A conjecture on the frame bounds of the multidimensional Gabor transform with complex–valued envelopes

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Summary

The discrete Gabor functions used for the n–D Gabor transform with constant complex–valued envelopes of the Gabor functions are assumed to form a frame. The approximate analytic expressions for the frame bounds of this frame are conjectured. These expressions can be used to analytically study both the discretization error of the continuous Gabor transform and the stability of the discrete Gabor transform in dependence on the regular but generally oblique lattice of the central points of the Gabor functions in the $2n$–D phase space. The algorithm of constructing the optimum phase–space lattice of the central points of the Gabor functions with constant complex–valued envelopes is then conjectured. The algorithm of constructing the phase–space lattice of the central points of the Gabor functions with slightly varying complex–valued envelopes is also conjectured.

Keywords

Gabor function, Gabor transform, frame bounds, discretization error, phase space, metric tensor.

1. Introduction

This paper is devoted to the n–D discrete Gabor transform with constant complex–valued envelopes of the Gabor functions and with the regular but generally oblique lattice of the central points of the Gabor functions in the $2n$–D phase space.

The discrete Gabor functions used for the Gabor transform are assumed to form a frame which we shall call the *Gabor transform frame*. For the general properties of frames refer to Daubechies (1992, sec. 3.2) or to brief overview by Klimeš (2008c, sec. 2). We are interested in the frame bounds of the Gabor transform frame, because both the discretization error of the continuous Gabor transform and the stability of the discrete Gabor transform depend on the frame bounds.

In Section 2, we conjecture approximate analytic expressions for the frame bounds of the Gabor transform frame in dependence on the regular but generally oblique lattice of the central points of the Gabor functions in the $2n$–D phase space.

In Section 3, we conjecture the algorithm of constructing the optimum phase–space lattice of the central points of the Gabor functions with constant complex–valued envelopes.

In Section 4, we conjecture the algorithm of constructing the phase–space lattice of the central points of the Gabor functions with slightly varying complex–valued envelopes. This construction is necessary for the wavefield inversion using the sensitivity

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Gaussian packets (Klimeš, 2008a, sec. 4.1). Its very simple version has been applied by Klimeš (2008d).

This brief note does not present any research result, we just summarize the research topics worth considerable attention.

2. A conjecture on the frame bounds of the multidimensional Gabor transform with complex–valued envelopes

2.1. Gabor functions

Gabor functions, indexed here by lower-case Greek superscripts, read

$$g^\alpha(\mathbf{x}) = a^\alpha \exp[\mathrm{i}\mathbf{k}^{\alpha\mathrm{T}}(\mathbf{x}-\mathbf{x}^\alpha) - \tfrac{1}{2}(\mathbf{x}-\mathbf{x}^\alpha)^\mathrm{T}\mathbf{K}^\alpha(\mathbf{x}-\mathbf{x}^\alpha)] \quad . \tag{1}$$

Gabor function $g^\alpha(\mathbf{x})$ is centred at spatial position $\mathbf{x}^\alpha$ and has structural wavenumber vector $\mathbf{k}^\alpha$. The envelope and wavefront curvature of Gabor function g^α is described by symmetric complex–valued $n \times n$ matrix $\mathbf{K}^\alpha$ with the positive–definite real part. Real–valued amplitude

$$a^\alpha = (\pi)^{-\frac{n}{4}} \, [\det(\mathrm{Re}\mathbf{K}^\alpha)]^{\frac{1}{4}} \quad , \tag{2}$$

of the envelope of the n–D Gabor function is so chosen that the L_2 norm $|g^\alpha|$ is unit (Klimeš, 2008b, eq. 20).

2.2. Space–wavenumber metric for Gabor functions

We define the covariant metric tensor

$$\mathbf{G}_{\bullet\bullet} = \begin{pmatrix} \mathbf{Y}+\mathbf{R}\mathbf{Y}^{-1}\mathbf{R} & -\mathbf{R}\mathbf{Y}^{-1} \\ -\mathbf{Y}^{-1}\mathbf{R} & \mathbf{Y}^{-1} \end{pmatrix} = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ -\mathbf{R} & \mathbf{1} \end{pmatrix}^\mathrm{T} \begin{pmatrix} \mathbf{Y} & \mathbf{0} \\ \mathbf{0} & \mathbf{Y}^{-1} \end{pmatrix} \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ -\mathbf{R} & \mathbf{1} \end{pmatrix} \tag{3}$$

in coordinates $(\mathbf{x},\mathbf{k})$, see Klimeš (2008b, eq. 16). Its inverse matrix

$$\mathbf{G}^{\bullet\bullet} = \begin{pmatrix} \mathbf{Y}^{-1} & \mathbf{Y}^{-1}\mathbf{R} \\ \mathbf{R}\mathbf{Y}^{-1} & \mathbf{Y}+\mathbf{R}\mathbf{Y}^{-1}\mathbf{R} \end{pmatrix} = \begin{pmatrix} \mathbf{1} & \mathbf{R} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}^\mathrm{T} \begin{pmatrix} \mathbf{Y}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{Y} \end{pmatrix} \begin{pmatrix} \mathbf{1} & \mathbf{R} \\ \mathbf{0} & \mathbf{1} \end{pmatrix} \tag{4}$$

is the contravariant metric tensor in coordinates $(\mathbf{x},\mathbf{k})$, see Klimeš (2008b, eq. 17). Here

$$\mathbf{Y} = \mathrm{Re}\mathbf{K}^\alpha \tag{5}$$

and

$$\mathbf{R} = -\mathrm{Im}\mathbf{K}^\alpha \quad . \tag{6}$$

Both the covariant and contravariant metric tensors are symplectic.

2.3. Matrix of lattice intervals

We consider the regular but generally oblique lattice of the central points of the Gabor functions in the $2n$–D phase space. We index the lattice points by $2n$ integers $m_{\underline{i}}$. We assume that there exist smooth functions $\mathbf{x}(m_{\underline{i}})$ and $\mathbf{k}(m_{\underline{i}})$ of real–valued vectors $m_{\underline{i}}$ so that the lattice in space–wavenumber domain is formed by discrete values

$$\mathbf{x}^\alpha = \mathbf{x}(m_{\underline{i}}) \quad , \qquad \mathbf{k}^\alpha = \mathbf{k}(m_{\underline{i}}) \tag{7}$$

obtained for integer $m_{\underline{i}}$. Functions (7) are linear for strictly regular phase–space lattice. We shall refer to real–valued $m_{\underline{i}}$ as the discretization coordinates.

We define $2n\times 2n$ matrix

$$\mathbf{D} = \begin{pmatrix} \mathbf{X} \\ \mathbf{L} \end{pmatrix} \tag{8}$$

of lattice intervals, composed of two $n\times 2n$ matrices

$$X_{i\underline{j}} = \frac{\partial x_i}{\partial m_{\underline{j}}} \quad , \qquad L_{i\underline{j}} = \frac{\partial k_i}{\partial m_{\underline{j}}} \quad . \tag{9}$$

Matrix $\mathbf{D}$ is constant for strictly regular phase–space lattice.

We refer to matrix $\mathbf{D}^{\mathrm{T}}\mathbf{G}_{\bullet\bullet}\mathbf{D}$ as the covariant discretization metric tensor, and to its inverse matrix $(\mathbf{D}^{\mathrm{T}}\mathbf{G}_{\bullet\bullet}\mathbf{D})^{-1}$ as the contravariant discretization metric tensor. Note that the discretization metric tensor depends on indexing of the lattice points. The discretization metric tensors corresponding to differently indexed equal lattices differ.

2.4. A conjecture on the frame bounds

The approximate upper frame bound of the Gabor transform frame is

$$\boxed{B = \frac{(2\pi)^n}{|\det(\mathbf{D})|} \sum_{\mathbf{j}} \exp\left[-\pi^2\, \mathbf{j}^{\mathrm{T}} (\mathbf{D}^{\mathrm{T}}\mathbf{G}_{\bullet\bullet}\mathbf{D})^{-1}\mathbf{j}\right] \quad ,} \tag{10}$$

where the summation is performed over all integer values of $2n$ variables $j_{\underline{i}}$ forming vector $\mathbf{j}$.

The approximate lower frame bound of the Gabor transform frame is

$$\boxed{A = 2\,\frac{(2\pi)^n}{|\det(\mathbf{D})|} \sum_{\mathbf{j}} \exp\left[-(2\pi)^2\, \mathbf{j}^{\mathrm{T}} (\mathbf{D}^{\mathrm{T}}\mathbf{G}_{\bullet\bullet}\mathbf{D})^{-1}\mathbf{j}\right] - B \quad ,} \tag{11}$$

where the summation is again performed over all integer values of $2n$ variables $j_{\underline{i}}$ forming vector $\mathbf{j}$.

Note that different indexing of the same lattice points yields different discretization metric tensors $\mathbf{D}^{\mathrm{T}}\mathbf{G}_{\bullet\bullet}\mathbf{D}$, which results in different indexing $\mathbf{j}$ in (10) and (11), but in equal values A and B of the frame bounds.

3. Optimum phase–space lattice of the central points of the Gabor functions with constant complex–valued envelopes

3.1. Redundancy ratio and the volume of phase–space lattice cells

We consider the regular but generally oblique lattice of the central points of the Gabor functions in the $2n$–D phase space. The volume of the $2n$–D phase–space parallelepiped lattice cell in the phase–space metric is κ^{2n}, with

$$\kappa = \sqrt{\frac{2\pi}{\text{1–D redundancy ratio}}} \quad , \tag{12}$$

where the 1–D redundancy ratio corresponds to the 1–D Gabor transform. For the discrete Gabor transform, the 1–D redundancy ratio should be slightly greater than 1 to keep the transform stable and to preserve the phase–space localization. For the discretized continuous Gabor transform, the 1–D redundancy ratio should be considerably greater than 1 (e.g., 3 or more) to keep the discretization error small. Note that the redundancy ratio of the n–D Gabor transform equals the 1–D redundancy ratio powered to n.

3.2. Regular lattice in the Euclidean space

We first create the regular lattice of points in the $2n$–D Euclidean space with identity metric tensor, and denote its $2n \times 2n$ matrix of lattice intervals by $\mathbf{E}$. We assume that the volume $\det(\mathbf{E})$ of the $2n$–D parallelepiped lattice cells is unit.

In the simplest case of the rectangular lattice, matrix $\mathbf{E}$ equals the identity matrix, but we create the oblique lattice which represents the $2n$–D generalization of the equilateral triangular lattice.

We define unit vectors $e_{\underline{ik}}$, $\underline{k} = 1, 2, ..., 2n$, in the following way: For $\underline{i} < \underline{k}$, we set the $\underline{i}$–th component of the $\underline{k}$–th vector to

$$e_{\underline{ik}} = \sqrt{\frac{1}{2\,i\,(i+1)}} \quad , \tag{13}$$

for $\underline{i} = \underline{k}$ to

$$e_{\underline{ik}} = \sqrt{\frac{i+1}{2\,i}} \quad , \tag{14}$$

and for $\underline{i} > \underline{k}$ to

$$e_{\underline{ik}} = 0 \quad . \tag{15}$$

The difference between any two distinct vectors $e_{\underline{ik}}$ is a unit vector, and their scalar product equals $\frac{1}{2}$. For example, the matrix of first six vectors $e_{\underline{ik}}$ reads

$$\begin{pmatrix} \frac{2}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{3}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} \\ 0 & 0 & \frac{4}{2\sqrt{6}} & \frac{1}{2\sqrt{6}} & \frac{1}{2\sqrt{6}} & \frac{1}{2\sqrt{6}} \\ 0 & 0 & 0 & \frac{5}{2\sqrt{10}} & \frac{1}{2\sqrt{10}} & \frac{1}{2\sqrt{10}} \\ 0 & 0 & 0 & 0 & \frac{6}{2\sqrt{15}} & \frac{1}{2\sqrt{15}} \\ 0 & 0 & 0 & 0 & 0 & \frac{7}{2\sqrt{21}} \end{pmatrix} \quad . \tag{16}$$

Since the volume of the $2n$–D parallelepiped formed by vectors $e_{\underline{ik}}$ is

$$\det(e_{\underline{ik}}) = \sqrt{\frac{2n+1}{2^{2n}}} \quad , \tag{17}$$

we choose matrix $\mathbf{E}$ as

$$E_{\underline{ik}} = \sqrt{2}\,(2n+1)^{-\frac{1}{4n}}\, e_{\underline{ik}} \quad . \tag{18}$$

3.3. Optimum matrix of lattice intervals

We now transform matrix $\mathbf{E}$ to the matrix of phase–space lattice intervals,

$$\mathbf{D} = \kappa \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{R} & \mathbf{1} \end{pmatrix} \begin{pmatrix} \mathbf{Y}^{-\frac{1}{2}} & \mathbf{0} \\ \mathbf{0} & \mathbf{Y}^{\frac{1}{2}} \end{pmatrix} \mathbf{U}\mathbf{E} \quad , \tag{19}$$

where $\mathbf{U}$ is an arbitrary orthonormal $2n \times 2n$ matrix.

Optional matrix $\mathbf{U}$ may be used to reorder the coordinate axes in the Euclidean space, or to arbitrarily rotate the coordinate axes in the Euclidean space including their rotation to the eigenvectors of matrix

$$\begin{pmatrix} \mathbf{Y}^{-\frac{1}{2}} & \mathbf{0} \\ \mathbf{0} & \mathbf{Y}^{\frac{1}{2}} \end{pmatrix} \quad .$$

If we do not need matrix $\mathbf{U}$, it may coincide with the identity matrix.

Equations (3) and (19) yield

$$\mathbf{D}^{\mathrm{T}}\mathbf{G}_{\bullet\bullet}\mathbf{D} = \kappa^2\mathbf{E}^{\mathrm{T}}\mathbf{E} \quad . \tag{20}$$

The diagonal elements of this $2n \times 2n$ matrix are $2\kappa^2(2n+1)^{-\frac{1}{2n}}$ and the offdiagonal elements are $\kappa^2(2n+1)^{-\frac{1}{2n}}$. Then the diagonal elements of $2n{\times}2n$ matrix $(\mathbf{D}^{\mathrm{T}}\mathbf{G}_{\bullet\bullet}\mathbf{D})^{-1}$ are $\frac{2n}{(2n+1)}\kappa^{-2}(2n+1)^{\frac{1}{2n}}$ and the offdiagonal elements are $-\frac{1}{(2n+1)}\kappa^{-2}(2n+1)^{\frac{1}{2n}}$.

4. Phase–space lattice of the central points of the Gabor functions with slightly varying complex–valued envelopes

In this section we consider matrix $\mathbf{K}^{\alpha}$, which determines the shape of Gabor functions (1), smoothly varying with spatial coordinates $\mathbf{x}^{\alpha}$ and wavenumbers $\mathbf{k}^{\alpha}$. Then also metric tensors defined by equations (3) and (4) smoothly vary with phase–space coordinates $\mathbf{x}^{\alpha}$ and $\mathbf{k}^{\alpha}$.

We again assume that there exist smooth functions (7) projecting the integer values of discretization coordinates $m_{\underline{i}}$ on the phase–space lattice points, but functions (7) are no longer linear and matrix (8) is no longer constant. We require that matrix $\mathbf{D}$ smoothly varies with phase–space coordinates, and call this lattice *locally regular*.

We assume that the dependence of matrix $\mathbf{K}^{\alpha}$ on phase–space coordinates $\mathbf{x}^{\alpha}$ and $\mathbf{k}^{\alpha}$ is known. We propose the following algorithm of generating the locally regular phase–space lattice of the central points of Gabor functions.

4.1. Algorithm of generating the phase–space lattice

We specify a $(2n-1)$–D hypersurface in the $2n$–D phase space corresponding to $m_{\underline{2n}}=0$. We then determine a one–parametric set of $(2n-1)$–D equidistant hypersurfaces parametrized by integer values of discretization coordinate $m_{\underline{2n}}$. Equidistant means that the distances of the hypersurfaces with respect to covariant metric tensor (3) equal κ. A convenient condition on the equidistant hypersurfaces requires that the phase–space gradient of $m_{\underline{2n}}$ with respect to contravariant metric tensor (4) equals $1/\kappa$.

We than specify a $(2n-2)$–D hypersurface corresponding to $m_{\underline{2n-1}} = 0$ in each $(2n-1)$–D hypersurface corresponding to constant $m_{\underline{2n}}$, so that these hypersurfaces are situated in a single smooth $(2n-1)$–D hypersurface $m_{\underline{2n-1}}=0$. We then determine a one–parametric set of $(2n-2)$–D equidistant hypersurfaces parametrized by integer values of discretization coordinate $m_{\underline{2n-1}}$ in each $(2n-1)$–D hypersurface corresponding to constant $m_{\underline{2n}}$.

We continue in this way until we obtain a one–parametric set of equidistant points parametrized by integer values of discretization coordinate $m_{\underline{1}}$ along each line corresponding to constant integer values of $m_{\underline{2}}$, $m_{\underline{3}}$, ..., $m_{\underline{2n}}$.

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