Scalar products of the structural Gabor functions

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Summary

The equations for efficient calculation of the Lebesgue scalar products of Gabor functions are presented.

Keywords

Gabor function, scalar product, norm, frame, Gabor transform, wavefield inversion.

1. Introduction

In numerical least–square decomposition of a given function into the given set of Gabor functions (Klimeš, 2008, sec. 2.3), we need to know the L_2 Lebesgue scalar products of the Gabor functions. These scalar products form the matrix of linear equations for the coefficients of the decomposition. The same Lebesgue scalar products of the Gabor functions are used in the wavefield inversion based on the sensitivity Gaussian packets (Klimeš, 2008, sec. 3.3). We present the equations for calculating these scalar products in this paper. The equations are used in program `sgfmat.for` of package `FORMS` (Bucha & Bulant, 2008) for calculating the matrix of L_2 Lebesgue scalar products of the given Gabor functions.

2. Structural Gabor functions

Complex–valued structural Gabor functions, indexed here by lower-case Greek superscripts, read

$$g^\alpha(\mathbf{x}) = a^\alpha \exp[\mathrm{i}\mathbf{k}^{\alpha\mathrm{T}}(\mathbf{x}-\mathbf{x}^\alpha) - \tfrac{1}{2}(\mathbf{x}-\mathbf{x}^\alpha)^\mathrm{T}\mathbf{K}^\alpha(\mathbf{x}-\mathbf{x}^\alpha)] \quad . \tag{1}$$

Gabor function $g^\alpha(\mathbf{x})$ is centred at spatial position $\mathbf{x}^\alpha$ and has structural wavenumber vector $\mathbf{k}^\alpha$. The envelope and wavefront curvature of Gabor function g^α is described by symmetric complex–valued 3×3 matrix $\mathbf{K}^\alpha$ with the positive–definite real part. Real–valued amplitude

$$a^\alpha = (\pi)^{-\frac{3}{4}}\,[\det(\mathrm{Re}\mathbf{K}^\alpha)]^{\frac{1}{4}} \tag{2}$$

of the envelope is so chosen that the L_2 norm $|g^\alpha|$ is unit, which will follow later from equation (20).

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3. Scalar products of the structural Gabor functions

Let us calculate the L_2 Lebesgue scalar product

$$(g^\alpha, g^\beta) = \int \mathrm{d}^3x \, [g^\alpha(\mathbf{x})]^* g^\beta(\mathbf{x}) \tag{3}$$

of two structural Gabor functions. Asterisk * denotes the complex–conjugacy. We insert (1) into (3),

$$(g^\alpha, g^\beta) = a^\alpha a^\beta \int \mathrm{d}^3x \exp\begin{bmatrix} \mathrm{i}\mathbf{k}^{\beta\mathrm{T}}(\mathbf{x}-\mathbf{x}^\beta) - \mathrm{i}\mathbf{k}^{\alpha\mathrm{T}}(\mathbf{x}-\mathbf{x}^\alpha) \\ -\frac{1}{2}(\mathbf{x}-\mathbf{x}^\beta)^\mathrm{T}\mathbf{K}^\beta(\mathbf{x}-\mathbf{x}^\beta)] - \frac{1}{2}(\mathbf{x}-\mathbf{x}^\alpha)^\mathrm{T}\mathbf{K}^{\alpha*}(\mathbf{x}-\mathbf{x}^\alpha) \end{bmatrix} \ . \tag{4}$$

We insert relations

$$\mathbf{x}^\alpha = \mathbf{x}^0 - \tfrac{1}{2}\Delta\mathbf{x} \quad , \tag{5}$$

$$\mathbf{x}^\beta = \mathbf{x}^0 + \tfrac{1}{2}\Delta\mathbf{x} \quad , \tag{6}$$

$$\mathbf{k}^\alpha = \mathbf{k}^0 - \tfrac{1}{2}\Delta\mathbf{k} \quad , \tag{7}$$

$$\mathbf{k}^\beta = \mathbf{k}^0 + \tfrac{1}{2}\Delta\mathbf{k} \tag{8}$$

into (4) and obtain

$$(g^\alpha, g^\beta) = a^\alpha a^\beta \int \mathrm{d}^3x \exp\begin{bmatrix} -\mathrm{i}\mathbf{k}^{0\mathrm{T}}\Delta\mathbf{x} + \mathrm{i}(\Delta\mathbf{k})^\mathrm{T}(\mathbf{x}-\mathbf{x}^0) - \frac{1}{2}(\mathbf{x}-\mathbf{x}^0)^\mathrm{T}(\mathbf{K}^\beta+\mathbf{K}^{\alpha*})(\mathbf{x}-\mathbf{x}^0) \\ +\frac{1}{2}(\Delta\mathbf{x})^\mathrm{T}(\mathbf{K}^\beta-\mathbf{K}^{\alpha*})(\mathbf{x}-\mathbf{x}^0) - \frac{1}{8}(\Delta\mathbf{x})^\mathrm{T}(\mathbf{K}^\beta+\mathbf{K}^{\alpha*})\Delta\mathbf{x} \end{bmatrix} \ . \tag{9}$$

We insert abbreviations

$$\mathbf{K}^\beta + \mathbf{K}^{\alpha*} = 2\mathbf{Y} \quad , \tag{10}$$

$$\mathbf{K}^\beta - \mathbf{K}^{\alpha*} = -2\mathrm{i}\mathbf{R} \quad , \tag{11}$$

into (9),

$$(g^\alpha, g^\beta) = a^\alpha a^\beta \int \mathrm{d}^3x \exp\begin{bmatrix} -\mathrm{i}\mathbf{k}^{0\mathrm{T}}\Delta\mathbf{x} + \mathrm{i}(\Delta\mathbf{k})^\mathrm{T}(\mathbf{x}-\mathbf{x}^0) - (\mathbf{x}-\mathbf{x}^0)^\mathrm{T}\mathbf{Y}(\mathbf{x}-\mathbf{x}^0) \\ -\mathrm{i}(\Delta\mathbf{x})^\mathrm{T}\mathbf{R}(\mathbf{x}-\mathbf{x}^0) - \frac{1}{4}(\Delta\mathbf{x})^\mathrm{T}\mathbf{Y}\Delta\mathbf{x} \end{bmatrix} \ . \tag{12}$$

We now express (12) in form

$$(g^\alpha, g^\beta) = a^\alpha a^\beta \int \mathrm{d}^3x \exp\begin{bmatrix} -\mathrm{i}\mathbf{k}^{0\mathrm{T}}\Delta\mathbf{x} - [\mathbf{x}-\mathbf{x}^0 - \mathrm{i}\frac{1}{2}\mathbf{Y}^{-1}(\Delta\mathbf{k}-\mathbf{R}\Delta\mathbf{x})]^\mathrm{T}\mathbf{Y} \\ \times[\mathbf{x}-\mathbf{x}^0 - \mathrm{i}\frac{1}{2}\mathbf{Y}^{-1}(\Delta\mathbf{k}-\mathbf{R}\Delta\mathbf{x})] \\ -\frac{1}{4}(\Delta\mathbf{k}+\mathbf{R}\Delta\mathbf{x})^\mathrm{T}\mathbf{Y}^{-1}(\Delta\mathbf{k}+\mathbf{R}\Delta\mathbf{x}) - \frac{1}{4}(\Delta\mathbf{x})^\mathrm{T}\mathbf{Y}\Delta\mathbf{x} \end{bmatrix} \tag{13}$$

and integrate it analytically,

$$\begin{aligned}(g^\alpha, g^\beta) = {} & a^\alpha a^\beta (\pi)^{\frac{3}{2}} (\det \mathbf{Y})^{-\frac{1}{2}} \exp(-\mathrm{i}\mathbf{k}^{0\mathrm{T}}\Delta\mathbf{x}) \\ & \times \exp\left\{-\tfrac{1}{4}\left[(\Delta\mathbf{x})^\mathrm{T}(\mathbf{Y}+\mathbf{R}\mathbf{Y}^{-1}\mathbf{R})\Delta\mathbf{x} - 2(\Delta\mathbf{k})^\mathrm{T}\mathbf{Y}^{-1}\mathbf{R}\Delta\mathbf{x} + (\Delta\mathbf{k})^\mathrm{T}\mathbf{Y}^{-1}\Delta\mathbf{k}\right]\right\} \ . \end{aligned} \tag{14}$$

Equation (14) can be expressed in form

$$\boxed{(g^\alpha, g^\beta) = a^\alpha a^\beta (\pi)^{\frac{3}{2}} (\det \mathbf{Y})^{-\frac{1}{2}} \exp(-\mathrm{i}\mathbf{k}^{0\mathrm{T}}\Delta\mathbf{x}) \exp\left[-\tfrac{1}{4}\binom{\Delta\mathbf{x}}{\Delta\mathbf{k}}^\mathrm{T}\mathbf{G}_{\bullet\bullet}\binom{\Delta\mathbf{x}}{\Delta\mathbf{k}}\right] \quad ,} \tag{15}$$

where

$$\mathbf{G}_{\bullet\bullet} = \begin{pmatrix} \mathbf{Y}+\mathbf{R}\mathbf{Y}^{-1}\mathbf{R} & -\mathbf{R}\mathbf{Y}^{-1} \\ -\mathbf{Y}^{-1}\mathbf{R} & \mathbf{Y}^{-1} \end{pmatrix} = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ -\mathbf{R} & \mathbf{1} \end{pmatrix}^\mathrm{T} \begin{pmatrix} \mathbf{Y} & \mathbf{0} \\ \mathbf{0} & \mathbf{Y}^{-1} \end{pmatrix} \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ -\mathbf{R} & \mathbf{1} \end{pmatrix} \tag{16}$$

is the covariant complex–valued matrix in phase space. Its inverse matrix

$$\mathbf{G}^{\bullet\bullet} = \begin{pmatrix} \mathbf{Y}^{-1} & \mathbf{Y}^{-1}\mathbf{R} \\ \mathbf{R}\mathbf{Y}^{-1} & \mathbf{Y}+\mathbf{R}\mathbf{Y}^{-1}\mathbf{R} \end{pmatrix} = \begin{pmatrix} \mathbf{1} & \mathbf{R} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} \mathbf{Y}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{Y} \end{pmatrix} \begin{pmatrix} \mathbf{1} & \mathbf{R} \\ \mathbf{0} & \mathbf{1} \end{pmatrix} \tag{17}$$

is the contravariant complex–valued matrix in phase space. Both the covariant and contravariant matrices are symplectic.

4. Conservative limits on the phase–space distances between the structural Gabor functions

Scalar product (15) decays with distance. In order to avoid calculation of too small scalar products, we wish to know the limits on the phase–space distances beyond which

$$|(g^\alpha, g^\beta)| \le \varepsilon\, |g^\alpha|\, |g^\beta| \quad , \tag{18}$$

where ε is a given small positive number. Equation (15) yields

$$|(g^\alpha, g^\beta)| = a^\alpha a^\beta (\pi)^{\frac{3}{2}} |\det \mathbf{Y}|^{-\frac{1}{2}} \exp\!\Big[-\tfrac{1}{4} \big(\begin{smallmatrix}\Delta\mathbf{x}\\ \Delta\mathbf{k}\end{smallmatrix}\big)^{\mathrm{T}} \mathrm{Re}(\mathbf{G}_{\bullet\bullet}) \big(\begin{smallmatrix}\Delta\mathbf{x}\\ \Delta\mathbf{k}\end{smallmatrix}\big)\Big] \quad . \tag{19}$$

The L_2 norm of the structural Gabor function is

$$|g^\alpha| = a^\alpha (\pi)^{\frac{3}{4}}\, [\det(\mathrm{Re}\mathbf{K}^\alpha)]^{-\frac{1}{4}} \quad . \tag{20}$$

Because of choice (2), the L_2 norms of all structural Gabor functions are unit,

$$|g^\alpha| = 1 \quad . \tag{21}$$

Since

$$|\det \mathbf{Y}| \ge \det(\mathrm{Re}\mathbf{Y}) \ge \sqrt{\det(\mathrm{Re}\mathbf{K}^\alpha)\ \det(\mathrm{Re}\mathbf{K}^\beta)} \tag{22}$$

for matrices $\mathbf{Y}$, $\mathbf{K}^\alpha$ and $\mathbf{K}^\beta$ with the positive–definite real parts, equations (19) and (20) yield inequality

$$|(g^\alpha, g^\beta)| \le |g^\alpha|\, |g^\beta| \exp\!\Big[-\tfrac{1}{4} \big(\begin{smallmatrix}\Delta\mathbf{x}\\ \Delta\mathbf{k}\end{smallmatrix}\big)^{\mathrm{T}} \mathrm{Re}(\mathbf{G}_{\bullet\bullet}) \big(\begin{smallmatrix}\Delta\mathbf{x}\\ \Delta\mathbf{k}\end{smallmatrix}\big)\Big] \quad . \tag{23}$$

Inequality in (18) is thus satisfied if

$$\big(\begin{smallmatrix}\Delta\mathbf{x}\\ \Delta\mathbf{k}\end{smallmatrix}\big)^{\mathrm{T}} \mathrm{Re}(\mathbf{G}_{\bullet\bullet}) \big(\begin{smallmatrix}\Delta\mathbf{x}\\ \Delta\mathbf{k}\end{smallmatrix}\big) \ge -4\, \ln(\varepsilon) \quad . \tag{24}$$

The equality in (24) describes the boundary of an ellipsoid in phase space. Let us consider the coordinate subspaces touching this ellipsoid. The squares of the coordinates defining these coordinate subspaces are equal to the diagonal elements of matrix

$$-4\, \ln(\varepsilon)\, [\mathrm{Re}(\mathbf{G}_{\bullet\bullet})]^{-1} \quad . \tag{25}$$

We decompose matrices $\mathbf{K}^\alpha$ and $\mathbf{K}^\beta$ into the real and imaginary parts,

$$\mathbf{K}^\alpha = \mathbf{Y}^\alpha - \mathrm{i}\mathbf{R}^\alpha \quad , \qquad \mathbf{K}^\beta = \mathbf{Y}^\beta - \mathrm{i}\mathbf{R}^\beta \quad . \tag{26}$$

Then

$$\mathbf{Y} = \mathbf{Y}_+ - \mathrm{i}\mathbf{R}_- \quad , \qquad \mathbf{R} = \mathbf{R}_+ + \mathrm{i}\mathbf{Y}_- \quad , \tag{27}$$

where

$$\mathbf{Y}_+ = \tfrac{1}{2}(\mathbf{Y}^\beta + \mathbf{Y}^\alpha) \quad , \qquad \mathbf{R}_+ = \tfrac{1}{2}(\mathbf{R}^\beta + \mathbf{R}^\alpha) \quad , \tag{28}$$

$$\mathbf{Y}_- = \tfrac{1}{2}(\mathbf{Y}^\beta - \mathbf{Y}^\alpha) \quad , \qquad \mathbf{R}_- = \tfrac{1}{2}(\mathbf{R}^\beta - \mathbf{R}^\alpha) \tag{29}$$

are real–valued matrices.

$$\mathrm{Re}(\mathbf{G}_{\bullet\bullet}) = \begin{pmatrix} \mathrm{Re}(\mathbf{Y})+\mathrm{Re}(\mathbf{R}\mathbf{Y}^{-1}\mathbf{R}) & -\mathrm{Re}(\mathbf{R}\mathbf{Y}^{-1}) \\ -\mathrm{Re}(\mathbf{Y}^{-1}\mathbf{R}) & \mathrm{Re}(\mathbf{Y}^{-1}) \end{pmatrix} \quad . \tag{30}$$

Since

$$\begin{pmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{B} & \mathbf{D} \end{pmatrix}^{-1} = \begin{pmatrix} (\mathbf{A}-\mathbf{C}\mathbf{D}^{-1}\mathbf{B})^{-1} & (\mathbf{B}-\mathbf{D}\mathbf{C}^{-1}\mathbf{A})^{-1} \\ (\mathbf{C}-\mathbf{A}\mathbf{B}^{-1}\mathbf{D})^{-1} & (\mathbf{D}-\mathbf{B}\mathbf{A}^{-1}\mathbf{C})^{-1} \end{pmatrix} \quad , \tag{31}$$

the diagonal submatrices of matrix

$$[\mathrm{Re}(\mathbf{G}_{\bullet\bullet})]^{-1} = \begin{pmatrix} \mathbf{G}^{\Delta\mathrm{x}\Delta\mathrm{x}} & \mathbf{G}^{\Delta\mathrm{x}\Delta\mathrm{k}} \\ \mathbf{G}^{\Delta\mathrm{k}\Delta\mathrm{x}} & \mathbf{G}^{\Delta\mathrm{k}\Delta\mathrm{k}} \end{pmatrix} \tag{32}$$

are

$$\mathbf{G}^{\Delta\mathrm{x}\Delta\mathrm{x}} = \{\mathrm{Re}(\mathbf{Y}) + \mathrm{Re}(\mathbf{R}\mathbf{Y}^{-1}\mathbf{R}) - \mathrm{Re}(\mathbf{R}\mathbf{Y}^{-1})[\mathrm{Re}(\mathbf{Y}^{-1})]^{-1}\mathrm{Re}(\mathbf{Y}^{-1}\mathbf{R})\}^{-1} \tag{33}$$

and

$$\mathbf{G}^{\Delta\mathrm{k}\Delta\mathrm{k}} = \{\mathrm{Re}(\mathbf{Y}^{-1}) - \mathrm{Re}(\mathbf{Y}^{-1}\mathbf{R})[\mathrm{Re}(\mathbf{Y}) + \mathrm{Re}(\mathbf{R}\mathbf{Y}^{-1}\mathbf{R})]^{-1}\mathrm{Re}(\mathbf{R}\mathbf{Y}^{-1})\}^{-1} \quad . \tag{34}$$

We now simplify equation (33) and then equation (34).

Equation (27) yields

$$\mathbf{Y}^{-1} = (\mathbf{Y}_+ - \mathrm{i}\mathbf{R}_-)^{-1}(\mathbf{Y}_+ + \mathrm{i}\mathbf{R}_-)(\mathbf{Y}_+ + \mathrm{i}\mathbf{R}_-)^{-1} \quad , \tag{35}$$

which reads

$$\mathbf{Y}^{-1} = (\mathbf{Y}_+ + \mathbf{R}_-\mathbf{Y}_+^{-1}\mathbf{R}_-)^{-1} + \mathrm{i}(\mathbf{R}_- + \mathbf{Y}_+\mathbf{R}_-^{-1}\mathbf{Y}_+)^{-1} \quad . \tag{36}$$

We thus know $[\mathrm{Re}(\mathbf{Y}^{-1})]^{-1}$ in (33) and calculate $\mathrm{Re}(\mathbf{Y}^{-1}\mathbf{R})$. From (27) and (36), we obtain equation

$$\mathrm{Re}(\mathbf{Y}^{-1}\mathbf{R}) = (\mathbf{Y}_+ + \mathbf{R}_-\mathbf{Y}_+^{-1}\mathbf{R}_-)^{-1}\mathbf{R}_+ - (\mathbf{R}_- + \mathbf{Y}_+\mathbf{R}_-^{-1}\mathbf{Y}_+)^{-1}\mathbf{Y}_- \quad , \tag{37}$$

which after simple manipulation reads

$$\mathrm{Re}(\mathbf{Y}^{-1}\mathbf{R}) = (\mathbf{Y}_+ + \mathbf{R}_-\mathbf{Y}_+^{-1}\mathbf{R}_-)^{-1}(\mathbf{R}_+ - \mathbf{R}_-\mathbf{Y}_+^{-1}\mathbf{Y}_-) \quad . \tag{38}$$

We now calculate term $\mathrm{Re}(\mathbf{R}\mathbf{Y}^{-1}\mathbf{R})$ on the right–hand side of (33). From (27) and (36), we obtain equation

$$\begin{aligned} \mathrm{Re}(\mathbf{R}\mathbf{Y}^{-1}\mathbf{R}) = {} & \mathbf{R}_+(\mathbf{Y}_+ + \mathbf{R}_-\mathbf{Y}_+^{-1}\mathbf{R}_-)^{-1}\mathbf{R}_+ - \mathbf{Y}_-(\mathbf{R}_- + \mathbf{Y}_+\mathbf{R}_-^{-1}\mathbf{Y}_+)^{-1}\mathbf{R}_+ \\ & - \mathbf{R}_+(\mathbf{R}_- + \mathbf{Y}_+\mathbf{R}_-^{-1}\mathbf{Y}_+)^{-1}\mathbf{Y}_- - \mathbf{Y}_-(\mathbf{Y}_+ + \mathbf{R}_-\mathbf{Y}_+^{-1}\mathbf{R}_-)^{-1}\mathbf{Y}_- \quad , \end{aligned} \tag{39}$$

which we convert to form

$$\begin{aligned} \mathrm{Re}(\mathbf{R}\mathbf{Y}^{-1}\mathbf{R}) = {} & \mathbf{R}_+(\mathbf{Y}_+ + \mathbf{R}_-\mathbf{Y}_+^{-1}\mathbf{R}_-)^{-1}\mathbf{R}_+ - \mathbf{Y}_-\mathbf{Y}_+^{-1}\mathbf{R}_-(\mathbf{Y}_+ + \mathbf{R}_-\mathbf{Y}_+^{-1}\mathbf{R}_-)^{-1}\mathbf{R}_+ \\ & - \mathbf{R}_+(\mathbf{Y}_+ + \mathbf{R}_-\mathbf{Y}_+^{-1}\mathbf{R}_-)^{-1}\mathbf{R}_-\mathbf{Y}_+^{-1}\mathbf{Y}_- - \mathbf{Y}_-(\mathbf{Y}_+ + \mathbf{R}_-\mathbf{Y}_+^{-1}\mathbf{R}_-)^{-1}\mathbf{Y}_- \ . \end{aligned} \tag{40}$$

Equations (36) and (38) yield

$$\begin{aligned} & \mathrm{Re}(\mathbf{R}\mathbf{Y}^{-1})[\mathrm{Re}(\mathbf{Y}^{-1})]^{-1}\mathrm{Re}(\mathbf{Y}^{-1}\mathbf{R}) \\ & \quad = (\mathbf{R}_+ - \mathbf{Y}_-\mathbf{Y}_+^{-1}\mathbf{R}_-)(\mathbf{Y}_+ + \mathbf{R}_-\mathbf{Y}_+^{-1}\mathbf{R}_-)^{-1}(\mathbf{R}_+ - \mathbf{R}_-\mathbf{Y}_+^{-1}\mathbf{Y}_-) \quad . \end{aligned} \tag{41}$$

We insert (27), (40) and (41) into (33) and obtain equation

$$\begin{aligned} \mathbf{G}^{\Delta\mathrm{x}\Delta\mathrm{x}} = {} & [\mathbf{Y}_+ - \mathbf{Y}_-(\mathbf{Y}_+ + \mathbf{R}_-\mathbf{Y}_+^{-1}\mathbf{R}_-)^{-1}\mathbf{Y}_- \\ & -\mathbf{Y}_-\mathbf{Y}_+^{-1}\mathbf{R}_-(\mathbf{Y}_+ + \mathbf{R}_-\mathbf{Y}_+^{-1}\mathbf{R}_-)^{-1}\mathbf{R}_-\mathbf{Y}_+^{-1}\mathbf{Y}_-]^{-1} \quad , \end{aligned} \tag{42}$$

which finally reads

$$\mathbf{G}^{\Delta\mathbf{x}\Delta\mathbf{x}} = (\mathbf{Y}_+ - \mathbf{Y}_-\mathbf{Y}_+^{-1}\mathbf{Y}_-)^{-1} \quad . \tag{43}$$

We consider identity

$$\begin{aligned}(\mathbf{Y}_+ - \mathbf{Y}_-\mathbf{Y}_+^{-1}\mathbf{Y}_-)^{-1} &= [\tfrac{1}{2}(\mathbf{Y}^\beta + \mathbf{Y}^\alpha)(\mathbf{Y}^\beta + \mathbf{Y}^\alpha)^{-1}(\mathbf{Y}^\beta + \mathbf{Y}^\alpha) \\ &\quad - \tfrac{1}{2}(\mathbf{Y}^\beta - \mathbf{Y}^\alpha)(\mathbf{Y}^\beta + \mathbf{Y}^\alpha)^{-1}(\mathbf{Y}^\beta - \mathbf{Y}^\alpha)]^{-1} \\ &= \tfrac{1}{2}[\mathbf{Y}^\beta(\mathbf{Y}^\beta + \mathbf{Y}^\alpha)^{-1}\mathbf{Y}^\alpha]^{-1} \\ &= \tfrac{1}{2}[(\mathbf{Y}^\beta)^{-1} + (\mathbf{Y}^\alpha)^{-1}]\end{aligned} \tag{44}$$

following from (28) and (29), and insert it into (43),

$$\boxed{\mathbf{G}^{\Delta\mathbf{x}\Delta\mathbf{x}} = \tfrac{1}{2}[(\mathbf{Y}^\beta)^{-1} + (\mathbf{Y}^\alpha)^{-1}] \quad .} \tag{45}$$

If the square $(x_i^\alpha - x_i^\beta)^2$ of the difference of the i–th spatial coordinate of Gabor functions g^α and g^β exceeds the i–th diagonal element of matrix

$$-4\,\ln(\varepsilon)\,\mathbf{G}^{\Delta\mathbf{x}\Delta\mathbf{x}} = -2\,\ln(\varepsilon)\,[(\mathbf{Y}^\beta)^{-1} + (\mathbf{Y}^\alpha)^{-1}] \quad , \tag{46}$$

the scalar product satisfies inequality (18) and need not be calculated. The form of matrix (46) is very important. We calculate and store the diagonal elements of matrices $-2\,\ln(\varepsilon)\,(\mathbf{Y}^\alpha)^{-1}$ for all Gabor functions g^α. The diagonal elements of matrix (46) are then the sums of the stored matrix elements and the decision that the scalar product need not be calculated is very fast.

We now employ equation (43) in simplifying equation (34). Equation (33) yields

$$\begin{aligned}&\mathrm{Re}(\mathbf{Y}^{-1}\mathbf{R})[\mathrm{Re}(\mathbf{Y}) + \mathrm{Re}(\mathbf{R}\mathbf{Y}^{-1}\mathbf{R})]^{-1}\mathrm{Re}(\mathbf{R}\mathbf{Y}^{-1}) \\ &\qquad = \{[\mathrm{Re}(\mathbf{R}\mathbf{Y}^{-1})]^{-1}(\mathbf{G}^{\Delta\mathbf{x}\Delta\mathbf{x}})^{-1}[\mathrm{Re}(\mathbf{Y}^{-1}\mathbf{R})]^{-1} + \mathrm{Re}(\mathbf{Y}^{-1})^{-1}\}^{-1} \quad .\end{aligned} \tag{47}$$

We insert (47) into (34),

$$\mathbf{G}^{\Delta\mathbf{k}\Delta\mathbf{k}} = [\![\mathrm{Re}(\mathbf{Y}^{-1}) - \{[\mathrm{Re}(\mathbf{R}\mathbf{Y}^{-1})]^{-1}(\mathbf{G}^{\Delta\mathbf{x}\Delta\mathbf{x}})^{-1}[\mathrm{Re}(\mathbf{Y}^{-1}\mathbf{R})]^{-1} + \mathrm{Re}(\mathbf{Y}^{-1})^{-1}\}^{-1}]\!]^{-1}. \tag{48}$$

We perform the subtraction in (48) and obtain

$$\begin{aligned}\mathbf{G}^{\Delta\mathbf{k}\Delta\mathbf{k}} &= [\![\mathrm{Re}(\mathbf{Y}^{-1})[\mathrm{Re}(\mathbf{R}\mathbf{Y}^{-1})]^{-1}(\mathbf{G}^{\Delta\mathbf{x}\Delta\mathbf{x}})^{-1}[\mathrm{Re}(\mathbf{Y}^{-1}\mathbf{R})]^{-1} \\ &\quad \times \{[\mathrm{Re}(\mathbf{R}\mathbf{Y}^{-1})]^{-1}(\mathbf{G}^{\Delta\mathbf{x}\Delta\mathbf{x}})^{-1}[\mathrm{Re}(\mathbf{Y}^{-1}\mathbf{R})]^{-1} + [\mathrm{Re}(\mathbf{Y}^{-1})]^{-1}\}^{-1}]\!]^{-1} \quad .\end{aligned} \tag{49}$$

We multiply the inverse matrices in (49) and arrive at

$$\mathbf{G}^{\Delta\mathbf{k}\Delta\mathbf{k}} = [\mathrm{Re}(\mathbf{Y}^{-1})]^{-1}\mathrm{Re}(\mathbf{Y}^{-1}\mathbf{R})\mathbf{G}^{\Delta\mathbf{x}\Delta\mathbf{x}}\mathrm{Re}(\mathbf{R}\mathbf{Y}^{-1})[\mathrm{Re}(\mathbf{Y}^{-1})]^{-1} + [\mathrm{Re}(\mathbf{Y}^{-1})]^{-1} \, . \tag{50}$$

We insert (36) and (38) into (50),

$$\mathbf{G}^{\Delta\mathbf{k}\Delta\mathbf{k}} = (\mathbf{R}_+ - \mathbf{R}_-\mathbf{Y}_+^{-1}\mathbf{Y}_-)\mathbf{G}^{\Delta\mathbf{x}\Delta\mathbf{x}}(\mathbf{R}_+ - \mathbf{Y}_-\mathbf{Y}_+^{-1}\mathbf{R}_-) + \mathbf{Y}_+ + \mathbf{R}_-\mathbf{Y}_+^{-1}\mathbf{R}_- \quad . \tag{51}$$

We insert (43) into (51),

$$\mathbf{G}^{\Delta\mathbf{k}\Delta\mathbf{k}} = (\mathbf{R}_+ - \mathbf{R}_-\mathbf{Y}_+^{-1}\mathbf{Y}_-)(\mathbf{Y}_+ - \mathbf{Y}_-\mathbf{Y}_+^{-1}\mathbf{Y}_-)^{-1}(\mathbf{R}_+ - \mathbf{Y}_-\mathbf{Y}_+^{-1}\mathbf{R}_-) + \mathbf{Y}_+ + \mathbf{R}_-\mathbf{Y}_+^{-1}\mathbf{R}_- \, . \tag{52}$$

Since

$$\begin{aligned}
&\mathbf{Y}_+^{-1}\mathbf{Y}_-(\mathbf{Y}_+ - \mathbf{Y}_-\mathbf{Y}_+^{-1}\mathbf{Y}_-)^{-1}\mathbf{Y}_-\mathbf{Y}_+^{-1} + \mathbf{Y}_+^{-1} \\
&\quad = \mathbf{Y}_+^{-1}\mathbf{Y}_-(\mathbf{Y}_+ - \mathbf{Y}_-\mathbf{Y}_+^{-1}\mathbf{Y}_-)^{-1}(\mathbf{Y}_-\mathbf{Y}_+^{-1} + \mathbf{Y}_+\mathbf{Y}_-^{-1} - \mathbf{Y}_-\mathbf{Y}_+^{-1}) \\
&\quad = \mathbf{Y}_+^{-1}\mathbf{Y}_-(\mathbf{Y}_+ - \mathbf{Y}_-\mathbf{Y}_+^{-1}\mathbf{Y}_-)^{-1}\mathbf{Y}_+\mathbf{Y}_-^{-1} \\
&\quad = (\mathbf{Y}_+ - \mathbf{Y}_-\mathbf{Y}_+^{-1}\mathbf{Y}_-)^{-1} \quad ,
\end{aligned} \tag{53}$$

equation (52) reads

$$\begin{aligned}
\mathbf{G}^{\Delta\mathbf{k}\Delta\mathbf{k}} &= \mathbf{R}_+(\mathbf{Y}_+ - \mathbf{Y}_-\mathbf{Y}_+^{-1}\mathbf{Y}_-)^{-1}\mathbf{R}_+ + \mathbf{R}_-(\mathbf{Y}_+ - \mathbf{Y}_-\mathbf{Y}_+^{-1}\mathbf{Y}_-)^{-1}\mathbf{R}_- \\
&+ \mathbf{R}_+(\mathbf{Y}_- - \mathbf{Y}_+\mathbf{Y}_-^{-1}\mathbf{Y}_+)^{-1}\mathbf{R}_- + \mathbf{R}_-(\mathbf{Y}_- - \mathbf{Y}_+\mathbf{Y}_-^{-1}\mathbf{Y}_+)^{-1}\mathbf{R}_+ + \mathbf{Y}_+ \quad .
\end{aligned} \tag{54}$$

We consider identity (44) and similar identity

$$\begin{aligned}
(\mathbf{Y}_- - \mathbf{Y}_+\mathbf{Y}_-^{-1}\mathbf{Y}_+)^{-1} &= [\tfrac{1}{2}(\mathbf{Y}^\beta - \mathbf{Y}^\alpha)(\mathbf{Y}^\beta - \mathbf{Y}^\alpha)^{-1}(\mathbf{Y}^\beta - \mathbf{Y}^\alpha) \\
&\quad - \tfrac{1}{2}(\mathbf{Y}^\beta + \mathbf{Y}^\alpha)(\mathbf{Y}^\beta - \mathbf{Y}^\alpha)^{-1}(\mathbf{Y}^\beta + \mathbf{Y}^\alpha)]^{-1} \\
&= [-2\mathbf{Y}^\beta(\mathbf{Y}^\beta - \mathbf{Y}^\alpha)^{-1}\mathbf{Y}^\alpha]^{-1} \\
&= \tfrac{1}{2}[(\mathbf{Y}^\beta)^{-1} - (\mathbf{Y}^\alpha)^{-1}]
\end{aligned} \tag{55}$$

following from (28) and (29), and insert them into (54),

$$\begin{aligned}
\mathbf{G}^{\Delta\mathbf{k}\Delta\mathbf{k}} &= \tfrac{1}{2}\mathbf{R}_+[(\mathbf{Y}^\alpha)^{-1} + (\mathbf{Y}^\beta)^{-1}]\mathbf{R}_+ + \tfrac{1}{2}\mathbf{R}_-[(\mathbf{Y}^\alpha)^{-1} + (\mathbf{Y}^\beta)^{-1}]\mathbf{R}_- \\
&+ \tfrac{1}{2}\mathbf{R}_+[(\mathbf{Y}^\beta)^{-1} - (\mathbf{Y}^\alpha)^{-1}]\mathbf{R}_- + \tfrac{1}{2}\mathbf{R}_-[(\mathbf{Y}^\beta)^{-1} - (\mathbf{Y}^\alpha)^{-1}]\mathbf{R}_+ + \mathbf{Y}_+ \quad .
\end{aligned} \tag{56}$$

We insert (28) and (29) into (56),

$$\begin{aligned}
\mathbf{G}^{\Delta\mathbf{k}\Delta\mathbf{k}} &= \tfrac{1}{4}\mathbf{R}^\alpha[(\mathbf{Y}^\alpha)^{-1} + (\mathbf{Y}^\beta)^{-1}]\mathbf{R}^\alpha + \tfrac{1}{4}\mathbf{R}^\beta[(\mathbf{Y}^\alpha)^{-1} + (\mathbf{Y}^\beta)^{-1}]\mathbf{R}^\beta \\
&+ \tfrac{1}{4}\mathbf{R}^\beta[(\mathbf{Y}^\beta)^{-1} - (\mathbf{Y}^\alpha)^{-1}]\mathbf{R}^\beta - \tfrac{1}{4}\mathbf{R}^\alpha[(\mathbf{Y}^\beta)^{-1} - (\mathbf{Y}^\alpha)^{-1}]\mathbf{R}^\alpha \\
&+ \tfrac{1}{2}(\mathbf{Y}^\alpha + \mathbf{Y}^\beta) \quad ,
\end{aligned} \tag{57}$$

which reads

$$\boxed{\mathbf{G}^{\Delta\mathbf{k}\Delta\mathbf{k}} = \tfrac{1}{2}[\mathbf{Y}^\alpha + \mathbf{R}^\alpha(\mathbf{Y}^\alpha)^{-1}\mathbf{R}^\alpha + \mathbf{Y}^\beta + \mathbf{R}^\beta(\mathbf{Y}^\beta)^{-1}\mathbf{R}^\beta] \quad . \qquad (58)}$$

If the square $(k_i^\alpha - k_i^\beta)^2$ of the difference of the i–th wavenumber–vector component of the Gabor functions g^α and g^β exceeds the i–th diagonal element of matrix

$$-4\ln(\varepsilon)\,\mathbf{G}^{\Delta\mathbf{k}\Delta\mathbf{k}} = -2\ln(\varepsilon)\,[\mathbf{Y}^\alpha + \mathbf{R}^\alpha(\mathbf{Y}^\alpha)^{-1}\mathbf{R}^\alpha + \mathbf{Y}^\beta + \mathbf{R}^\beta(\mathbf{Y}^\beta)^{-1}\mathbf{R}^\beta] \quad , \tag{59}$$

the scalar product satisfies inequality (18) and need not be calculated. The form of matrices (46) and (59) is very important. We calculate and store the diagonal elements of matrices $-2\ln(\varepsilon)\,(\mathbf{Y}^\alpha)^{-1}$ and $-2\ln(\varepsilon)\,[\mathbf{Y}^\alpha + \mathbf{R}^\alpha(\mathbf{Y}^\alpha)^{-1}\mathbf{R}^\alpha]$ for all Gabor functions g^α. The diagonal elements of matrices (46) and (59) are then the sums of the stored matrix elements and the decision that the scalar product need not be calculated is very fast, which considerably speeds up the calculation of the extensively sparse matrix of the scalar products.

Because matrix $(\mathbf{K}^\alpha)^{-1}$ plays the same role in the wavenumber domain as matrix $\mathbf{K}^\alpha$ in the spatial domain and because

$$\mathbf{Y}^\alpha + \mathbf{R}^\alpha(\mathbf{Y}^\alpha)^{-1}\mathbf{R}^\alpha = [\mathrm{Im}(\mathbf{K}^\alpha)^{-1}]^{-1} \quad , \tag{60}$$

the forms of matrices (46) and (59) are equivalent.

5. Scalar products of the real–valued structural Gabor functions

To work with the real–valued coefficients and functions instead of complex–valued ones, we may wish to replace the complex–valued structural Gabor functions (1) by real–valued structural Gabor functions $2\mathrm{Re}g^{\alpha}(\mathbf{x})$ and $2\mathrm{Im}g^{\alpha}(\mathbf{x})$ (Klimeš, 2008, sec. 2.2). In this case, we need to calculate the corresponding scalar products.

We express the scalar products of the complex–valued structural Gabor functions in terms of the real–valued structural Gabor functions,

$$(g^{\alpha},g^{\beta}) = (\mathrm{Re}g^{\alpha},\mathrm{Re}g^{\beta}) + (\mathrm{Im}g^{\alpha},\mathrm{Im}g^{\beta}) + \mathrm{i}(\mathrm{Re}g^{\alpha},\mathrm{Im}g^{\beta}) - \mathrm{i}(\mathrm{Im}g^{\alpha},\mathrm{Re}g^{\beta}) \quad , \tag{61}$$

$$(g^{\alpha *},g^{\beta}) = (\mathrm{Re}g^{\alpha},\mathrm{Re}g^{\beta}) - (\mathrm{Im}g^{\alpha},\mathrm{Im}g^{\beta}) + \mathrm{i}(\mathrm{Re}g^{\alpha},\mathrm{Im}g^{\beta}) + \mathrm{i}(\mathrm{Im}g^{\alpha},\mathrm{Re}g^{\beta}) \quad . \tag{62}$$

From equations (61) and (62), we obtain the conversion equations from the scalar products of the complex–valued structural Gabor functions to the scalar products of the real–valued structural Gabor functions,

$$(2\mathrm{Re}g^{\alpha},2\mathrm{Re}g^{\beta}) = 2[\mathrm{Re}(g^{\alpha},g^{\beta}) + \mathrm{Re}(g^{\alpha *},g^{\beta})] \quad , \tag{63}$$

$$(2\mathrm{Im}g^{\alpha},2\mathrm{Im}g^{\beta}) = 2[\mathrm{Re}(g^{\alpha},g^{\beta}) - \mathrm{Re}(g^{\alpha *},g^{\beta})] \quad , \tag{64}$$

$$(2\mathrm{Re}g^{\alpha},2\mathrm{Im}g^{\beta}) = 2[\mathrm{Im}(g^{\alpha},g^{\beta}) + \mathrm{Im}(g^{\alpha *},g^{\beta})] \quad , \tag{65}$$

$$(2\mathrm{Im}g^{\alpha},2\mathrm{Re}g^{\beta}) = 2[-\mathrm{Im}(g^{\alpha},g^{\beta}) + \mathrm{Im}(g^{\alpha *},g^{\beta})] \quad . \tag{66}$$

It is thus sufficient to calculate complex–valued scalars products (g^{α},g^{β}) and $(g^{\alpha *},g^{\beta})$ using the equations of Section 3, and convert them using equations (63)–(66). Gabor function $g^{\alpha *}(\mathbf{x})$ complex–conjugate to $g^{\alpha}(\mathbf{x})$ differs from Gabor function $g^{\alpha}(\mathbf{x})$ by the opposite signs of $\mathbf{k}^{\alpha}$ and $\mathrm{Im}\mathbf{K}^{\alpha}$ only.

The norms of the real–valued structural Gabor functions are

$$|2\mathrm{Re}g^{\alpha}| = \sqrt{2 + 2\mathrm{Re}(g^{\alpha *},g^{\alpha})} \tag{67}$$

and

$$|2\mathrm{Im}g^{\alpha}| = \sqrt{2 - 2\mathrm{Re}(g^{\alpha *},g^{\alpha})} \quad . \tag{68}$$

For $(g^{\alpha *},g^{\alpha})$, equation (14) yields

$$(g^{\alpha *},g^{\alpha}) = (a^{\alpha})^2(\pi)^{\frac{3}{2}}(\det \mathbf{K}^{\alpha})^{-\frac{1}{2}} \exp[-(\mathbf{k}^{\alpha})^{\mathrm{T}}(\mathbf{K}^{\alpha})^{-1}\mathbf{k}^{\alpha}] \quad . \tag{69}$$

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