)\,s\,(z^2+q^2+s^2)^N\,{}_2\mathrm{F}_1(1,-N;\tfrac{3}{2};\tfrac{s^2}{z^2+q^2+s^2}) \\ & - \tfrac{(z^2+q^2+s^2)^{1+N}}{s} + \tfrac{(z^2+q^2)^{1+N}}{s} \quad , \end{aligned}} \tag{278} $$

which is suitable for $s^2 \lesssim z^2+q^2$. For very small s, we insert identity

$$ \begin{aligned} (1-z)^{1+N} - 1 &= \sum_{n=0}^{+\infty} \frac{(-1)^n\,(-1-N)_n\,(-z)^n}{n!} - 1 \\ &= \sum_{n=1}^{+\infty} \frac{(-1-N)_n\,z^n}{n!} \\ &= (-1-N)\,z \sum_{n=0}^{+\infty} \frac{(-N)_n\,z^n}{(2)_n} \\ &= -(1+N)\,z\,{}_2\mathrm{F}_1(1,-N;2;z) \end{aligned} \tag{279} $$

into (278) and obtain

$$ I(z,0,s) = (1+N)\,s\,(z^2+s^2)^N\,\left[2\,{}_2\mathrm{F}_1(1,-N;\tfrac{3}{2};\tfrac{s^2}{z^2+s^2}) - {}_2\mathrm{F}_1(1,-N;2;\tfrac{s^2}{z^2+s^2})\right] \quad . \tag{280} $$

For $z^2+q^2 \lesssim s^2$, we insert (154) into (278) and arrive at

$$ \boxed{\begin{aligned} I(\sqrt{z^2+q^2},0,s) =& \tfrac{2(1+N)}{1+2N}\,s\,(z^2+q^2+s^2)^N\,{}_2\mathrm{F}_1(1,-N;\tfrac{1}{2}-N;\tfrac{z^2+q^2}{z^2+q^2+s^2}) - \tfrac{(z^2+q^2+s^2)^{1+N}}{s} \\ & - \tfrac{2(1+N)}{1+2N}\,\tfrac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}-N)}{\Gamma(-N)}\,\tfrac{|s|}{s}\,(z^2+q^2)^{\frac{1}{2}+N} + \tfrac{(z^2+q^2)^{1+N}}{s} \quad . \end{aligned}} \tag{281} $$

Integrals on the right–hand side of

$$ \begin{aligned} \tfrac{\partial^2}{\partial q^2} I(\sqrt{z^2+q^2},0,s) &= 2(1+N)\int_0^s \mathrm{d}s\,\frac{(z^2+q^2+s^2)^N - (z^2+q^2)^N}{s^2} \\ &+4N(1+N)q^2\int_0^s \mathrm{d}s\,\frac{(z^2+q^2+s^2)^{-1+N} - (z^2+q^2)^{-1+N}}{s^2} \end{aligned} \tag{282} $$

can analogously be expressed as

$$ \begin{aligned} &\int_0^s \mathrm{d}s\,\frac{(z^2+q^2+s^2)^N - (z^2+q^2)^N}{s^2} \\ &\quad = -s\,\tfrac{z^2+q^2}{s^2}\,(z^2+q^2+s^2)^{N-1}\,{}_2\mathrm{F}_1(1,1-N;\tfrac{1}{2};\tfrac{s^2}{z^2+q^2+s^2}) \\ &\quad + s\,(z^2+q^2+s^2)^{N-1}\,{}_2\mathrm{F}_1(1,1-N;\tfrac{3}{2};\tfrac{s^2}{z^2+q^2+s^2}) \\ &\quad + \tfrac{(z^2+q^2)^N}{s|q|} \end{aligned} \tag{283} $$

and

$$\int_0^s \mathrm{d}s \, \frac{(z^2+q^2+s^2)^{-1+N} - (z^2+q^2)^{-1+N}}{s^2}$$
$$= -s\,\tfrac{z^2+q^2}{s^2}\,(z^2+q^2+s^2)^{N-2}\,{}_2\mathrm{F}_1(1,2-N;\tfrac{1}{2};\tfrac{s^2}{z^2+q^2+s^2})$$
$$+ s\,(z^2+q^2+s^2)^{N-2}\,{}_2\mathrm{F}_1(1,2-N;\tfrac{3}{2};\tfrac{s^2}{z^2+q^2+s^2})$$
$$+ \tfrac{(z^2+q^2)^{N-1}}{s|q|} \quad . \tag{284}$$

The hypergeometric functions in (283) and (284) can be converted into the hypergeometric functions in (275) using the Gauss' relation

$$(c-b-a)_2\mathrm{F}_1(a,b;c;x) - (c-a)_2\mathrm{F}_1(a-1,b;c;x) + b(1-x)_2\mathrm{F}_1(a,b+1;c;x) = 0 \tag{285}$$

for contiguous hypergeometric functions by Abramowitz & Stegun (1964, eq. 15.2.18). We consider just special case

$$(c-b-1)_2\mathrm{F}_1(1,b;c;x) - (c-1)_2\mathrm{F}_1(0,b;c;x) + b(1-x)_2\mathrm{F}_1(1,b+1;c;x) = 0 \tag{286}$$

of (285) for $a = 1$, and obtain conversion identity

$${}_2\mathrm{F}_1(1,b+1;c;x) = [b(1-x)]^{-1}[(c-1) - (c-b-1)_2\mathrm{F}_1(1,b;c;x)] = 0 \tag{287}$$

applicable to (283) and (284).

Calculation of (273) thus requires only one evaluation of the hypergeometric function of a single variable, which should be fast compared to hypergeometric functions of two variables. Calculation of error estimate (253) then requires 4 evaluations of the hypergeometric function of a single variable. If the error is acceptable, we may use approximation (238) for nearly parallel line segments, which requires 8 evaluations of the hypergeometric function of a single variable.

7.4. Other approximations for nearly parallel line segments

There are also other possible approximations for nearly parallel line segments, alternative to (238). Although we recommend approximation (238), we examine here two other approximations for completeness and comparison.

7.4.1. First alternative approximation for nearly parallel line segments

The first approximation for nearly parallel line segments, alternative to (238), is

$$\widehat{\Theta} = \tfrac{\pm 1}{2(1+N)}\tfrac{1}{2}\{\overline{s}_{A1B2}[\mp I(\sqrt{z^2+q_{A1}^2},0,s_{B2A1}) + I(\sqrt{z^2+q_{B2}^2},0,s_{A1B2})]$$
$$+\overline{s}_{B1A2}[\mp I(\sqrt{z^2+q_{B1}^2},0,s_{A2B1}) + I(\sqrt{z^2+q_{A2}^2},0,s_{B1A2})]$$
$$-\overline{s}_{A1A2}[\mp I(\sqrt{z^2+q_{A1}^2},0,s_{A2A1}) + I(\sqrt{z^2+q_{A2}^2},0,s_{A1A2})]$$
$$-\overline{s}_{B1B2}[\mp I(\sqrt{z^2+q_{B1}^2},0,s_{B2B1}) + I(\sqrt{z^2+q_{B2}^2},0,s_{B1B2})]\} \quad . \tag{288}$$

Inserting (166a)–(167d) into (288),

$$\widehat{\Theta} = \tfrac{\pm 1}{2(1+N)}\tfrac{1}{2}[\overline{s}_{A1B2}\, I(\sqrt{z^2+q_{A1}^2},0,\overline{s}_{A1B2} - \delta s_{A1B2})$$
$$+\overline{s}_{A1B2}\, I(\sqrt{z^2+q_{B2}^2},0,\overline{s}_{A1B2} + \delta s_{A1B2})$$
$$+\overline{s}_{B1A2}\, I(\sqrt{z^2+q_{B1}^2},0,\overline{s}_{B1A2} - \delta s_{B1A2})$$

$$
\begin{aligned}
&+\overline{s}_{B1A2}\, I(\sqrt{z^2+q_{A2}^2}, 0, \overline{s}_{B1A2}+\delta s_{B1A2})\\
&-\overline{s}_{A1A2}\, I(\sqrt{z^2+q_{A1}^2}, 0, \overline{s}_{A1A2}-\delta s_{A1A2})\\
&-\overline{s}_{A1A2}\, I(\sqrt{z^2+q_{A2}^2}, 0, \overline{s}_{A1A2}+\delta s_{A1A2})\\
&-\overline{s}_{B1B2}\, I(\sqrt{z^2+q_{B1}^2}, 0, \overline{s}_{B1B2}-\delta s_{B1B2})\\
&-\overline{s}_{B1B2}\, I(\sqrt{z^2+q_{B2}^2}, 0, \overline{s}_{B1B2}+\delta s_{B1B2})] \quad .
\end{aligned} \tag{289}
$$

Applying the second–order Taylor expansion with respect to s,

$$
\begin{aligned}
\widehat{\Theta}-\overline{\Theta} = \tfrac{\pm 1}{2(1+N)}\tfrac{1}{4}\{&\overline{s}_{A1B2}(\delta s_{A1B2})^2[I_{ss}(\sqrt{z^2+q_{A1}^2},0,\overline{s}_{A1B2})+I_{ss}(\sqrt{z^2+q_{B2}^2},0,\overline{s}_{A1B2})]\\
&+\overline{s}_{B1A2}(\delta s_{B1A2})^2[I_{ss}(\sqrt{z^2+q_{B1}^2},0,\overline{s}_{B1A2})+I_{ss}(\sqrt{z^2+q_{A2}^2},0,\overline{s}_{B1A2})]\\
&-\overline{s}_{A1A2}(\delta s_{A1A2})^2[I_{ss}(\sqrt{z^2+q_{A1}^2},0,\overline{s}_{A1A2})+I_{ss}(\sqrt{z^2+q_{A2}^2},0,\overline{s}_{A1A2})]\\
&-\overline{s}_{B1B2}(\delta s_{B1B2})^2[I_{ss}(\sqrt{z^2+q_{B1}^2},0,\overline{s}_{B1B2})+I_{ss}(\sqrt{z^2+q_{B2}^2},0,\overline{s}_{B1B2})]\}\\
-\tfrac{\pm 1}{2(1+N)}\tfrac{1}{2}\{&\overline{s}_{A1B2}\,\delta s_{A1B2}\,[I_{s}(\sqrt{z^2+q_{A1}^2},0,\overline{s}_{A1B2})-I_{s}(\sqrt{z^2+q_{B2}^2},0,\overline{s}_{A1B2})]\\
&+\overline{s}_{B1A2}\,\delta s_{B1A2}\,[I_{s}(\sqrt{z^2+q_{B1}^2},0,\overline{s}_{B1A2})-I_{s}(\sqrt{z^2+q_{A2}^2},0,\overline{s}_{B1A2})]\\
&-\overline{s}_{A1A2}\,\delta s_{A1A2}\,[I_{s}(\sqrt{z^2+q_{A1}^2},0,\overline{s}_{A1A2})-I_{s}(\sqrt{z^2+q_{A2}^2},0,\overline{s}_{A1A2})]\\
&-\overline{s}_{B1B2}\,\delta s_{B1B2}\,[I_{s}(\sqrt{z^2+q_{B1}^2},0,\overline{s}_{B1B2})-I_{s}(\sqrt{z^2+q_{B2}^2},0,\overline{s}_{B1B2})]\}\,.
\end{aligned} \tag{290}
$$

We apply the zero–order Taylor expansion of I_{ss} and the first–order Taylor expansion of I_s with respect to q at $\overline{q}$ in (290). Equations (168a)–(169b) yield

$$
q_{X1} - q_{Y2} = (s_{X1} \mp s_{Y2})\,\sin\alpha \quad , \tag{291}
$$

where $X = A, B$ and $Y = A, B$. Equations (164a)–(164d) then further yield

$$
q_{X1} - q_{Y2} \approx \overline{s}_{X1Y2}\,\sin\alpha \quad . \tag{292}
$$

Equations (165a)–(165d) analogously yield

$$
\delta s_{X1Y2} \approx \tfrac{1}{2}\overline{q}\sin\alpha \quad . \tag{293}
$$

Applying approximations (292) and (293) to (290) with the Taylor expansions, the error $\Theta - \widehat{\Theta}$ of (288) can be approximated by (253) with $\overline{D}(z,q,s)$ replaced by

$$
\widehat{D}(z,q,s) = \overline{D}(z,q,s) - \tfrac{1}{2}q^2 s\, I_{ss}(\sqrt{z^2+q^2},0,s) + q\, s^2 \tfrac{\partial}{\partial q} I_s(\sqrt{z^2+q^2},0,s) \quad . \tag{294}
$$

7.4.2. Second alternative approximation for nearly parallel line segments

The second approximation for nearly parallel line segments, alternative to (238), is

$$
\begin{aligned}
\widetilde{\Theta} = \tfrac{\pm 1}{2(1+N)}\tfrac{1}{2}[&s_{B2A1}\, I(\sqrt{z^2+q_{A1}^2},0,s_{B2A1}) + s_{A1B2}\, I(\sqrt{z^2+q_{B2}^2},0,s_{A1B2})\\
&+s_{A2B1}\, I(\sqrt{z^2+q_{B1}^2},0,s_{A2B1}) + s_{B1A2}\, I(\sqrt{z^2+q_{A2}^2},0,s_{B1A2})\\
&-s_{A2A1}\, I(\sqrt{z^2+q_{A1}^2},0,s_{A2A1}) - s_{A1A2}\, I(\sqrt{z^2+q_{A2}^2},0,s_{A1A2})\\
&-s_{B2B1}\, I(\sqrt{z^2+q_{B1}^2},0,s_{B2B1}) - s_{B1B2}\, I(\sqrt{z^2+q_{B2}^2},0,s_{B1B2})]\,.
\end{aligned} \tag{295}
$$

Inserting (166a)–(167d) into (295),

$$\begin{aligned}\widetilde{\Theta} = \tfrac{\pm 1}{2(1+N)}\tfrac{1}{2}[&(\overline{s}_{A1B2}-\delta s_{A1B2})\,I(\sqrt{z^2+q_{A1}^2},0,\overline{s}_{A1B2}-\delta s_{A1B2})\\ +&(\overline{s}_{A1B2}+\delta s_{A1B2})\,I(\sqrt{z^2+q_{B2}^2},0,\overline{s}_{A1B2}+\delta s_{A1B2})\\ +&(\overline{s}_{B1A2}-\delta s_{B1A2})\,I(\sqrt{z^2+q_{B1}^2},0,\overline{s}_{B1A2}-\delta s_{B1A2})\\ +&(\overline{s}_{B1A2}+\delta s_{B1A2})\,I(\sqrt{z^2+q_{A2}^2},0,\overline{s}_{B1A2}+\delta s_{B1A2})\\ -&(\overline{s}_{A1A2}-\delta s_{A1A2})\,I(\sqrt{z^2+q_{A1}^2},0,\overline{s}_{A1A2}-\delta s_{A1A2})\\ -&(\overline{s}_{A1A2}+\delta s_{A1A2})\,I(\sqrt{z^2+q_{A2}^2},0,\overline{s}_{A1A2}+\delta s_{A1A2})\\ -&(\overline{s}_{B1B2}-\delta s_{B1B2})\,I(\sqrt{z^2+q_{B1}^2},0,\overline{s}_{B1B2}-\delta s_{B1B2})\\ -&(\overline{s}_{B1B2}+\delta s_{B1B2})\,I(\sqrt{z^2+q_{B2}^2},0,\overline{s}_{B1B2}+\delta s_{B1B2})] \quad . \end{aligned} \tag{296}$$

Applying the second–order Taylor expansion of the right–hand side of (296) with respect to s,

$$\begin{aligned}\widetilde{\Theta}-\widehat{\Theta} = \tfrac{\pm 1}{2(1+N)}\tfrac{1}{2}\{&(\delta s_{A1B2})^2[I_s(\sqrt{z^2+q_{A1}^2},0,\overline{s}_{A1B2})+I_s(\sqrt{z^2+q_{B2}^2},0,\overline{s}_{A1B2})]\\ +&(\delta s_{B1A2})^2[I_s(\sqrt{z^2+q_{B1}^2},0,\overline{s}_{B1A2})+I_s(\sqrt{z^2+q_{A2}^2},0,\overline{s}_{B1A2})]\\ -&(\delta s_{A1A2})^2[I_s(\sqrt{z^2+q_{A1}^2},0,\overline{s}_{A1A2})+I_s(\sqrt{z^2+q_{A2}^2},0,\overline{s}_{A1A2})]\\ -&(\delta s_{B1B2})^2[I_s(\sqrt{z^2+q_{B1}^2},0,\overline{s}_{B1B2})+I_s(\sqrt{z^2+q_{B2}^2},0,\overline{s}_{B1B2})]\}\\ -\tfrac{\pm 1}{2(1+N)}\tfrac{1}{2}\{&\delta s_{A1B2}\,[I(\sqrt{z^2+q_{A1}^2},0,\overline{s}_{A1B2})-I(\sqrt{z^2+q_{B2}^2},0,\overline{s}_{A1B2})]\\ +&\delta s_{B1A2}\,[I(\sqrt{z^2+q_{B1}^2},0,\overline{s}_{B1A2})-I(\sqrt{z^2+q_{A2}^2},0,\overline{s}_{B1A2})]\\ -&\delta s_{A1A2}\,[I(\sqrt{z^2+q_{A1}^2},0,\overline{s}_{A1A2})-I(\sqrt{z^2+q_{A2}^2},0,\overline{s}_{A1A2})]\\ -&\delta s_{B1B2}\,[I(\sqrt{z^2+q_{B1}^2},0,\overline{s}_{B1B2})-I(\sqrt{z^2+q_{B2}^2},0,\overline{s}_{B1B2})]\}\,. \end{aligned} \tag{297}$$

We apply the zero–order Taylor expansion of I_s and the first–order Taylor expansion of I with respect to q at $\overline{q}$ in (297). Applying then approximations (292) and (293), the error $\Theta-\widetilde{\Theta}$ of (295) can be approximated by (253) with $\overline{D}(z,q,s)$ replaced by

$$\widetilde{D}(z,q,s) = \widehat{D}(z,q,s) - q^2 I_s(\sqrt{z^2+q^2},0,s) + q\,s\,\tfrac{\partial}{\partial q}I(\sqrt{z^2+q^2},0,s) \quad . \tag{298}$$

8. Special cases

8.1. Coplanar line segments (2-D)

If $\sin\alpha$ is not small, we use equation (95) with $z = 0$. If $z = 0$, equation (131) yields for $s^2 \lesssim q^2$

$$I(0,q,s) = s\,(q^2+s^2)^N\,{}_2\mathrm{F}_1(1,-N;\tfrac{3}{2};\tfrac{s^2}{q^2+s^2}) \quad , \tag{299}$$

and equation (157) yields for $s^2 \gtrsim q^2$

$$I(0,q,s) = \frac{1}{1+2N}\,s\,(q^2+s^2)^N\,{}_2\mathrm{F}_1(1,-N;\tfrac{1}{2}-N;\tfrac{q^2}{q^2+s^2}) - \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}-N)}{(1+2N)\Gamma(-N)}\,s\,\tfrac{|q|}{|s|}\,(q^2)^N . \tag{300}$$

If $\sin\alpha$ is small, we use equations (238), (278) and (281) with $z = 0$.

8.2. Parallel line segments

For $z = 0$ and $q_{A1} = q_{B1} = q_{A2} = q_{B2} = q$, equation (238) reads

$$\begin{aligned}\overline{\Theta} = \tfrac{\pm 1}{2(1+N)}[& s_{A1B2}\,I(q,0,s_{A1B2}) \\ & + s_{B1A2}\,I(q,0,s_{B1A2}) \\ & - s_{A1A2}\,I(q,0,s_{A1A2}) \\ & - s_{B1B2}\,I(q,0,s_{B1B2})] \quad .\end{aligned} \tag{301}$$

This equation formally corresponds to equation (95) with $z = q$ and $q_{A1} = q_{B1} = q_{A2} = q_{B2} = 0$. It is important to know that we cannot formally insert $q_{A1} = q_{B1} = q_{A2} = q_{B2} = q \neq 0$ into equation (95) because we would obtain incorrect result, but we may formally insert $q_{A1} = q_{B1} = q_{A2} = q_{B2} = 0$ into equation (95).

Functions $I(q,0,s)$ can be calculated using equations (278) and (281) with $z = 0$.

8.3. Colinear line segments (1-D, well logs)

We consider line segments A_1B_1 and A_2B_2 situated on the same straight line. These line segments may, for example, represent two segments of a well.

For $z = 0$ and $q = 0$, distances s_{X1}, s_{Y2}, s_{X1Y2} and s_{Y2X1}, where $X = A, B$ and $Y = A, B$, satisfy relations

$$s_{X1Y2} = s_{X1} \mp s_{Y2} \quad , \tag{302}$$

$$s_{Y2X1} = s_{Y2} \mp s_{X1} \quad , \tag{303}$$

and consequently

$$s_{Y2X1} = \mp s_{X1Y2} \quad . \tag{304}$$

Inserting (304) into (95), and considering (302) and (303) with $z = 0$ and $q = 0$, we obtain

$$\begin{aligned}\Theta = \tfrac{1}{2(1+N)}\,[& \pm s_{A1B2}\,I(0,0,s_{A1B2}) \pm s_{B1A2}\,I(0,0,s_{B1A2}) \\ & \mp s_{A1A2}\,I(0,0,s_{A1A2}) \mp s_{B1B2}\,I(0,0,s_{B1B2})] \quad ,\end{aligned} \tag{305}$$

which is identical to (301) with $q = 0$.

If $z = 0$ and $q = 0$ both equations (281) and (300) yield

$$I(0,0,s) = \frac{s\,|s|^{2N}}{1+2N} \quad . \tag{306}$$

Then

$$\Theta = \tfrac{\pm 1}{(1+2N)(2+2N)} \left[(s_{A1B2}^2)^{1+N} + (s_{B1A2}^2)^{1+N} - (s_{A1A2}^2)^{1+N} - (s_{B1B2}^2)^{1+N}\right] \quad . \tag{307}$$

Since the upper sign in (302)–(307) corresponds to equal orientations of line segments A_1B_1 and A_2B_2, equation (307) reads

$$\Theta = \tfrac{1}{(1+2N)(2+2N)} \left|(s_{A1B2}^2)^{1+N} + (s_{B1A2}^2)^{1+N} - (s_{A1A2}^2)^{1+N} - (s_{B1B2}^2)^{1+N}\right| \quad , \tag{308}$$

where s_{A1B2}, s_{B1A2}, s_{A1A2} and s_{B1B2} are the distances of the endpoints of line segments A_1B_1 and A_2B_2. Equation (308) can also be simply derived by directly integrating equation (19) along the straight line. For coinciding line segments $A_1B_1 \equiv A_2B_2 \equiv AB$, equation (308) reduces to

$$\Theta = \tfrac{2}{(1+2N)(2+2N)} (s_{AB}^2)^{1+N} \tag{309}$$

(Klimeš, 2002c, eq. 30).

8.4. Parallel sides of a rectangle

We consider line segments forming two parallel sides of a rectangle. Their length is s and their distance is z. Integral (19) for these parallel sides of a rectangle reads

$$\Theta = \int_0^s \mathrm{d}s_1 \int_0^s \mathrm{d}s_2 \, [(s_1 - s_2)^2 + z^2]^N \quad . \tag{310}$$

Integrand of double integral (310) over square of size s is symmetric with respect to diagonal line $s_1 - s_2 = 0$ and is constant along lines $s_1 - s_2 = \varrho$, where ϱ is a constant. We thus switch to integration with respect to ϱ over the triangle $s_1 \geq s_2$,

$$\Theta = 2 \int_0^s \mathrm{d}\varrho \, (s - \varrho) \, (\varrho^2 + z^2)^N \quad . \tag{311}$$

Considering (98) with $b = -N$, we integrate (311) and obtain equation

$$\Theta = 2s^2 \, (z^2)^N \, {}_2\mathrm{F}_1(\tfrac{1}{2}, -N; \tfrac{3}{2}; -\tfrac{s^2}{z^2}) - \tfrac{1}{1+N} (z^2 + s^2)^{1+N} + \tfrac{1}{1+N} (z^2)^{1+N} \quad . \tag{312}$$

We convert the negative variable of the hypergeometric function in equation (312) to a positive variable using identity (103),

$$\Theta = 2s^2 \, (z^2 + s^2)^N \, {}_2\mathrm{F}_1(1, -N; \tfrac{3}{2}; \tfrac{s^2}{z^2+s^2}) - \tfrac{1}{1+N} (z^2 + s^2)^{1+N} + \tfrac{1}{1+N} (z^2)^{1+N} \; . \tag{313}$$

The Gauss hypergeometric series corresponding to the hypergeometric function in equation (313) rapidly converges if $s \lesssim z$. For $z \lesssim s$, we convert the variable of the hypergeometric function in equation (313) using identity (116),

$$\begin{aligned} \Theta = {} & 2 \, \tfrac{1}{1+2N} \, s^2 \, (z^2 + s^2)^N \, {}_2\mathrm{F}_1(1, -N; \tfrac{1}{2} - N; \tfrac{z^2}{z^2+s^2}) \\ & + 2 \, \tfrac{\Gamma(\frac{3}{2})\Gamma(-\frac{1}{2}-N)}{\Gamma(-N)} \, s^2 \sqrt{\tfrac{z^2}{z^2+s^2}} \, (z^2)^N \, {}_2\mathrm{F}_1(\tfrac{1}{2}, \tfrac{3}{2}+N; \tfrac{3}{2}+N; \tfrac{z^2}{z^2+s^2}) \\ & - \tfrac{1}{1+N} (z^2 + s^2)^{1+N} + \tfrac{1}{1+N} (z^2)^{1+N} \quad . \end{aligned} \tag{314}$$

Considering (125), we obtain for $z \lesssim s$ equation

$$\begin{aligned} \Theta = {} & 2 \, \tfrac{1}{1+2N} \, s^2 \, (z^2 + s^2)^N \, {}_2\mathrm{F}_1(1, -N; \tfrac{1}{2} - N; \tfrac{z^2}{z^2+s^2}) \\ & - 2 \, \tfrac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}-N)}{(1+2N)\Gamma(-N)} \, s^2 \, \tfrac{|z|}{|s|} \, (z^2)^N - \tfrac{1}{1+N} (z^2 + s^2)^{1+N} + \tfrac{1}{1+N} (z^2)^{1+N} \quad . \end{aligned} \tag{315}$$

We now compare equations (313) and (315) derived above with more general equations of Section 7. For the considered special case of the parallel sides of a rectangle, equations (195) or (199) read

$$\Theta = \tfrac{1}{1+N}\, s\, I(z,0,s) \quad , \tag{316}$$

which can be used to express function

$$I(z,0,s) = \tfrac{1+N}{s}\Theta \quad . \tag{317}$$

For $s \lesssim z$, we insert (313) into (317) and obtain equation

$$I(z,0,s) = 2\,(1+N)\, s\,(z^2+s^2)^N\, {}_2\mathrm{F}_1(1,-N;\tfrac{3}{2};\tfrac{s^2}{z^2+s^2}) - \tfrac{1}{s}\,(z^2+s^2)^{1+N} + \tfrac{1}{s}\,(z^2)^{1+N} \,, \tag{318}$$

which represents a special case of equation (278) for $q=0$. For $z \lesssim s$, we insert (315) into (317) and obtain equation

$$\begin{aligned} I(z,0,s) = {} & 2\,\tfrac{1+N}{1+2N}\, s\,(z^2+s^2)^N\, {}_2\mathrm{F}_1(1,-N;\tfrac{1}{2}-N;\tfrac{z^2}{z^2+s^2}) \\ & - 2\,\tfrac{1+N}{1+2N}\,\tfrac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}-N)}{\Gamma(-N)}\, s\,\tfrac{|z|}{|s|}\,(z^2)^N - \tfrac{1}{s}\,(z^2+s^2)^{1+N} + \tfrac{1}{s}\,(z^2)^{1+N} \quad , \end{aligned} \tag{319}$$

which represents a special case of equation (281) for $q=0$. The special case of equation (319) for $z=0$ yields equation (306).

Acknowledgements

I am grateful to Václav Vavryčuk for directing my attention to hypergeometric functions, and to Peter Cigánik for calculating integrals (97), (98) and many other using program Mathematica and for coding and numerically testing the proposed algorithms to calculate the geometrical covariances of travel times.

The research has been supported by the Grant Agency of the Czech Republic under contract 205/07/0032, by the Ministry of Education of the Czech Republic within research project MSM0021620860, and by the members of the consortium "Seismic Waves in Complex 3–D Structures" (see "http://sw3d.cz").

References

Abramowitz, M. & Stegun, I.A. (1964): *Handbook of Mathematical Functions.* National Bureau of Standards, Washington D.C..

Bucha, V. & Bulant, P. (eds.) (2008): SW3D–CD–12 (DVD ROM). In: *Seismic Waves in Complex 3-D Structures, Report 18*, pp. 245–245, Dep. Geophys., Charles Univ., Prague.

Franklin, J.N. (1970): Well–posed stochastic extensions of ill-posed linear problems. *J. math. Anal. Appl.*, **31**, 682–716.

Gradshteyn, I.S. & Ryzhik, I.M. (Jeffrey, A. & Zwillinger, D., eds.) (2000): *Table of Integrals, Series, and Products. Sixth Edition.* Academic Press, San Diego.

Klimeš, L. (2002a): Application of the medium covariance functions to travel–time tomography. *Pure appl. Geophys.*, **159**, 1791–1810, online at "http://sw3d.cz".

Klimeš, L. (2002b): Correlation functions of random media. *Pure appl. Geophys.*, **159**, 1811–1831, online at "http://sw3d.cz".

Klimeš, L. (2002c): Estimating the correlation function of a self–affine random medium. *Pure appl. Geophys.*, **159**, 1833–1853, online at "http://sw3d.cz".

Klimeš, L. (2008): Computer representation of the model covariance function resulting from travel–time tomography. In: *Seismic Waves in Complex 3-D Structures, Report 18*, pp. 17–26, Dep. Geophys., Charles Univ., Prague.

Tarantola, A. (1987): *Inverse Problem Theory.* Elsevier, Amsterdam.