Frame bounds and discretization error of the 1-D Gabor transform

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Summary

Discrete Gabor functions used for the Gabor transform form a frame. Approximate analytic expressions for the frame bounds of this frame are derived. These expressions can be used to study analytically both the discretization error of the continuous Gabor transform and the stability of the discrete Gabor transform in dependence on the phase–space lattice intervals.

Keywords

Gabor function, Gabor transform, frame bounds, discretization error.

1. Introduction

In this paper, we study the standard 1–D discrete Gabor transform with constant real–valued Gaussian envelopes of the Gabor functions and with the regular rectangular lattice of the central points of the Gabor functions in phase space.

We are especially interested in following two cases:
(a) Discretized continuous Gabor transform. In this case, we are interested in the dependence of the discretization error on the lattice intervals. The cells of the phase–space lattice should be considerably smaller than 2π in order to keep the discretization error small.
(b) Discrete Gabor transform. In this case, the cells of the phase–space lattice may be only slightly smaller than 2π, but the transform becomes unstable and the phase–space localization breaks down when the area of the phase–space lattice approaches 2π (Daubechies, 1992, sec. 3.4.4 B). We are interested in the dependence of the stability of the discrete Gabor transform on the lattice intervals.

The discrete Gabor functions used for the Gabor transform form a frame which we shall call the *Gabor transform frame.* Both the discretization error of the continuous Gabor transform and the stability of the discrete Gabor transform depend on the frame bounds of the Gabor transform frame.

In this paper, we derive approximate analytic expressions for the frame bounds of the Gabor transform frame in dependence on the phase–space lattice intervals by combining the general theory by Daubechies (1992, sec. 3.2) with equations by Klimeš (1986).

The Einstein summation over repetitive indices is used throughout the paper.

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2. Frames

A set of functions $\varphi_j(x)$ in a Hilbert space constitute a *frame* if there exist positive reals A and finite B so that, for all functions f in the Hilbert space,

$$A \|f\|^2 \leq \sum_j |(\varphi_j, f)|^2 \leq B \|f\|^2 \tag{1}$$

(Daubechies, 1992, eq. 3.2.1). The constants A and B are called *frame bounds.* If equality holds,

$$B = A \quad , \tag{2}$$

we call the frame a *tight frame.*

Since the functions in the frame do not need to be linearly independent and the frame may thus contain more functions than a basis, the decomposition

$$f(x) = c_j \varphi_j(x) \tag{3}$$

of a function into the frame may not be unique. We shall thus impose the condition that coefficients c_j minimize their norm

$$\sqrt{c_j c_j} \quad . \tag{4}$$

Coefficients c_j can numerically be determined by simultaneously solving the ill–conditioned system

$$(\varphi_i, \varphi_j)\, c_j = (\varphi_i, f) \tag{5}$$

of equations and minimizing (4).

For each frame $\varphi_j(x)$ there is a unique *dual frame* $\widetilde{\varphi}_j(x)$ in the Hilbert space, with frame bounds

$$\widetilde{A} = B^{-1} \quad , \qquad \widetilde{B} = A^{-1} \tag{6}$$

(Daubechies, 1992, eq. 3.2.6). The dual frame to $\widetilde{\varphi}_j(x)$ is again $\varphi_j(x)$. If frame $\varphi_j(x)$ is tight, then the dual frame is a multiple of $\varphi_j(x)$,

$$\widetilde{\varphi}_j(x) = A^{-1}\, \varphi_j(x) \quad . \tag{7}$$

Otherwise, dual frame $\widetilde{\varphi}_j(x)$ differs from $\varphi_j(x)$ and it is often difficult to determine it (e.g., Daubechies, 1992, fig. 3.6). Coefficients c_j minimizing (4) can be obtained as scalar products

$$c_j = (\widetilde{\varphi}_j, f) \tag{8}$$

(Daubechies, 1992, eq. 3.2.8). If frame $\varphi_j(x)$ is tight, then by inserting (7) into (8) we get

$$c_j = A^{-1}(\varphi_j, f) \quad . \tag{9}$$

If frame bounds A and B are close to each other, we may use approximation

$$c_j \approx \tfrac{2}{A+B}\, (\varphi_j, f) \tag{10}$$

(Daubechies, 1992, eq. 3.2.11). The relative error of approximation (10) satisfies inequality

$$\frac{\|f - c_j \varphi_j\|}{\|f\|} \leq \frac{B - A}{B + A} \tag{11}$$

(Daubechies, 1992), where A and B are the frame bounds.

2.1. Nearly tight frames

Nearly tight frames are frames with frame bounds A and B so close to each other that relative error (11) is acceptable. These frames often represent discretized versions of integral transforms. In this case, the coefficients c_j of decomposition (3) are approximately obtained using equation (10) or using equation

$$c_j \approx \tfrac{1}{C}(\varphi_j, f) \quad , \tag{12}$$

where C is obtained by discretizing the integral transform. For the discretized continuous Gabor transform,

$$C = \frac{2\pi}{\Delta x\,\Delta k} \quad . \tag{13}$$

Frames corresponding to discretized versions of integral transforms are usually considerably oversampled. The "redundancy ratio" is approximately determined by $\frac{A+B}{2}$ or by C. In this case, we are interested especially in the *relative root–mean–square discretization error*. For approximate decomposition (10), this error is determined by (11). For approximate decomposition (12), this error satisfies inequality

$$\frac{\| f - c_j \varphi_j \|}{\| f \|} \leq \max(\tfrac{C-A}{C}, \tfrac{B-C}{C}) \tag{14}$$

(Daubechies, 1992). For $C = \frac{A+B}{2}$, inequality (14) reduces to (11). The discretization error (14) of the Gabor transform will be estimated in Section 3.4.

2.2. Loose frames

We shall refer to frames with considerably different frame bounds A and B as the *loose frames*. These frames often have their "redundancy ratio" only slightly greater than one. In this case, the coefficients c_j of decomposition (3) are usually determined numerically. Then there is no discretization error, and we are interested in *relative numerical instability of the decomposition.*

For φ_j forming an orthonormal basis,

$$c_j c_j = \| f \|^2 \quad . \tag{15}$$

For frame φ_j, normalized so as $\|\varphi_j\| = 1$, it follows from (1), (6) and (8) that

$$c_j c_j \leq A^{-1} \| f \|^2 \quad . \tag{16}$$

We may thus measure the relative numerical instability of the decomposition in terms of A^{-1}. In this case, we are interested in the lower frame bound A which should be reasonably greater than zero. We estimate the lower frame bound A of the Gabor transform on a coarse lattice in Section 3.5.

3. Gabor transform frame

We consider constant envelope

$$g(x) = \pi^{-\frac{1}{4}} \exp(-\tfrac{x^2}{2}) \tag{17}$$

of the Gabor functions. The shifted envelopes are

$$g_n(x) = g(x - n\Delta x) \quad . \tag{18}$$

The individual Gabor functions forming the frame are

$$g_{mn}(x) = \exp(\mathrm{i}m\Delta kx)\, g_n(x) \quad . \tag{19}$$

These functions represent a special case of general frame functions $\varphi_j(x)$ from the previous section.

The sum of the squares of the scalar products of arbitrary function $f(x)$ with the frame functions is

$$\sum_{m,n} |(g_{mn}, f)|^2 = \sum_{m,n} \int \mathrm{d}x_1 \int \mathrm{d}x_2\, f^*(x_1)\, g_{mn}(x_1)\, g^*_{mn}(x_2)\, f(x_2) \quad . \tag{20}$$

Inserting (19) into (20),

$$\sum_{m,n} |(g_{mn}, f)|^2 = \sum_{m,n} \int \mathrm{d}x_1 \int \mathrm{d}x_2\, f^*(x_1) f(x_2)\, \exp[(\mathrm{i}m\Delta k(x_1 - x_2)]\, g_n(x_1) g^*_n(x_2) \quad . \tag{21}$$

We insert identity

$$\sum_m \exp(\mathrm{i}m\Delta kx) = \frac{2\pi}{\Delta k} \sum_m \delta(x + m\tfrac{2\pi}{\Delta k}) \quad , \tag{22}$$

where $\delta(x)$ is the Dirac distribution, into (21),

$$\sum_{m,n} |(g_{mn}, f)|^2 = \frac{2\pi}{\Delta k} \sum_{m,n} \int \mathrm{d}x_1 \int \mathrm{d}x_2\, f^*(x_1) f(x_2)\, \delta(x_1 - x_2 + m\tfrac{2\pi}{\Delta k})\, g_n(x_1) g^*_n(x_2) \quad . \tag{23}$$

We integrate (23) with respect to x_2,

$$\sum_{m,n} |(g_{mn}, f)|^2 = \frac{2\pi}{\Delta k} \sum_{m,n} \int \mathrm{d}x_1\, f^*(x_1) f(x_1 + m\tfrac{2\pi}{\Delta k})\, g_n(x_1) g^*_n(x_1 + m\tfrac{2\pi}{\Delta k}) \quad , \tag{24}$$

and shift integration variable x_1,

$$\sum_{m,n} |(g_{mn}, f)|^2 = \frac{2\pi}{\Delta k} \sum_{m,n} \int \mathrm{d}x\, f^*(x - m\tfrac{\pi}{\Delta k}) f(x + m\tfrac{\pi}{\Delta k})\, g_n(x - m\tfrac{\pi}{\Delta k}) g^*_n(x + m\tfrac{\pi}{\Delta k}) \quad . \tag{25}$$

We apply identity

$$g_n(x - m\tfrac{\pi}{\Delta k}) g^*_n(x + m\tfrac{\pi}{\Delta k}) = \pi^{-\frac{1}{2}} \exp[-(x - n\Delta x)^2]\, \exp[-(\tfrac{m\pi}{\Delta k})^2] \tag{26}$$

following from (17) with (18), and identity

$$\sum_n \exp[-(x - n\Delta x)^2] = \frac{\sqrt{\pi}}{\Delta x} \sum_n \cos(n\tfrac{2\pi}{\Delta x}x)\, \exp[-(\tfrac{n\pi}{\Delta x})^2] \tag{27}$$

(Klimeš, 1986, eq. 40). Inserting (26) with (27) into (25), we obtain

$$\sum_{m,n} |(g_{mn}, f)|^2 = \frac{2\pi}{\Delta x \Delta k} \sum_{m,n} B_{mn}\, \exp[-(\tfrac{n\pi}{\Delta x})^2]\, \exp[-(\tfrac{m\pi}{\Delta k})^2] \quad , \tag{28}$$

where

$$B_{mn} = \int \mathrm{d}x\, f^*(x - m\tfrac{\pi}{\Delta k}) f(x + m\tfrac{\pi}{\Delta k})\, \cos(n\tfrac{2\pi}{\Delta x}x) \quad . \tag{29}$$

Since

$$B_{-mn} = (B_{mn})^* \quad , \tag{30}$$

the imaginary part of B_{mn} vanishes in the summation and we may express

$$\sum_{m,n} |(g_{mn}, f)|^2 = \frac{2\pi}{\Delta x \Delta k} \sum_{m,n} C_{mn}\, \exp[-(\tfrac{n\pi}{\Delta x})^2]\, \exp[-(\tfrac{m\pi}{\Delta k})^2] \quad , \tag{31}$$

where

$$C_{mn} = \mathrm{Re}(B_{mn}) \quad . \tag{32}$$

Inserting (29) into (32), we arrive at

$$C_{mn} = \int \mathrm{d}x\, [\mathrm{Re}f(x - m\tfrac{\pi}{\Delta k})\mathrm{Re}f(x + m\tfrac{\pi}{\Delta k}) + \mathrm{Im}f(x - m\tfrac{\pi}{\Delta k})\mathrm{Im}f(x + m\tfrac{\pi}{\Delta k})]\, \cos(n\tfrac{2\pi}{\Delta x}x) \; . \tag{33}$$

Since

$$C_{00} = \|f\|^2 \quad , \tag{34}$$

we may express (31) in form

$$\sum_{m,n} |(g_{mn}, f)|^2 = \frac{2\pi}{\Delta x \Delta k} \left[\|f\|^2 + \sum_{m^2+n^2>0} C_{mn}\, \exp[-(\tfrac{n\pi}{\Delta x})^2]\, \exp[-(\tfrac{m\pi}{\Delta k})^2] \right] \quad . \tag{35}$$

3.2. Proved conservative frame bounds

To express the cosine in (33) in terms of the complex–valued exponential function with unit amplitude,

$$C_{mn} = \mathrm{Re}\left[\int \mathrm{d}x\, [\mathrm{Re}f(x - m\tfrac{\pi}{\Delta k})\mathrm{Re}f(x + m\tfrac{\pi}{\Delta k}) + \mathrm{Im}f(x - m\tfrac{\pi}{\Delta k})\mathrm{Im}f(x + m\tfrac{\pi}{\Delta k})]\, \exp(\mathrm{i}n\tfrac{2\pi}{\Delta x}x) \right] . \tag{36}$$

We introduce functions $f^{\mathrm{R}+}(x)$ and $f^{\mathrm{R}-}(x)$,

$$f^{\mathrm{R}\pm}(x) = \mathrm{Re}f(x \pm m\tfrac{\pi}{\Delta k})\, \exp(\pm\mathrm{i}n\tfrac{\pi}{\Delta x}x) \quad , \tag{37}$$

and functions $f^{\mathrm{I}+}(x)$ and $f^{\mathrm{I}-}(x)$,

$$f^{\mathrm{I}\pm}(x) = \mathrm{Im}f(x \pm m\tfrac{\pi}{\Delta k})\, \exp(\pm\mathrm{i}n\tfrac{\pi}{\Delta x}x) \quad . \tag{38}$$

Equation (36) then reads

$$C_{mn} = \mathrm{Re}[(f^{\mathrm{R}-}, f^{\mathrm{R}+}) + (f^{\mathrm{I}-}, f^{\mathrm{I}+})] \quad . \tag{39}$$

Since

$$|\mathrm{Re}(f^{\mathrm{R}-}, f^{\mathrm{R}+})| \le |(f^{\mathrm{R}-}, f^{\mathrm{R}+})| \quad , \qquad |\mathrm{Re}(f^{\mathrm{I}-}, f^{\mathrm{I}+})| \le |(f^{\mathrm{I}-}, f^{\mathrm{I}+})| \quad , \tag{40}$$

equation (39) yields inequality

$$|C_{mn}| \le |(f^{\mathrm{R}-}, f^{\mathrm{R}+})| + |(f^{\mathrm{I}-}, f^{\mathrm{I}+})| \quad . \tag{41}$$

Since

$$|(f^{\mathrm{R}-}, f^{\mathrm{R}+})| \le \|f^{\mathrm{R}-}\|\|f^{\mathrm{R}+}\| \quad , \qquad |(f^{\mathrm{I}-}, f^{\mathrm{I}+})| \le \|f^{\mathrm{I}-}\|\|f^{\mathrm{I}+}\| \quad , \tag{42}$$

inequality (41) yields inequality

$$|C_{mn}| \leq \|f^{\mathrm{R}-}\|\|f^{\mathrm{R}+}\| + \|f^{\mathrm{I}-}\|\|f^{\mathrm{I}+}\| \quad . \tag{43}$$

Since

$$\|f^{\mathrm{R}-}\|\|f^{\mathrm{R}+}\| + \|f^{\mathrm{I}-}\|\|f^{\mathrm{I}+}\| = \|\mathrm{Re}f\|^2 + \|\mathrm{Im}f\|^2 = \|f\|^2 \quad , \tag{44}$$

inequality (43) reads

$$|C_{mn}| \leq \|f\|^2 \quad . \tag{45}$$

Then for any m, n,

$$-\|f\|^2 \leq C_{mn} \leq \|f\|^2 \quad . \tag{46}$$

Inserting (46) for $m^2 + n^2 > 0$ into (35), we obtain inequality

$$A\,\|f\|^2 \leq \sum_{m,n} |(g_{mn}, f)|^2 \leq B\,\|f\|^2 \quad , \tag{47}$$

with frame bounds

$$A = \frac{2\pi}{\Delta x \Delta k}\,(1 - E) \tag{48}$$

and

$$B = \frac{2\pi}{\Delta x \Delta k}\,(1 + E) \quad , \tag{49}$$

where

$$E = \sum_{m^2+n^2>0} \exp[-(\tfrac{n\pi}{\Delta x})^2]\, \exp[-(\tfrac{m\pi}{\Delta k})^2] \quad . \tag{50}$$

We may express double sum (50) in form

$$E = E_x + E_k + E_x E_k \quad , \tag{51}$$

where the single sums read

$$E_x = 2\sum_{n=1}^{\infty} \exp[-(\tfrac{n\pi}{\Delta x})^2] \tag{52}$$

and

$$E_k = 2\sum_{m=1}^{\infty} \exp[-(\tfrac{m\pi}{\Delta k})^2] \quad . \tag{53}$$

The proved upper bound on relative error (11) is then

$$\frac{B - A}{B + A} = E \quad . \tag{54}$$

3.3. Approximate frame bounds

We see in (31) that the most important coefficients for the determination of frame bounds are C_{10} and C_{01}. Distribution $f(x)$, such that simultaneously $C_{10} = \|f\|^2$ and $C_{01} = \|f\|^2$, exists only if $\frac{2\pi}{\Delta x \Delta k}$ is integer. Analogously, distribution $f(x)$, such that simultaneously $C_{10} = -\|f\|^2$ and $C_{01} = -\|f\|^2$, exists only if $\frac{2\pi}{\Delta x \Delta k}$ is integer.

For integer $\frac{2\pi}{\Delta x \Delta k}$, there exists $f(x)$ such that

$$C_{mn} = \|f\|^2 \tag{55}$$

for all m, n, and (49) is the exact upper frame bound. In other cases, there is no function $f(x)$ satisfying (55) for all m, n, and the exact upper frame bound is smaller than (49), but *I believe* that (49) is a good approximation to the exact upper frame bound even in these cases.

If $C_{10} = -\|f\|^2$ for some $f(x)$, then $C_{m0} = (-1)^m \|f\|^2$ for this $f(x)$. If $C_{01} = -\|f\|^2$ for some $f(x)$, then $C_{0n} = (-1)^n \|f\|^2$ for this $f(x)$. There is thus no function $f(x)$ such that $C_{mn} = -\|f\|^2$ for all m, n.

For integer $\frac{2\pi}{\Delta x \Delta k}$, there exists function $f(x)$ for which simultaneously $C_{10} = -\|f\|^2$ and $C_{01} = -\|f\|^2$. For this function,

$$C_{mn} = \begin{cases} \|f\|^2 & \text{for both } m \text{ and } n \text{ even,} \\ -\|f\|^2 & \text{otherwise,} \end{cases} \tag{56}$$

if $\frac{2\pi}{\Delta x \Delta k}$ is odd, and

$$C_{mn} = (-1)^{m+n} \|f\|^2 \tag{57}$$

if $\frac{2\pi}{\Delta x \Delta k}$ is even.

To distinguish between odd m and even m in (31), we decompose sum (52) as

$$E_x = E_{x1} + E_{x2} \quad , \tag{58}$$

where

$$E_{x1} = 2 \sum_{n=1}^{\infty} \exp[-(\tfrac{(2n-1)\pi}{\Delta x})^2] \tag{59}$$

and

$$E_{x2} = 2 \sum_{n=1}^{\infty} \exp[-(\tfrac{2n\pi}{\Delta x})^2] \quad . \tag{60}$$

To distinguish between odd n and even n in (31), we analogously decompose sum (53) as

$$E_k = E_{k1} + E_{k2} \quad , \tag{61}$$

where

$$E_{k1} = 2 \sum_{n=1}^{\infty} \exp[-(\tfrac{(2n-1)\pi}{\Delta k})^2] \tag{62}$$

and

$$E_{k2} = 2 \sum_{n=1}^{\infty} \exp[-(\tfrac{2n\pi}{\Delta k})^2] \quad . \tag{63}$$

We now insert (56) and (57) into (35) to obtain the estimates of the lower frame bound for integer $\frac{2\pi}{\Delta x \Delta k}$.

For odd $\frac{2\pi}{\Delta x \Delta k}$,

$$\sum_{m,n} |(g_{mn}, f)|^2 = A^{\rm odd} \, \|f\|^2 \quad , \tag{64}$$

where

$$A^{\rm odd} = \frac{2\pi}{\Delta x \Delta k} \left(1 - E^{\rm odd}\right) \quad , \tag{65}$$

with

$$E^{\rm odd} = E_{x1} - E_{x2} + E_{k1} - E_{k2} + E_{x1}E_{k1} + E_{x1}E_{k2} + E_{x2}E_{k1} - E_{x2}E_{k2} \quad . \tag{66}$$

This equation may also be expressed in form

$$E^{\rm odd} = E_{x1} - E_{x2} + E_{k1} - E_{k2} + [E_{x1} + E_{x2}][E_{k1} + E_{k2}] - 2E_{x2}E_{k2} \quad . \tag{67}$$

The estimate (65) of the lower frame bound is greater than the too small conservative estimate (48).

For even $\frac{2\pi}{\Delta x \Delta k}$,

$$\sum_{m,n} |(g_{mn}, f)|^2 = A^{\rm even} \, \|f\|^2 \quad , \tag{68}$$

where

$$A^{\rm even} = \frac{2\pi}{\Delta x \Delta k} \left(1 - E^{\rm even}\right) \tag{69}$$

with

$$E^{\rm even} = E_{x1} - E_{x2} + E_{k1} - E_{k2} - E_{x1}E_{k1} + E_{x1}E_{k2} + E_{x2}E_{k1} - E_{x2}E_{k2} \quad . \tag{70}$$

This equation may also be expressed in form

$$E^{\rm even} = E_{x1} - E_{x2} + E_{k1} - E_{k2} - [E_{x1} - E_{x2}][E_{k1} - E_{k2}] \quad . \tag{71}$$

The estimate (69) of the lower frame bound is greater than estimate (65), and is thus obviously incorrect for odd $\frac{2\pi}{\Delta x \Delta k}$.

I hope that (65) is a good approximation to the exact lower frame bound. For the limiting case of $\Delta x \Delta k = 2\pi$, approximation (65) can be calculated analytically and gives $A^{\rm odd} = 0$, which is equal to the exact lower bound.

Refer to Tables 1a–1e for the values of frame bounds (48), (65), (69) and (49) at different steps Δx and Δk.

If the relative error (11) of approximation (10) is small, the differences between estimates (48), (65) and (69) are negligible. In this case, the frame bounds can be approximated by

$$A^0 = \tfrac{2\pi}{\Delta x \Delta k} \left(1 - E^0\right) \quad , \qquad B^0 = \tfrac{2\pi}{\Delta x \Delta k} \left(1 + E^0\right) \quad , \tag{72}$$

where

$$E^0 = 2\exp[-(\tfrac{\pi}{\Delta x})^2] + 2\exp[-(\tfrac{\pi}{\Delta k})^2] \tag{73}$$

is the approximation of relative error (11).

Table 1a. Estimates of frame bounds for $\Delta x \Delta k = 0.5\,\pi$.

Δx	A	A^{odd}	A^{even}	B
0.500	0.909	1.203	$1.2\underline{03}$	7.091
1.000	3.853	3.853	$3.85\underline{3}$	4.147
1.500	3.899	3.899	3.899	4.101
2.000	3.321	3.322	3.322	4.679
2.500	2.336	2.365	2.365	5.664
3.000	1.228	1.427	1.427	6.772

Table 1b. Estimates of frame bounds for $\Delta x \Delta k = 0.75\,\pi$.

Δx	A	A^{odd}	A^{even}	B
1.000	1.761	1.769	1.769	3.573
1.500	2.500	2.500	2.505	2.833
2.000	2.209	2.210	2.211	3.124
2.500	1.557	1.577	1.577	3.776
3.000	0.819	0.951	0.951	4.515

Table 1c. Estimates of frame bounds for $\Delta x \Delta k = 1.0\,\pi$.

Δx	A	A^{odd}	A^{even}	B
1.000	0.454	0.601	0.601	3.546
1.500	1.518	1.519	1.540	2.482
2.000	1.575	1.575	1.600	2.425
2.500	1.157	1.172	1.178	2.843
3.000	0.613	0.713	0.713	3.387

Table 1d. Estimates of frame bounds for $\Delta x \Delta k = 1.5\,\pi$.

Δx	A	A^{odd}	A^{even}	B
1.000	-0.879	0.027	0.028	3.545
1.500	0.244	0.342	0.391	2.422
2.000	0.577	0.582	0.735	2.089
2.500	0.544	0.554	0.690	2.123
3.000	0.327	0.393	0.458	2.340
3.500	0.011	0.224	0.244	2.656
4.000	-0.347	0.105	0.110	3.014

Table 1e. Estimates of frame bounds for $\Delta x \Delta k = 1.9\,\pi$.

Δx	A	A^{odd}	A^{even}	B
1.500	-0.317	0.031	0.088	$2.4\underline{22}$
2.000	0.031	0.082	0.318	2.074
2.500	0.084	$0.09\underline{6}$	0.403	2.021
3.000	0.028	0.081	0.314	2.077
3.500	-0.113	0.055	0.181	2.218
4.000	-0.327	0.031	0.085	2.432

Tables 1a–1d. Values of the frame bounds for different steps Δx and Δk. Proved estimate A of the lower frame bound given by equation (48) is too conservative. Estimate A^{odd} of the lower frame bound given by equation (65) coincides with frame bound A in Table 3.3 by Daubechies (1992) and seems to represent a good approximation of the lower frame bound. Estimate A^{even} of the lower frame bound given by equation (69) coincides with frame bound A_{exact} in Table 3.3 by Daubechies (1992) and is applicable to even values of $2\pi/(\Delta x \Delta k)$ only. Estimate B of the upper frame bound given by equation (49) coincides with frame bounds B and B_{exact} in Table 3.3 by Daubechies (1992) and represents a good approximation of the upper frame bound. The digits differing from Table 3.3 by Daubechies (1992) are underlined.

3.4. Discretized continuous Gabor transform

For discretized continuous Gabor transform

$$f(x) \approx c_{mn}\, g_{mn}(x) \quad , \tag{74}$$

we have

$$c_{mn} = \tfrac{\Delta x \Delta k}{2\pi} (g_{mn}, f) \tag{75}$$

instead of (10). Relative error (14) is then

$$\tfrac{\Delta x \Delta k}{2\pi} B - 1 = E \quad , \tag{76}$$

no matter whether we consider estimate (48), (65) or (69) of the lower frame bound. If this relative error is small, it may be approximated by (73).

3.5. Coarse lattices

For fine lattices, $\Delta x \le \sqrt{\pi}$ and $\Delta k \le \sqrt{\pi}$, sums (59), (60), (62) and (63) in (66) converge rapidly.

For $\Delta x > \sqrt{\pi}$, we express

$$E_{x1} = E_x - E_{x2} \quad , \tag{77}$$

and apply identity (27) with $x = 0$ to sum (52),

$$1 + E_x = \frac{\Delta x}{\sqrt{\pi}} \left[1 + 2 \sum_{n=1}^{\infty} \exp[-(n\Delta x)^2] \right] \quad . \tag{78}$$

Analogously for $\Delta k > \sqrt{\pi}$,

$$E_{k1} = E_k - E_{k2} \tag{79}$$

and

$$1 + E_k = \frac{\Delta k}{\sqrt{\pi}} \left[1 + 2 \sum_{m=1}^{\infty} \exp[-(m\Delta k)^2] \right] \quad . \tag{80}$$

For $\Delta x > 2\sqrt{\pi}$, we apply identity (27) with $x = 0$ also to sum (60),

$$1 + E_{x2} = \frac{\Delta x}{2\sqrt{\pi}} \left[1 + 2 \sum_{n=1}^{\infty} \exp[-(\tfrac{n\Delta x}{2})^2] \right] \quad . \tag{81}$$

For $\Delta k > 2\sqrt{\pi}$, we apply identity (27) with $x = 0$ also to sum (63),

$$1 + E_{k2} = \frac{\Delta k}{2\sqrt{\pi}} \left[1 + 2 \sum_{m=1}^{\infty} \exp[-(\tfrac{m\Delta k}{2})^2] \right] \quad . \tag{82}$$

We express (66) in form

$$1 - E^{\text{odd}} = 2\,(1 + E_{x2})\,(1 + E_{k2}) - (1 + E_x)\,(1 + E_k) \quad . \tag{83}$$

For $\Delta x > \sqrt{\pi}$ and $\Delta k > \sqrt{\pi}$, we insert (83) with (78) and (80) into (65),

$$\begin{aligned} A^{\text{odd}} &= \tfrac{2\pi}{\Delta x \Delta k}\, 2\,(1 + E_{x2})\,(1 + E_{k2}) \\ &- 2 \left[1 + 2 \sum_{n=1}^{\infty} \exp[-(n\Delta x)^2] \right] \left[1 + 2 \sum_{m=1}^{\infty} \exp[-(m\Delta k)^2] \right] \quad . \end{aligned} \tag{84}$$

Neglecting the terms smaller than $4\exp(-4\pi)$ and leaving only the terms greater than $4\exp(-4\pi)$,

$$A^{\rm odd} \approx \tfrac{2\pi}{\Delta x\Delta k}\,2\big[1+2\exp[-(\tfrac{2\pi}{\Delta x})^2]\big]\,\big[1+2\exp[-(\tfrac{2\pi}{\Delta k})^2]\big] \\ -\,2\big[1+2\exp[-(\Delta x)^2]\big]\,\big[1+2\exp[-(\Delta k)^2]\big] \quad . \tag{85}$$

Neglecting also the terms smaller than $8\exp(-2\Delta x\Delta k)$,

$$A^{\rm odd} \approx 2\big[\tfrac{2\pi}{\Delta x\Delta k}-1\big] + \tfrac{2\pi}{\Delta x\Delta k}\,4\exp[-(\tfrac{2\pi}{\Delta x})^2] - 4\exp[-(\Delta k)^2] \\ +\,\tfrac{2\pi}{\Delta x\Delta k}\,4\exp[-(\tfrac{2\pi}{\Delta k})^2] - 4\exp[-(\Delta x)^2] \quad . \tag{86}$$

We see that lower frame bound $A^{\rm odd}$ has its maximum at fixed $\Delta x\Delta k$ for $\Delta x=\Delta k$.

For $\sqrt{\pi}<\Delta x\le 2\sqrt{\pi}$ and $\Delta k\le\sqrt{\pi}$, it is slightly better to insert (83) with (78) only into (65). For $\Delta x\le\sqrt{\pi}$ and $\sqrt{\pi}<\Delta k\le 2\sqrt{\pi}$, it is slightly better to insert (83) with (80) only into (65). However, for coarse lattices, the differences in convergence from (84) are so small as we may extend applicability of (84), (85) and (86) to $\Delta x\le 2\sqrt{\pi}$ and $\Delta k\le 2\sqrt{\pi}$.

Refer to Tables 2a–2d for comparison of lower frame bound $A^{\rm odd}$ with its approximations (85) and (86).

For $\Delta x>2\sqrt{\pi}$, we insert (83) with (78) and (81) into (65),

$$A^{\rm odd} = \frac{\sqrt{\pi}}{\Delta k}2\Bigg\{\Big[1+E_{k2}\Big]\Big[1+2\sum_{n=1}^{\infty}\exp[-(\tfrac{n\Delta x}{2})^2]\Big] - \Big[1+E_k\Big]\Big[1+2\sum_{n=1}^{\infty}\exp[-(n\Delta x)^2]\Big]\Bigg\} \; . \tag{87}$$

Neglecting the terms smaller than $\frac{\sqrt{\pi}}{\Delta k}4\exp(-4\pi)$,

$$A^{\rm odd} \approx \tfrac{\sqrt{\pi}}{\Delta k}\,4\big\{\exp[-(\tfrac{\Delta x}{2})^2] - \exp[-(\tfrac{\pi}{\Delta k})^2]\big\} \quad . \tag{88}$$

Refer to Tables 3a–3d for comparison of lower frame bound $A^{\rm odd}$ with its approximation (88).

For $\Delta k>2\sqrt{\pi}$, we insert (83) with (80) and (82) into (65),

$$A^{\rm odd} = \frac{\sqrt{\pi}}{\Delta x}2\Bigg\{\Big[1+E_{x2}\Big]\Big[1+2\sum_{n=1}^{\infty}\exp[-(\tfrac{n\Delta k}{2})^2]\Big] - \Big[1+E_x\Big]\Big[1+2\sum_{n=1}^{\infty}\exp[-(n\Delta k)^2]\Big]\Bigg\} \; . \tag{89}$$

Neglecting terms smaller than $\frac{\sqrt{\pi}}{\Delta x}4\exp(-4\pi)$,

$$A^{\rm odd} \approx \tfrac{\sqrt{\pi}}{\Delta x}\,4\big\{\exp[-(\tfrac{\Delta k}{2})^2] - \exp[-(\tfrac{\pi}{\Delta x})^2]\big\} \quad . \tag{90}$$

Refer to Tables 4a–4d for comparison of lower frame bound $A^{\rm odd}$ with its approximation (90).

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Table 2a. Estimates of frame bounds for $\Delta x \Delta k = 1.80\,\pi$.

Δx	A^{odd}	$A^{\mathrm{odd}}_{(85)}$	$A^{\mathrm{odd}}_{(86)}$
1.500	0.076535	0.076962	0.076963
2.000	0.179647	0.179647	0.179693
2.500	0.200433	0.200433	0.200518
3.000	0.162526	0.162529	0.162555
3.500	0.105173	0.105278	0.105281
4.000	0.055911	0.057030	0.057030

Table 2b. Estimates of frame bounds for $\Delta x \Delta k = 1.85\,\pi$.

Δx	A^{odd}	$A^{\mathrm{odd}}_{(85)}$	$A^{\mathrm{odd}}_{(86)}$
1.500	0.051985	0.052361	0.052362
2.000	0.128564	0.128565	0.128592
2.500	0.147116	0.147116	0.147175
3.000	0.121806	0.121807	0.121827
3.500	0.080583	0.080637	0.080639
4.000	0.043865	0.044501	0.044501

Table 2c. Estimates of frame bounds for $\Delta x \Delta k = 1.90\,\pi$.

Δx	A^{odd}	$A^{\mathrm{odd}}_{(85)}$	$A^{\mathrm{odd}}_{(86)}$
1.500	0.031391	0.031688	0.031689
2.000	0.081721	0.081722	0.081736
2.500	0.095873	0.095873	0.095909
3.000	0.081005	0.081006	0.081020
3.500	0.054769	0.054794	0.054795
4.000	0.030533	0.030856	0.030857

Table 2d. Estimates of frame bounds for $\Delta x \Delta k = 1.95\,\pi$.

Δx	A^{odd}	$A^{\mathrm{odd}}_{(85)}$	$A^{\mathrm{odd}}_{(86)}$
1.500	0.014224	0.014400	0.014400
2.000	0.038933	0.038934	0.038940
2.500	0.046808	0.046808	0.046826
3.000	0.040337	0.040337	0.040345
3.500	0.027860	0.027868	0.027869
4.000	0.015906	0.016030	0.016030

Tables 2a–2d. Approximate lower frame bounds for different steps Δx and Δk on coarse lattices. Estimate A^{odd} of the lower frame bound is given by equation (65) which represents slowly converging series on coarse lattices. The same estimate A^{odd} can efficiently be calculated using rapidly converging series (84). Estimate $A^{\mathrm{odd}}_{(85)}$ of the lower frame bound represents approximation (85) of A^{odd}. Estimate $A^{\mathrm{odd}}_{(86)}$ of the lower frame bound represents approximation (86) of $A^{\mathrm{odd}}_{(85)}$. Recommended regions of applicability of approximations $A^{\mathrm{odd}}_{(85)}$ and $A^{\mathrm{odd}}_{(86)}$ are $1.595 < \Delta x < 3.545$ for Table 2a, $1.639 < \Delta x < 3.545$ for Table 2b, $1.683 < \Delta x < 3.545$ for Table 2c and $1.728 < \Delta x < 3.545$ for Table 2d.

Table 3a. Estimates of frame bounds for $\Delta x \Delta k = 1.80\,\pi$.

Δx	$A^{\rm odd}$	$A^{\rm odd}_{(88)}$
3.000	0.162526	0.162572
3.500	0.105173	0.105173
4.000	0.055911	0.055911
5.000	0.009308	0.009308
6.000	0.000816	0.000816
7.000	0.000040	0.000040

Table 3b. Estimates of frame bounds for $\Delta x \Delta k = 1.85\,\pi$.

Δx	$A^{\rm odd}$	$A^{\rm odd}_{(88)}$
3.000	0.121806	0.121850
3.500	0.080583	0.080584
4.000	0.043865	0.043865
5.000	0.007673	0.007673
6.000	0.000705	0.000705
7.000	0.000036	0.000036

Table 3c. Estimates of frame bounds for $\Delta x \Delta k = 1.90\,\pi$.

Δx	$A^{\rm odd}$	$A^{\rm odd}_{(88)}$
3.000	0.081005	0.081043
3.500	0.054769	0.054770
4.000	0.030533	0.030533
5.000	0.005629	0.005629
6.000	0.000547	0.000547
7.000	0.000029	0.000029

Table 3d. Estimates of frame bounds for $\Delta x \Delta k = 1.95\,\pi$.

Δx	$A^{\rm odd}$	$A^{\rm odd}_{(88)}$
3.000	0.040337	0.040361
3.500	0.027860	0.027860
4.000	0.015906	0.015906
5.000	0.003097	0.003097
6.000	0.000320	0.000320
7.000	0.000018	0.000018

Tables 3a–3d. Approximate lower frame bounds for different steps Δx and Δk on coarse lattices. Estimate $A^{\rm odd}$ of the lower frame bound is given by equation (65) which represents slowly converging series on coarse lattices. The same estimate $A^{\rm odd}$ can efficiently be calculated using rapidly converging series (87). Estimate $A^{\rm odd}_{(88)}$ of the lower frame bound represents approximation (88) of $A^{\rm odd}$. Recommended region of applicability of approximation $A^{\rm odd}_{(88)}$ is $\Delta x > 3.544$.

Table 4a. Estimates of frame bounds for $\Delta x \Delta k = 1.80\,\pi$.

Δx	$A^{\rm odd}$	$A^{\rm odd}_{(90)}$
0.750	0.000006	0.000006
1.000	0.002025	0.002025
1.250	0.023773	0.023773
1.500	0.076535	0.076535
1.750	0.136352	0.136360
2.000	0.179647	0.179800

Table 4b. Estimates of frame bounds for $\Delta x \Delta k = 1.85\,\pi$.

Δx	$A^{\rm odd}$	$A^{\rm odd}_{(90)}$
0.750	0.000002	0.000003
1.000	0.001158	0.001158
1.250	0.015256	0.015256
1.500	0.051985	0.051985
1.750	0.095637	0.095641
2.000	0.128564	0.128649

Table 4c. Estimates of frame bounds for $\Delta x \Delta k = 1.90\,\pi$.

Δx	$A^{\rm odd}$	$A^{\rm odd}_{(90)}$
0.750	0.000001	0.000001
1.000	0.000593	0.000593
1.250	0.008720	0.008721
1.500	0.031391	0.031391
1.750	0.059596	0.059598
2.000	0.081721	0.081763

Table 4d. Estimates of frame bounds for $\Delta x \Delta k = 1.95\,\pi$.

Δx	$A^{\rm odd}$	$A^{\rm odd}_{(90)}$
0.750	0.000000	0.000000
1.000	0.000230	0.000230
1.250	0.003749	0.003749
1.500	0.014224	0.014224
1.750	0.027843	0.027844
2.000	0.038933	0.038949

Tables 4a–4d. Approximate lower frame bounds for different steps Δx and Δk on coarse lattices. Estimate $A^{\rm odd}$ of the lower frame bound is given by equation (65) which represents slowly converging series on coarse lattices. The same estimate $A^{\rm odd}$ can efficiently be calculated using rapidly converging series (89). Estimate $A^{\rm odd}_{(90)}$ of the lower frame bound represents approximation (90) of $A^{\rm odd}$. Recommended regions of applicability of approximation $A^{\rm odd}_{(90)}$ are $\Delta x < 1.596$ for Table 4a, $\Delta x < 1.640$ for Table 4b, $\Delta x < 1.684$ for Table 4c and $\Delta x < 1.729$ for Table 4d.