Initial conditions for paraxial ray tracing in inhomogeneous anisotropic media

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Summary

Paraxial ray methods have found broad applications in the seismic ray method and in numerical modelling and interpretation of high-frequency seismic wave fields propagating in inhomogeneous, isotropic or anisotropic structures. The basic procedure in paraxial ray methods consists in dynamic ray tracing. We derive the initial conditions for dynamic ray equations, for rays initiated at three types of initial manifolds given in a three-dimensional medium: 1) curved surfaces (surface source), 2) isolated points (point source), and 3) curved, planar and non-planar lines (line source). These initial conditions are very general, valid for homogeneous or inhomogeneous, isotropic or anisotropic media, and for both a constant and a variable initial travel time along the initial manifold. The results presented in the paper considerably extend the possible applications of the paraxial ray method.

Keywords: computational seismology, theoretical seismology, seismic anisotropy, wave propagation

1 Introduction

By standard kinematic ray tracing, we can determine the ray as a spatial trajectory, and the travel time, slowness vector and the ray-velocity vector at any point of the trajectory. We cannot, however, determine these quantities in the vicinity of the ray; for this purpose it would be necessary to compute new rays. In addition, certain important ray quantities cannot be determined along a single ray without computing vicinal rays. For example, this applies to such important quantities as geometrical spreading and ray amplitudes.

The limitations of kinematic ray tracing can be overcome by applying paraxial ray methods. Paraxial ray methods are based on an additional procedure, called dynamic (or paraxial) ray tracing, which consists in computing the partial derivatives of position and slowness vectors at any point of the ray with respect to the parameters specifying the ray

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(ray parameters). Once dynamic ray tracing is performed, geometrical spreading can be determined along the ray, and paraxial rays, paraxial slowness vectors, and paraxial ray-velocity vectors can be determined in the vicinity of the reference ray. In addition, dynamic ray equations have found many other important applications. Generally, paraxial theory enables the local study of a ray field and of properties such as convergence, divergence, and the occurrence of caustics. A comprehensive theory for paraxial ray theory in inhomogeneous anisotropic media is presented in Červený (2001, chap. 4) and Moser and Červený (2007), together with many applications and references to previous work.

In this paper, we consider the dynamic ray equations in general three-dimensional inhomogeneous anisotropic media. They represent a system of linear, ordinary differential equations of the first order, which can be solved together with the kinematic ray tracing system, or along a known ray. They can be expressed in various coordinate systems. The most important are the Cartesian coordinate system and the ray-centred coordinate system. For dynamic ray equations in Cartesian coordinates see Červený (1972, 2001), Gajewski and Pšenčík (1987, 1990), Norris (1987), Kendall and Thomson (1989), Farra (1989), Kendall et al. (1992), Farra and Le Bégat (1995), Pšenčík and Teles (1996), Hanyga et al. (2001), Gjøystdal et al. (2002), Chapman (2004), Iversen (2004), Moser (2004), Moser and Červený (2007), Červený and Moser (2007), and many others. The dynamic ray equations in ray-centred coordinates have been commonly used in isotropic media (Popov and Pšenčík, 1978a, 1978b; Hubral et al., 1992, 1993, 1995). For anisotropic media, see Hanyga (1982), Klimeš (1994), Bakker (1996), Červený (2001, sec. 4.14), etc.

To solve the dynamic ray equations, the initial conditions at an initial point of the ray under consideration must be known. The aim of this paper is to derive and discuss the initial conditions for dynamic ray equations in Cartesian coordinates in general three-dimensional inhomogeneous anisotropic media. We consider situations where the initial point S of the ray is situated on an initial surface (surface source), on an initial line (line source), or at an isolated point (point source). In the first case, we obtain a single ray from S, in the second case a one-parametric system of rays from S, and in the third case a two-parametric system of rays from S.

For dynamic ray equations in ray-centred coordinates in isotropic media, a comprehensive treatment of the initial conditions for all the three situations listed above is given by Červený (2001, sec. 4.5). The authors do not know of a general discussion of the initial conditions for dynamic ray equations in anisotropic media; only special cases have appeared in the seismological literature. For example, the initial conditions for dynamic ray equations at a point source in an anisotropic medium were given by Pšenčík and Teles (1996). Point-source solutions were also treated by Hanyga et al. (2001), who considered a very general anisotropic medium with an arbitrary number of interfaces, possibly intersecting at edges and vertices.

The discussed initial conditions for dynamic ray equations may find many applications in seismology and seismic exploration. They have to be known in all cases when paraxial ray methods are used intentionally. The initial conditions at a surface point S can be applied in seismic modelling experiments as, for instance, the exploding reflector (Loewenthal et al., 1976; Gray et al., 2001), in plane wave migration (Treitel et al., 1982), in wavefront construction (Vinje et al., 1993; Lambaré et al., 1996; Gjøystdal et al., 2002), in the isochrone method (Iversen, 2004), etc. The initial conditions for an initial point S on a line are useful in modelling the edge diffraction cone, in accordance with the geometrical theory of diffraction (Keller, 1962). Finally, the initial conditions for a point source are always needed when Green's function in anisotropic media is calculated. References to this case are too numerous to be given here. The standard approach to defining initial conditions by differentiating the ray position vector and ray slowness vector along the initial line or surface can be very laborious. One of the key results presented in Moser (2004) and in Moser and Červený (2007) is a factorization of the 6×6 interface propagator matrix in Cartesian coordinates into two matrices. One of these matrices allows the initial matrices for paraxial ray tracing from a known travel time distribution along an initial surface to be expressed in a convenient way. The relevant equations in this paper are (29) and (33).

For a smooth initial surface, the expressions for the initial matrices for paraxial ray tracing are relatively simple to derive. Consequently, we shall present the case of a smooth initial surface first. For an initial point or line, some modifications to the equations are needed, which we obtain by taking the limiting cases from the smooth surface to the point or line, respectively.

The paper is organized as follows. In Section 2, we review very briefly some material from kinematic and dynamical ray tracing, needed for the subsequent sections. In Section 3, we define the initial smooth surface and its parametrization. The initial conditions for kinematic and dynamic ray tracing on a smooth surface are discussed in Section 4, with Section 4.3 dedicated to some special cases of practical interest. Section 5 presents the initial conditions on a spherical surface and takes the limit to a point. Finally, in Section 6 a tubular surface is introduced for the purpose of taking the limit to a smooth initial line. Some concluding remarks are presented in Section 6.

For any general details on paraxial ray theory, we refer to Červený (2001, chapter 4), and on paraxial ray theory in Cartesian coordinates to Moser and Červený (2007). In the notation, we closely follow Moser and Červený (2007). All 3×1 vectors are denoted in lower-case upright boldface ($\mathbf{a}$), all 3×3 matrices in upper-case upright boldface ($\mathbf{A}$). All 2×2 matrices are in upper-case italic boldface ($\boldsymbol{A}$). Greek indices $\mu, \nu, ...$ run over 1 and 2, Latin indices $i, j, k, l, ...$ over 1, 2, and 3. The Kronecker symbol δ_{ij} equals 1 for $i = j$ and 0 for $i \neq j$. The summation convention is assumed for repeated indices in the same product. Derivatives or partial derivatives are denoted by subscripts as indicated when they are introduced.

2 Kinematic and dynamic ray tracing

We consider a high-frequency wave field, propagating in a smoothly heterogeneous, isotropic or anisotropic medium. The elastodynamic equation then yields a non-linear partial differential equation of the first order for the travel time $T(\mathbf{x})$, called the eikonal equation. In Hamiltonian form, we can express the eikonal equation as follows:

$$\mathcal{H}(\mathbf{x}, \mathbf{p}) = 0 \ . \tag{1}$$

Here $\mathcal{H}(\mathbf{x},\mathbf{p})$ is called the Hamiltonian function (or simply the Hamiltonian), $\mathbf{x}$ is the position vector, expressed in Cartesian coordinates x_i, $(i = 1, 2, 3)$, and $\mathbf{p} = \nabla T$ is the slowness vector.

For isotropic heterogeneous media, we express the Hamiltonian in the following form:

$$\mathcal{H}(\mathbf{x},\mathbf{p}) = \tfrac{1}{2}[V^2(\mathbf{x})\mathbf{p}^T\mathbf{p} - 1] \ , \tag{2}$$

where $V(\mathbf{x})$ is the spatially variable velocity of P or S waves. For an anisotropic heterogeneous medium, a suitable form of the Hamiltonian is as follows:

$$\mathcal{H}(\mathbf{x},\mathbf{p}) = \tfrac{1}{2}[G_m(\mathbf{x},\mathbf{p}) - 1] \ , \tag{3}$$

where $G_m(\mathbf{x},\mathbf{p})$ is a selected eigenvalue of the 3×3 Christoffel matrix

$$\Gamma_{ik}(x_m, p_n) = a_{ijkl}(\mathbf{x})p_j p_l \ . \tag{4}$$

The functions $a_{ijkl}(\mathbf{x})$ represent the spatially variable density-normalized elastic stiffnesses.

The eikonal equation (1) can be solved for $T(\mathbf{x})$ by the method of characteristics, which represent rays. The parameter τ, specifying the position of points along a selected ray, depends on the form of the Hamiltonian. Here we shall use Hamiltonians which are homogeneous functions of the second degree in p_i (with a possible additional constant). The parameter τ then has the physical meaning of the travel time of the wave under consideration along the ray.

Every ray satisfies the kinematic ray equations (ray tracing system):

$$\mathbf{x}_\tau = \mathcal{H}_\mathbf{p} \ , \qquad \mathbf{p}_\tau = -\mathcal{H}_\mathbf{x} \ . \tag{5}$$

The kinematic ray equations (5) represent a system of six coupled non-linear ordinary differential equations. The initial conditions for (5) have to satisfy the eikonal equation (1), which is then satisfied along the whole ray. In numerical ray tracing, equation (1) can be used to check and control the accuracy of ray computation.

We consider a two-parametric system of rays (orthonomic ray field):

$$\mathbf{x}(\gamma_1, \gamma_2, \tau) \ , \qquad \mathbf{p}(\gamma_1, \gamma_2, \tau) \ , \tag{6}$$

satisfying equation (5). Here γ_1 and γ_2 are parameters specifying the rays, for example the take-off angles from a point source, the coordinates at the initial surface, etc. It is useful to combine γ_1, γ_2, τ into one vector, $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \gamma_3 = \tau)$, and introduce the 3×3 paraxial matrices $\mathbf{Q}$ and $\mathbf{P}$ as follows:

$$\mathbf{Q} = \partial\mathbf{x}/\partial\boldsymbol{\gamma} = (\mathbf{x}_{\gamma_1}, \mathbf{x}_{\gamma_2}, \mathbf{x}_{\gamma_3}) \ , \quad \mathbf{P} = \partial\mathbf{p}/\partial\boldsymbol{\gamma} = (\mathbf{p}_{\gamma_1}, \mathbf{p}_{\gamma_2}, \mathbf{p}_{\gamma_3}) \ . \tag{7}$$

Matrices $\mathbf{Q}$ and $\mathbf{P}$ can be computed along the ray using the dynamic ray equations

$$\begin{aligned} \mathbf{Q}_\tau &= \mathcal{H}_{\mathbf{px}}\mathbf{Q} + \mathcal{H}_{\mathbf{pp}}\mathbf{P} \ , \\ \mathbf{P}_\tau &= -\mathcal{H}_{\mathbf{xx}}\mathbf{Q} - \mathcal{H}_{\mathbf{xp}}\mathbf{P} \ . \end{aligned} \tag{8}$$

In equations (5) and (8), subscripts $\mathbf{x}$ and $\mathbf{p}$ denote first-order partial derivatives with respect to these quantities, double subscripts $\mathbf{xx}, \mathbf{xp}, \mathbf{px}$ and $\mathbf{pp}$ second-order partial derivatives. The subscript τ denotes the first-order ordinary derivative with respect to τ along the ray. Equations (8) can be solved simultaneously with (5), or along a known ray.

For homogeneous media, the dynamic ray equations (8) can be solved analytically. Taking into account that $\mathcal{H}_{\mathbf{xx}} = \mathcal{H}_{\mathbf{px}} = \mathcal{H}_{\mathbf{xp}} = \mathbf{O}$ in this case, we obtain

$$\mathbf{Q}(\tau) = \mathbf{Q}(\tau_0) + (\tau - \tau_0)\mathcal{H}_{\mathbf{pp}}\mathbf{P}(\tau_0) \ , \quad \mathbf{P}(\tau) = \mathbf{P}(\tau_0) \ , \tag{9}$$

where τ_0 corresponds to the initial point of the ray. Equation (9) clearly demonstrates the great importance of the initial conditions for dynamic ray equations, $\mathbf{Q}(\tau_0)$ and $\mathbf{P}(\tau_0)$, in paraxial ray methods, particularly in computing the geometrical spreading and ray amplitudes. In homogeneous media, the 3×3 matrix $\mathbf{P}(\tau)$ remains constant along the whole (straight line) ray, but the 3×3 matrix $\mathbf{Q}(\tau)$ of geometrical spreading is considerably influenced by $\mathbf{P}(\tau_0)$. Equation (9) also indicates the basic role played by the tensor $\mathcal{H}_{\mathbf{pp}}$ in paraxial ray methods. This tensor is also called the *wave propagation metric tensor* and has many interesting properties, particularly in anisotropic media. For a detailed discussion, see Klimeš (2002) and Červený (2002).

The computed 3×3 paraxial matrices $\mathbf{Q}$ and $\mathbf{P}$ can be used to determine various paraxial quantities in the neighbourhood of the reference ray, without the need to trace new rays. For example, for $\det \mathbf{Q} \neq 0$, we can compute the 3×3 symmetric matrix of second derivatives $\mathbf{M}$ of the travel-time field with respect to Cartesian coordinates:

$$\mathbf{M}(\tau) = \mathbf{P}(\tau)\mathbf{Q}^{-1}(\tau) \ . \tag{10}$$

The initial conditions for the dynamic ray equations (8) must satisfy the constraint relation, which follows form (1),

$$\mathcal{H}_{\mathbf{x}}^T\mathbf{Q} + \mathcal{H}_{\mathbf{p}}^T\mathbf{P} = 0 \ . \tag{11}$$

The constraint relation (11) is satisfied along the whole ray, once it is satisfied at the initial point of the ray. Using (5) and (7), we can also express the constraint relation (11) in the following form:

$$\mathbf{p}_\tau^T\mathbf{x}_{\gamma_n} = \mathbf{x}_\tau^T\mathbf{p}_{\gamma_n} \ , \tag{12}$$

where $n = 1, 2, 3$.

3 Initial surface Σ

Consider a smooth initial surface Σ, given by the parametric equation

$$\mathbf{x} = \mathbf{g}(u_1, u_2) \ , \tag{13}$$

where u_1 and u_2 are the initial surface coordinates. They may represent, e.g., the Gaussian coordinates of the surface Σ. Alternatively, we can also use a local Cartesian coordinate system u_1, u_2 in the plane tangent to Σ at an arbitrarily selected initial point $S \in \Sigma$, with its origin at S. This local choice is very popular in the seismic ray method, see Červený (2001), but varies from point to point along Σ. The initial travel time T^0 along Σ may be constant, or may vary with surface coordinates u_1, u_2;

$$T^0 = T^0(u_1, u_2) \ . \tag{14}$$

We assume that the initial surface Σ, given by (13), and the initial travel time T^0 along Σ, given by (14), are continuous and have continuous first and second partial derivatives with respect to u_1 and u_2. We denote

$$\mathbf{g}_\mu = \partial \mathbf{g}/\partial u_\mu \ , \qquad \mathbf{g}_{\mu\nu} = \partial^2 \mathbf{g}/\partial u_\mu \partial u_\nu \ , \tag{15}$$

and similarly,

$$T^0_\mu = \partial T^0/\partial u_\mu \ , \quad T^0_{\mu\nu} = \partial^2 T^0/\partial u_\mu \partial u_\nu \ . \tag{16}$$

Here $\mu, \nu = 1, 2$. Note that $\mathbf{g}_1$ and $\mathbf{g}_2$ are vectors tangent to Σ, and $\mathbf{g}_{\mu\nu}$ are related to the curvature of Σ. The unit vector $\mathbf{n}^\Sigma$ perpendicular to Σ is given by the relation

$$\mathbf{n}^\Sigma(u_1, u_2) = (\mathbf{g}_1 \times \mathbf{g}_2)/||\mathbf{g}_1 \times \mathbf{g}_2|| \ . \tag{17}$$

Finally, at an arbitrary initial point S of Σ, we can introduce the 3×3 matrix $\mathbf{X}(S)$ by the relation

$$\mathbf{X}(S) = (\mathbf{g}_1(S), \mathbf{g}_2(S), \mathbf{g}_3(S)) \ . \tag{18}$$

At the initial point $S \in \Sigma$, the 3×3 matrix $\mathbf{X}$ combines the vectors $\mathbf{g}_1$ and $\mathbf{g}_2$, tangential to Σ, with the vector $\mathbf{g}_3 = \mathbf{x}_\tau$, tangent to the ray. In the following Section 4, we assume that $\mathbf{X}$ is regular ($\det \mathbf{X} \neq 0$), and hence that 1) the initial ray does not graze the surface, and 2) the parametrization (13) is regular, i.e. the tangent vectors $\mathbf{g}_1$ and $\mathbf{g}_2$ are linearly independent. Cases where the tangent vectors are not linearly independent, such as the initial point and the initial line, will be discussed in Sections 5 and 6.

4 Initial conditions for kinematic and dynamic ray tracing at a surface

In this section, we select an arbitrary point S on a surface Σ, and construct the initial conditions for kinematic and dynamic ray equations, (5) and (8). The surface Σ may represent a structural interface, the free surface of the Earth, an isochrone, or merely a formal surface in a smooth medium. The initial conditions for kinematic ray equations (5) must satisfy eikonal equation (1), the initial conditions for dynamic ray equations must satisfy the constraint relations (11). In Section 4.1, we discuss the initial conditions for kinematic ray equations, in Section 4.2 the initial conditions for dynamic ray equations, and in Section 4.3 we present some special cases.

4.1 Kinematic ray equations

This case was discussed in detail in Červený (2001, sec. 4.5.1). For this reason, we will be brief and present, without derivation, the final equations. We start with the determination of the slowness vector $\mathbf{p}$ at any point S of the surface Σ.

The tangential component $\mathbf{p}^\Sigma$ of the slowness vector is given by the relation

$$\mathbf{p}^\Sigma = A_{\mu\nu} T^0_\nu \mathbf{g}_\mu \ , \tag{19}$$

where the 2×2 matrix $\boldsymbol{A}$ reads

$$\boldsymbol{A} = (EG - F^2)^{-1} \begin{pmatrix} G & -F \\ -F & E \end{pmatrix} \ , \tag{20}$$

with

$$E = \mathbf{g}_1^T \mathbf{g}_1 \ , \quad F = \mathbf{g}_1^T \mathbf{g}_2 \ , \quad G = \mathbf{g}_2^T \mathbf{g}_2 \tag{21}$$

(see Červený 2001, Section 4.5.1). The initial slowness vector $\mathbf{p}$ is then given by the relation

$$\mathbf{p} = \sigma \mathbf{n}^\Sigma + \mathbf{p}^\Sigma \ , \tag{22}$$

where σ should be determined from the eikonal equation $\mathcal{H}(\mathbf{x}, \mathbf{p}) = 0$, and from the choice of the initial wave mode (P, S1, S2). For general anisotropy and for $\mathbf{p}^\Sigma \neq \mathbf{0}$, the following algebraic equation of the sixth degree for σ is obtained:

$$\det[a_{ijkl}(\sigma n_j^\Sigma + p_j^\Sigma)(\sigma n_l^\Sigma + p_l^\Sigma) - \delta_{ik}] = 0 \ . \tag{23}$$

The six roots σ of (23) correspond to P, S1 and S2 waves which propagate in the halfspace under consideration, outwards and towards Σ. Proper criteria are needed to select from these six waves the P, S1 and S2 waves propagating outwards from Σ. For certain simpler cases, the algebraic equations of the sixth degree can be factorized into two equations; one of the fourth degree and the other of the second degree. For isotropic media, the equation for σ reduces to a quadratic equation, both for P and S waves, which yields simple solutions:

$$\sigma_{1,2} = \pm\sqrt{1/V^2 - ||\mathbf{p}^\Sigma||^2} \ . \tag{24}$$

Here V is the velocity of P or S waves.

Finally, the initial values of $\mathbf{x}_\tau$ and $\mathbf{p}_\tau$ can be found by differentiation:

$$\mathbf{x}_\tau = \mathcal{H}_{\mathbf{p}}(\mathbf{x}, \mathbf{p}) \ , \quad \mathbf{p}_\tau = -\mathcal{H}_{\mathbf{x}}(\mathbf{x}, \mathbf{p}) \ . \tag{25}$$

We call the vector $\mathbf{x}_\tau$, which is tangent to the ray at the initial point $S \in \Sigma$, the *ray-velocity vector.*

4.2 Dynamic ray equations

Now we derive the initial conditions for the dynamic ray equations (8) in Cartesian coordinates from an initial surface Σ, in terms of the 3×3 matrices $\mathbf{Q}$ and $\mathbf{P}$. We assume that the 3×3 matrix $\mathbf{X} = (\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3 = \mathbf{x}_\tau)$ is regular at S and determine its inverse $\mathbf{X}^{-1}$.

The ray parameters γ_1 and γ_2 may be chosen arbitrarily to uniquely specify individual rays. We choose them, without loss of generality, to be equal to the initial surface coordinates u_1, u_2:

$$\gamma_1 = u_1 \ , \qquad \gamma_2 = u_2 \ . \tag{26}$$

Now we will compute the initial matrices $\mathbf{Q}$ and $\mathbf{P}$ at the surface Σ. We have to remember that the paraxial derivatives (15) and (16) are computed along Σ, but the partial derivatives $\mathbf{x}_{\gamma_\mu}$ and $\mathbf{p}_{\gamma_\mu}$ ($\mu = 1, 2$), involved in $\mathbf{Q}$ and $\mathbf{P}$, are computed along the wavefront, see (7). We use the relation

$$\mathbf{g}_\mu = \left.\frac{\partial \mathbf{x}}{\partial u_\mu}\right|_\tau + \left.\frac{\partial \mathbf{x}}{\partial \tau}\right|_{u_\mu} \frac{\partial T^0}{\partial u_\mu} \ , \tag{27}$$

which can be expressed as follows:

$$\mathbf{g}_\mu = \mathbf{x}_{\gamma_\mu} + \mathbf{x}_\tau T^0_\mu \ , \quad \mu = 1, 2 \ . \tag{28}$$

Consequently,

$$\mathbf{Q} = \mathbf{X}\mathbf{U} \ , \tag{29}$$

where the matrix $\mathbf{U}$ is given by

$$\mathbf{U} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -T^0_1 & -T^0_2 & 1 \end{pmatrix} \ . \tag{30}$$

This is the final expression for the initial condition for the 3×3 matrix $\mathbf{Q}$. Note that the matrix $\mathbf{U}$ was derived in Červený (2001, section 4.5.2) and is denoted $\mathbf{Q}^{(u)}$ in Moser and Červený (2007).

The determination of $\mathbf{P}$ is more involved. We use the relation for the matrix $\mathbf{M}$ of the second derivatives of the travel-time field with respect to Cartesian coordinates, derived by Moser and Červený (2007):

$$\mathbf{M} = \mathbf{X}^{-T}(\mathbf{M}^{(u)} + \mathbf{W})\mathbf{X}^{-1} \ , \tag{31}$$

where $\mathbf{W}$ and $\mathbf{M}^{(u)}$ are symmetric 3×3 matrices, given by the relations

$$\mathbf{W} = \begin{pmatrix} -\mathbf{g}^T_{11}\mathbf{p} & -\mathbf{g}^T_{12}\mathbf{p} & \mathbf{g}^T_1\mathbf{p}_\tau \\ -\mathbf{g}^T_{21}\mathbf{p} & -\mathbf{g}^T_{22}\mathbf{p} & \mathbf{g}^T_2\mathbf{p}_\tau \\ \mathbf{g}^T_1\mathbf{p}_\tau & \mathbf{g}^T_2\mathbf{p}_\tau & \mathbf{p}^T_\tau\mathbf{x}_\tau \end{pmatrix} \ , \qquad \mathbf{M}^{(u)} = \begin{pmatrix} T^0_{11} & T^0_{12} & 0 \\ T^0_{21} & T^0_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} \ . \tag{32}$$

This yields, using (10) and (29):

$$\mathbf{P} = \mathbf{M}\mathbf{Q} = \mathbf{X}^{-T}(\mathbf{M}^{(u)} + \mathbf{W})\mathbf{X}^{-1}\mathbf{Q} = \mathbf{X}^{-T}(\mathbf{M}^{(u)} + \mathbf{W})\mathbf{U} \ . \tag{33}$$

Equations (29) and (33) represent the final expressions for matrices $\mathbf{Q}$ and $\mathbf{P}$ at the initial surface Σ. The expression (33) for $\mathbf{P}$, however, may be expressed in many alternative forms. For example, we can use the explicit analytical expression for $\mathbf{X}^{-T}$. We denote

$$\mathbf{X}^{-T} = (\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3) \ . \tag{34}$$

The vectors $\mathbf{h}_1$, $\mathbf{h}_2$ and $\mathbf{h}_3$, can be calculated numerically from $\mathbf{g}_1$, $\mathbf{g}_2$ and $\mathbf{g}_3 = \mathbf{x}_\tau$. In certain situations, it is also useful to know them analytically. For instance, this will allow us to perform a limiting process for some sources for which matrix $\mathbf{X}$ is degenerate (point source, line source). Analytical expressions for $\mathbf{h}_i$ in terms of $\mathbf{g}_1$, $\mathbf{g}_2$, $\mathbf{x}_\tau$ are as follows,

$$\mathbf{h}_1 = \mathbf{g}_2 \times \mathbf{x}_\tau / \det \mathbf{X} \ , \quad \mathbf{h}_2 = \mathbf{x}_\tau \times \mathbf{g}_1 / \det \mathbf{X} \ , \quad \mathbf{h}_3 = \mathbf{g}_1 \times \mathbf{g}_2 / \det \mathbf{X} \ , \tag{35}$$

where $\det \mathbf{X} = \mathbf{x}_\tau^T(\mathbf{g}_1 \times \mathbf{g}_2)$.

Actually, if we take $(\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3 \equiv \mathbf{x}_\tau)$ as contravariant basis vectors of a non-orthogonal coordinate system, then $(\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3)$ represent the covariant basis vectors of this system. $\mathbf{g}_1$ and $\mathbf{g}_2$ may be arbitrary non-parallel vectors tangent to Σ of different lengths.

Using (32)–(34), we obtain

$$\mathbf{P} = (\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3) \begin{pmatrix} T^0_{11} - \mathbf{g}^T_{11}\mathbf{p} & T^0_{12} - \mathbf{g}^T_{12}\mathbf{p} & \mathbf{g}^T_1 \mathbf{p}_\tau \\ T^0_{21} - \mathbf{g}^T_{21}\mathbf{p} & T^0_{22} - \mathbf{g}^T_{22}\mathbf{p} & \mathbf{g}^T_2 \mathbf{p}_\tau \\ \mathbf{g}^T_1 \mathbf{p}_\tau & \mathbf{g}^T_2 \mathbf{p}_\tau & \mathbf{x}^T_\tau \mathbf{p}_\tau \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -T^0_1 & -T^0_2 & 1 \end{pmatrix} . \tag{36}$$

The third column of matrix $\mathbf{P}$, P_{k3} $(k = 1, 2, 3)$, can be simplified considerably. Denoting again $\mathbf{g}_3 = \mathbf{x}_\tau$, we get

$$P_{k3} = h_{ki} g_{ij} p_{\tau j} = \delta_{kj} p_{\tau j} = p_{\tau k} \ , \tag{37}$$

where $p_{\tau j}$ is the j-th component of the vector $\mathbf{p}_\tau$. Thus, the third columns of matrices $\mathbf{Q}$ and $\mathbf{P}$ are simple, $\mathbf{x}_\tau$ and $\mathbf{p}_\tau$. They are known from kinematic ray tracing and need not be recomputed for dynamical ray tracing.

Let us summarize the final explicit equations for the initial conditions for $\mathbf{Q}$ and $\mathbf{P}$ at the initial surface Σ. In vectorial form, the 3×3 matrices $\mathbf{Q}$ and $\mathbf{P}$ can be expressed as follows (7):

$$\mathbf{Q} = \partial \mathbf{x} / \partial \boldsymbol{\gamma} = (\mathbf{x}_{\gamma_1}, \mathbf{x}_{\gamma_2}, \mathbf{x}_{\gamma_3}) \ , \quad \mathbf{P} = \partial \mathbf{p} / \partial \boldsymbol{\gamma} = (\mathbf{p}_{\gamma_1}, \mathbf{p}_{\gamma_2}, \mathbf{p}_{\gamma_3}) \ , \tag{38}$$

where

$$\begin{aligned} \mathbf{x}_{\gamma_\mu} &= \mathbf{g}_\mu - T^0_\mu \mathbf{x}_\tau \ , & \mathbf{x}_{\gamma_3} &= \mathbf{x}_\tau \ , \\ \mathbf{p}_{\gamma_\mu} &= \mathbf{h}_\nu (T^0_{\mu\nu} - \mathbf{g}^T_{\mu\nu}\mathbf{p}) + \mathbf{h}_3 (\mathbf{g}^T_\mu \mathbf{p}_\tau) - T^0_\mu \mathbf{p}_\tau \ , & \mathbf{p}_{\gamma_3} &= \mathbf{p}_\tau \ , \end{aligned} \tag{39}$$

$(\mu, \nu = 1, 2)$.

Equation (38) with (39) are valid for arbitrarily chosen surficial coordinates u_1 and u_2 along Σ. In practical applications, however, it is common to introduce a local Cartesian coordinate system, with its origin at $S \in \Sigma$. The basis vectors $\mathbf{g}_1$ and $\mathbf{g}_2$, tangent to Σ at S, are then taken unit and mutually perpendicular. The third basis vector $\mathbf{g}_3$ equals the unit normal $\mathbf{n}^\Sigma$ at S, see (17). The triplet of mutually perpendicular unit vectors $\mathbf{g}_1$, $\mathbf{g}_2$, $\mathbf{n}^\Sigma$ is right-handed. For local Cartesian coordinates, it is possible to simplify the expression $\mathbf{g}^T_{\mu\nu}\mathbf{p}$ and the expressions (35) for $\mathbf{h}_i$ in (36) and (39). ¿From (22) we obtain $\mathbf{g}^T_{\mu\nu}\mathbf{p} = \sigma \mathbf{g}^T_{\mu\nu}\mathbf{n}^\Sigma + \mathbf{g}^T_{\mu\nu}\mathbf{p}^\Sigma$. The second term vanishes, as $\mathbf{p}^\Sigma$ is tangent to Σ at S, and we get

$$\mathbf{g}^T_{\mu\nu}\mathbf{p} = -\sigma D_{\mu\nu} \ , \tag{40}$$

where $D_{\mu\nu} = -\mathbf{g}^T_{\mu\nu}\mathbf{n}^\Sigma$ is a 2×2 curvature matrix of the surface Σ at S (Červený, 2001). The parameter σ is known from kinematic ray tracing (23). The expressions for $\mathbf{h}_i$ in local Cartesian coordinates read:

$$\mathbf{h}_\nu = \mathbf{g}_\nu - \mathbf{n}^\Sigma(\mathbf{x}^T_\tau \mathbf{g}_\nu)/(\mathbf{x}^T_\tau \mathbf{n}^\Sigma) \ , \quad \mathbf{h}_3 = \mathbf{n}^\Sigma/(\mathbf{x}^T_\tau \mathbf{n}^\Sigma) \ . \tag{41}$$

4.3 Special cases

The expressions for the initial values for dynamic ray equations, presented in matrix form by (29) and (33), and in vectorial form by (38) and (39), are very general. They can be used for any three-dimensional smooth, regular initial surface Σ, for homogeneous or inhomogeneous, isotropic or anisotropic media, and for any distribution of the initial travel time T^0 along Σ. In this section, we specify these equations for several special cases. Since the equations for $\mathbf{x}_{\gamma_\mu}$ are straightforward, we shall consider only the equations (39) for $\mathbf{p}_{\gamma_\mu}$. In the whole section, we consider a local Cartesian coordinate system with its origin at S, defined by the equations (40) and (41).

4.3.1 Exploding surface

Here we consider the initial travel time T^0 along Σ to be constant. Consequently, $T^0_\mu = T^0_{\mu\nu} = 0$, $\mu, \nu = 1, 2$. This corresponds to the exploding reflector experiment in seismic migration (Loewenthal et al., 1976; Gray et al., 2001). Equations (39) with (40) then symplify:

$$\mathbf{p}_{\gamma_\mu} = \mathbf{h}_\nu \sigma D_{\mu\nu} + \mathbf{h}_3(\mathbf{g}^T_\mu \mathbf{p}_\tau) \ . \tag{42}$$

4.3.2 Planar surface

Next, we consider the case of a planar initial surface Σ, $D_{\mu\nu} = 0$. The Cartesian coordinate system, with basis vectors $\mathbf{g}_1$, $\mathbf{g}_2$, $\mathbf{g}_3 = \mathbf{n}^\Sigma$, now has a global character. Equations (39) yield

$$\mathbf{p}_{\gamma_\mu} = \mathbf{h}_\nu T^0_{\mu\nu} + \mathbf{h}_3(\mathbf{g}^T_\mu \mathbf{p}_\tau) - T^0_\mu \mathbf{p}_\tau \ . \tag{43}$$

4.3.3 Homogeneous medium

In a homogeneous medium, isotropic or anisotropic, $\mathbf{p}_\tau = \mathbf{0}$. In this case, we obtain a particularly simple result from (39):

$$\mathbf{p}_{\gamma_\mu} = \mathbf{h}_\nu (T^0_{\mu\nu} + \sigma D_{\mu\nu}) \ . \tag{44}$$

Note that the dynamic ray equations can be solved analytically in a homogeneous medium, see (9).

4.3.4 Isotropic medium

In isotropic media, we have

$$\begin{array}{lll} \sigma = \sqrt{1/V^2 - \mathbf{p}^{\Sigma T}\mathbf{p}^{\Sigma}} \ , & \mathbf{p}^{\Sigma} = \mathbf{g}_{\nu} T^0_{\nu} \ , & \mathbf{p}^T \mathbf{p} = 1/V^2 \ , \\ \mathbf{p} = \sigma \mathbf{n}^{\Sigma} + \mathbf{g}_{\nu} T^0_{\nu} \ , & \mathbf{x}_{\tau} = V^2(\sigma \mathbf{n}^{\Sigma} + \mathbf{g}_{\nu} T^0_{\nu}) \ , & \mathbf{p}_{\tau} = -V^{-1} \nabla V \ , \\ \mathbf{h}_{\nu} = \mathbf{g}_{\nu} - \mathbf{n}^{\Sigma} T^0_{\nu}/\sigma \ , & \mathbf{h}_3 = \mathbf{n}^{\Sigma}/V^2 \sigma \ . & \end{array} \tag{45}$$

Here V is the velocity of P or S waves. The final expression for $\mathbf{p}_{\gamma_\mu}$ from (39) and (40) is then as follows:

$$\mathbf{p}_{\gamma_\mu} = (\mathbf{g}_{\nu} - \mathbf{n}^{\Sigma} T^0_{\nu}/\sigma)(T^0_{\mu\nu} + \sigma D_{\mu\nu}) + \mathbf{n}^{\Sigma}(\mathbf{g}^T_{\mu} \mathbf{p}_{\tau}) - T^0_{\mu} \mathbf{p}_{\tau} \ . \tag{46}$$

Several special cases of (46) can again be considered: the exploding surface ($T^0_{\nu} = T^0_{\mu\nu} = 0$), the planar surface Σ ($D_{\mu\nu} = 0$), the homogeneous medium ($\mathbf{p}_{\tau} = \mathbf{0}$). The resulting expressions are straightforward.

5 Spherical initial surface and point source

For a point source, matrix $\mathbf{X}$ is degenerate, and the approach of Section 4.2 breaks down. This difficulty, however, can be overcome if we use a spherical initial surface Σ of the radius a, and perform a simple limiting process: $a \to 0$.

Consider an initial spherical surface Σ with its centre at S^0 and with radius a,

$$\mathbf{g}(u_1, u_2) = \mathbf{x}(S^0) + a\mathbf{n}(u_1, u_2) \ . \tag{47}$$

Here $\mathbf{n}(u_1, u_2)$ is a unit normal to Σ, $\mathbf{n}^T \mathbf{n} = 1$, and u_1, u_2 are arbitrary parameters specifying the direction of $\mathbf{n}$ (for example, the take-off angles at S^0).

We assume that the initial travel time $T^0(u_1, u_2)$ along Σ is constant: $T^0_{\mu} = T^0_{\mu\nu} = 0$ ($\mu, \nu = 1, 2$). Note that it is important for the limiting process to take the radius a independent of the parameters u_1, and u_2, in order to avoid any preferential directions in the final equations. Also we assume that $\mathbf{n}(u_1, u_2)$ is continuous and has continuous first and second derivatives with respect to u_1 and u_2.

Vectors $\mathbf{g}_{\mu}$ and $\mathbf{g}_{\mu\nu}$ are given by relations:

$$\mathbf{g}_{\mu} = a\mathbf{n}_{\mu} \ , \quad \mathbf{g}_{\mu\nu} = a\mathbf{n}_{\mu\nu} \ . \tag{48}$$

5.1 Kinematic ray equations

Let us first discuss the initial conditions for kinematic ray tracing. Since the initial travel time along Σ is constant, the initial slowness vector $\mathbf{p}$ has no component along Σ and is given by

$$\mathbf{p} = \sigma \mathbf{n} = \mathcal{C}^{-1} \mathbf{n} \ . \tag{49}$$

Here $\mathcal{C}$ is the phase velocity (in the direction of $\mathbf{p}$). The expression (49) is valid even for $a = 0$ and may be used, together with the coordinates of S^0, as initial conditions for ray tracing from a point source S^0. Using (25), we also obtain $\mathbf{x}_\tau$ and $\mathbf{p}_\tau$ for a point source.

For $a = 0$, the system of initial slowness vectors from the point S_0 is two-parametric. The loci of the end-points of the initial slowness vector $\mathbf{p}$ form a closed surface, called the *slowness surface*. For isotropic media, the slowness surface is a sphere with its centre at S_0. In anisotropic media, it may display singular behaviour (parabolic lines, kiss singularities, point singularity). At any regular point of the slowness surface, the normal to the slowness surface determines the relevant ray-velocity vector $\mathbf{x}_\tau$.

The ray-velocity vectors $\mathbf{x}_\tau$ are tangent to the rays constructed from the point S_0. The two-parametric system of rays from the point S_0 is called the central ray field. The loci of the end-points of the ray-velocity vectors from S_0 form a surface, usually called the *wave surface*. The slowness surface and wave surface are mutually related by so-called polar reciprocity (Helbig, 1994). The wave surface is usually more complicated than the slowness surface. It may be multivalued, with loops, cuspoidal points, etc. It actually shows the form of the wavefront of the wave generated by a point source at S_0. At any regular point of the wave surface, the normal to the wave surface has the direction of the slowness vector $\mathbf{p}$.

5.2 Dynamic ray equations

To obtain the initial conditions for dynamic ray tracing for a spherical surface, we use the expressions (35) for $\mathbf{h}_i$:

$$\mathbf{h}_1 = \mathbf{n}_2 \times \mathbf{x}_\tau / ad \ , \quad \mathbf{h}_2 = \mathbf{x}_\tau \times \mathbf{n}_1 / ad \ , \quad \mathbf{h}_3 = \mathbf{n}_1 \times \mathbf{n}_2 / d \ , \tag{50}$$

where $d = \mathbf{x}_\tau^T(\mathbf{n}_1 \times \mathbf{n}_2)$. Inserting these expressions into (39), yields

$$\mathbf{x}_{\gamma_1} = a\mathbf{n}_1 \ , \quad \mathbf{x}_{\gamma_2} = a\mathbf{n}_2 \ , \quad \mathbf{x}_{\gamma_3} = \mathbf{x}_\tau \ , \tag{51}$$

and

$$\begin{aligned}
\mathbf{p}_{\gamma_1} &= -[(\mathbf{n}_2 \times \mathbf{x}_\tau)(\mathbf{n}_{11}^T\mathbf{p}) + (\mathbf{x}_\tau \times \mathbf{n}_1)(\mathbf{n}_{12}^T\mathbf{p}) - a(\mathbf{n}_1 \times \mathbf{n}_2)(\mathbf{n}_1^T\mathbf{p}_\tau)]/d \ , \\
\mathbf{p}_{\gamma_2} &= -[(\mathbf{n}_2 \times \mathbf{x}_\tau)(\mathbf{n}_{12}^T\mathbf{p}) + (\mathbf{x}_\tau \times \mathbf{n}_1)(\mathbf{n}_{22}^T\mathbf{p}) - a(\mathbf{n}_1 \times \mathbf{n}_2)(\mathbf{n}_2^T\mathbf{p}_\tau)]/d \ , \\
\mathbf{p}_{\gamma_3} &= \mathbf{p}_\tau \ .
\end{aligned} \tag{52}$$

These are the final general equations for the initial conditions for dynamic ray equations from a *spherical surface* with radius a. They are valid for arbitrarily chosen parameters $u_1 = \gamma_1$ and $u_2 = \gamma_2$, with the exception of singular situations, when $d = 0$.

Taking the limit $a \to 0$, we obtain the initial conditions for dynamic ray equations from a *point source*:

$$\mathbf{x}_{\gamma_1} = \mathbf{0} \ , \quad \mathbf{x}_{\gamma_2} = \mathbf{0} \ , \quad \mathbf{x}_{\gamma_3} = \mathbf{x}_\tau \ , \tag{53}$$

and

$$\begin{aligned}
\mathbf{p}_{\gamma_1} &= -[(\mathbf{n}_2 \times \mathbf{x}_\tau)(\mathbf{n}_{11}^T\mathbf{p}) + (\mathbf{x}_\tau \times \mathbf{n}_1)(\mathbf{n}_{12}^T\mathbf{p})]/d \ , \\
\mathbf{p}_{\gamma_2} &= -[(\mathbf{n}_2 \times \mathbf{x}_\tau)(\mathbf{n}_{12}^T\mathbf{p}) + (\mathbf{x}_\tau \times \mathbf{n}_1)(\mathbf{n}_{22}^T\mathbf{p})]/d \ , \\
\mathbf{p}_{\gamma_3} &= \mathbf{p}_\tau \ .
\end{aligned} \tag{54}$$

These are the final general equations for $\mathbf{Q}$ and $\mathbf{P}$ for a point source. They are valid for arbitrarily chosen parameters $u_1 = \gamma_1$ and $u_2 = \gamma_2$, again for $d \neq 0$. In the following subsections, we shall present examples and deal with the case $d = 0$.

The expressions for $\mathbf{p}_{\gamma_1}$ and $\mathbf{p}_{\gamma_2}$ simplify if we choose the parameters u_1, u_2 in such a way that $\mathbf{n}_{12}^T \mathbf{n}^\Sigma = 0$. Then

$$\mathbf{p}_{\gamma_1} = -(\mathbf{n}_{11}^T \mathbf{p})(\mathbf{n}_2 \times \mathbf{x}_\tau)/d \ , \quad \mathbf{p}_{\gamma_2} = -(\mathbf{n}_{22}^T \mathbf{p})(\mathbf{x}_\tau \times \mathbf{n}_1)/d \ . \tag{55}$$

5.3 Special cases

We shall now present three examples of parametrizations in u_1 and u_2.

5.3.1 Parametrization in spherical polar coordinates

In the first example, we consider take-off angles φ_0 and δ_0 as the parameters u_1, u_2, and specify $\mathbf{n}$ by the following relation:

$$\mathbf{n}(\varphi_0, \delta_0) = \mathbf{i}_1 \cos\varphi_0 \cos\delta_0 + \mathbf{i}_2 \sin\varphi_0 \cos\delta_0 + \mathbf{i}_3 \sin\delta_0 \ . \tag{56}$$

Here $\mathbf{i}_1$, $\mathbf{i}_2$ and $\mathbf{i}_3$ are arbitrarily chosen fixed mutually perpendicular unit vectors. For $\mathbf{n}_1 = \partial\mathbf{n}/\partial\varphi_0$ and $\mathbf{n}_2 = \partial\mathbf{n}/\partial\delta_0$, we obtain

$$\begin{aligned}
\mathbf{n}_1 &= -\mathbf{i}_1 \sin\varphi_0 \cos\delta_0 + \mathbf{i}_2 \cos\varphi_0 \cos\delta_0 \ , \\
\mathbf{n}_2 &= -\mathbf{i}_1 \cos\varphi_0 \sin\delta_0 - \mathbf{i}_2 \sin\varphi_0 \sin\delta_0 + \mathbf{i}_3 \cos\delta_0 \ .
\end{aligned} \tag{57}$$

Note that $\mathbf{n}$ and $\mathbf{n}_2$ are unit vectors, but $\mathbf{n}_1$ is not: $||\mathbf{n}_1|| = |\cos\delta_0|$. For $\delta_0 \to \pm\pi/2$, that is close to the poles, the length of the vector $\mathbf{n}_1$ goes to zero. At the poles themselves, $\delta_0 = \pm\pi/2$, and the approach based on the take-off angles ϕ_0, δ_0 breaks down, since $d = \mathbf{x}_\tau^T(\mathbf{n}_1 \times \mathbf{n}_2)$ goes to zero. For this reason, we assume that δ_0 is not close to $\pm\pi/2$. Then we obtain $\mathbf{n}_1^T \mathbf{n}_2 = 0$, $\mathbf{n}_{11}^T \mathbf{p} = -\cos^2\delta_0/\mathcal{C}$, $\mathbf{n}_{12}^T \mathbf{p} = 0$, $\mathbf{n}_{22}^T \mathbf{p} = -1/\mathcal{C}$, and $d = \mathbf{x}_\tau^T(\mathbf{n}_1 \times \mathbf{n}_2) = \cos\delta_0(\mathbf{x}_\tau^T \mathbf{n})$, where $\mathcal{C}$ is the phase velocity in the direction of $\mathbf{n}$. Equations (55) then simplify to:

$$\mathbf{p}_{\gamma_1} = \cos\delta_0(\mathbf{n}_2 \times \mathbf{x}_\tau)/\mathcal{C}(\mathbf{x}_\tau^T \mathbf{n}) \ , \quad \mathbf{p}_{\gamma_2} = (\mathbf{x}_\tau^T \times \mathbf{n}_1)/\mathcal{C}(\mathbf{x}_\tau^T \mathbf{n}) \cos\delta_0 \ . \tag{58}$$

The vectors $\mathbf{n}_1$, $\mathbf{n}_2$ and $\mathbf{n}$ are mutually perpendicular, and are related as follows: $\mathbf{n}_1 = (\mathbf{n}_2 \times \mathbf{n}) \cos\delta_0$, $\mathbf{n}_2 = (\mathbf{n} \times \mathbf{n}_1)/\cos\delta_0$. Using these expressions, we can alternatively express (58) in the form

$$\mathbf{p}_{\gamma_1} = -\mathcal{C}^{-1}[\mathbf{n}_1 - \mathbf{n}(\mathbf{x}_\tau^T \mathbf{n}_1)/(\mathbf{x}_\tau^T \mathbf{n})] \ , \quad \mathbf{p}_{\gamma_2} = \mathcal{C}^{-1}[\mathbf{n}_2 - \mathbf{n}(\mathbf{x}_\tau^T \mathbf{n}_2)/(\mathbf{x}_\tau^T \mathbf{n})] \ . \tag{59}$$

Since $\mathbf{x}_\tau$ is the ray velocity, $\mathbf{x}_\tau^T \mathbf{n}$ the phase velocity, and $\mathbf{n}/(\mathbf{x}_\tau^T \mathbf{n}) = \mathbf{n}/\mathcal{C} = \mathbf{p}$, the equations (59) yield,

$$\mathbf{p}_{\gamma_1} = -\mathcal{C}^{-1}[\mathbf{n}_1 - \mathbf{p}(\mathbf{x}_\tau^T \mathbf{n}_1)] \ , \quad \mathbf{p}_{\gamma_2} = \mathcal{C}^{-1}[\mathbf{n}_2 - \mathbf{p}(\mathbf{x}_\tau^T \mathbf{n}_2)] \ . \tag{60}$$

Equations (60) with (57) fully coincide with those given by Pšenčík and Teles (1996, Eqs. A.4), obtained by a quite different method.

It is not difficult to show that the cross product $\mathbf{p}_{\gamma_1} \times \mathbf{p}_{\gamma_2}$ is tangent to $\mathbf{x}_\tau$ at the point source. Actually, for our choice of $\mathbf{n}(\varphi_0, \delta_0)$ given by (56), we obtain

$$\mathbf{p}_{\gamma_1} \times \mathbf{p}_{\gamma_2} = \mathbf{x}_\tau \cos\delta_0 / \mathcal{C}^2 \mathbf{x}_\tau^T \mathbf{n} \ . \tag{61}$$

Both $\mathbf{p}_{\gamma_1}$ and $\mathbf{p}_{\gamma_2}$ are perpendicular to $\mathbf{x}_\tau$; $\mathbf{x}_\tau^T \mathbf{p}_{\gamma_\mu} = 0$ at the point source. Along the ray, this relation remains valid only in homogeneous media, but not in inhomogeneous media, see (12).

5.3.2 Factoring out the polar singularity of spherical polar coordinates

In the second example, we again use the take-off angles $u_1 = \phi_0$, $u_2 = \delta_0$, but try to avoid the difficulties with δ_0 close to $\pm\pi/2$. We formally express the vector $\mathbf{n}_1$, given by (57), in the following form:

$$\mathbf{n}_1 = \cos\delta_0 \bar{\mathbf{n}}_1 \ , \quad \bar{\mathbf{n}}_1 = \mathbf{i}_1 \sin\phi_0 + \mathbf{i}_2 \cos\phi_0 \ . \tag{62}$$

Here $\bar{\mathbf{n}}_1$ is a unit vector. Using (62), we obtain

$$\mathbf{p}_{\gamma_1} = -(\cos\delta_0/\mathcal{C})\bar{\mathbf{p}}_{\gamma_1} \ , \quad \mathbf{p}_{\gamma_2} = (1/\mathcal{C})\bar{\mathbf{p}}_{\gamma_2} \ , \tag{63}$$

where

$$\bar{\mathbf{p}}_{\gamma_1} = \bar{\mathbf{n}}_1 - \mathbf{p}(\mathbf{x}_\tau^T \bar{\mathbf{n}}_1) \ , \quad \bar{\mathbf{p}}_{\gamma_2} = \mathbf{n}_2 - \mathbf{p}(\mathbf{x}_\tau^T \mathbf{n}_2) \ . \tag{64}$$

In (63), $\cos\delta_0/\mathcal{C}$ and $1/\mathcal{C}$ are constants. As the dynamic ray equations are linear, solutions with the initial conditions (63) can be obtained from solutions with the initial conditions (64) by merely multiplying the latter solutions by the constants $\cos\delta_0/\mathcal{C}$ and $1/\mathcal{C}$. The dynamic ray equations with the initial conditions (64) behave well, even for δ_0 close or equal to $\pm\pi/2$. Moreover, they have certain fascinating properties. For example, if we compute the ray-theory Green functions in an anisotropic inhomogeneous medium, we need to know the geometrical spreading, calculated from dynamic ray equations with the initial conditions (64), not (63). Červený (2001) calls this geometrical spreading the relative geometrical spreading. It can be proved that the relative geometrical spreading is reciprocal, but the standard geometrical spreading is not. For details, see Kendall et al. (1992), Pšenčík and Teles (1996), and Červený (2001).

5.3.3 Azimuthal equidistant projection

In the third example, we can fully avoid the problem with one of the poles, $u_2 \to \pi/2$, using the so-called *azimuthal equidistant projection* of a unit sphere. It is parametrized by two Cartesian coordinates u_1, u_2, centred at the projection point. The unit vector $\mathbf{n}$, tangent to the initial slowness vector $\mathbf{p}$, expressed in the local Cartesian coordinate system with suitable basis vectors $\mathbf{i}_k$, is given by

$$\mathbf{n}_1 = \mathbf{i}_1 u_1 \sin r/r \ , \quad \mathbf{n}_2 = \mathbf{i}_2 u_2 \sin r/r \ , \quad \mathbf{n}_3 = \mathbf{i}_3 \cos r \ , \tag{65}$$

with $r = \sqrt{u_1^2 + u_2^2}$. The application of the azimuthal equidistant projection of a unit sphere is particularly suitable in seismic reflection studies. The determination of the initial conditions for dynamic ray equations is straightforward and not presented here. The azimuthal equidistant projection is included in the program package CRT (Complete Ray Tracing), available online at "http://sw3d.mff.cuni.cz".

6 Tubular surface and its limit to the initial line

Let us consider a fully general, three-dimensional, smooth initial line C^0, specified by a parametric equation $\mathbf{x} = \mathbf{x}(u_1)$, where u_1 is an arbitrary monotonic parameter along C^0. In this case, matrix $\mathbf{X}$ is degenerate and the equations of Section 4.2 cannot be used. This difficulty is overcome if we use an auxiliary tubular initial surface Σ, with the curve C^0 along its axis, and with radius a. We can then use all equations of Section 4.2, and from these equations we obtain the line source C^0 by a limiting process $a \to 0$.

The tubular surface Σ may be specified as follows:

$$\mathbf{g}(u_1, u_2) = \mathbf{x}(u_1) + a\mathbf{f}(u_1, u_2) \ . \tag{66}$$

Here $\mathbf{f}(u_1, u_2)$ is a unit vector perpendicular to C^0 for any u_1 and u_2: $\mathbf{f}^T\mathbf{x}_1 = 0$. The second parameter u_2 specifies the direction of $\mathbf{f}$ in the plane perpendicular to C^0 at u_1 ; $\mathbf{f}$ may be considered periodic in u_2: $\mathbf{f}(u_1, u_2 + 2\pi) = \mathbf{f}(u_1, u_2)$. Further, $\mathbf{f}$ is continuous and has continuous first and second derivatives with respect to u_1 and u_2. The travel time along Σ is a function of u_1 only, $T^0 = T^0(u_1)$ and $T^0_2 = T^0_{12} = T^0_{22} = 0$.

In the following treatment of a tubular surface Σ, we shall consider only the leading terms in powers of a. The reason is that all other terms will vanish in the limiting process $a \to 0$. We obtain:

$$\begin{array}{lll} \mathbf{g}_1 = \mathbf{x}_1 \ , & \mathbf{g}_2 = a\mathbf{f}_2 \ , & \\ \mathbf{g}_{11} = \mathbf{x}_{11} \ , & \mathbf{g}_{12} = a\mathbf{f}_{12} \ , & \mathbf{g}_{22} = a\mathbf{f}_{22} \ . \end{array} \tag{67}$$

Here, the subscript 1 denotes the partial derivative with respect to u_1, and the subscript 2 the partial derivative with respect to u_2.

6.1 Kinematic ray equations

We first determine the initial conditions for kinematic ray equations, particularly for the initial slowness vector $\mathbf{p}$. The unit normal $\mathbf{n}^\Sigma$ to Σ and the tangential component $\mathbf{p}^\Sigma$ of the slowness vector $\mathbf{p}$ are given by (17) and (19). Taking into account that $\mathbf{x}_1^T\mathbf{f}_2 = 0$, we obtain:

$$\mathbf{n}^\Sigma = (\mathbf{x}_1 \times \mathbf{f}_2)/||\mathbf{x}_1 \times \mathbf{f}_2|| = -\mathbf{f} \ , \quad \mathbf{p}^\Sigma = T^0_1\mathbf{x}_1/||\mathbf{x}_1||^2 \ . \tag{68}$$

The slowness vector $\mathbf{p}$ is then given by the expression (22), which is also valid for $a \to 0$. Equation (22) is sufficient to start the kinematic ray tracing. We can also determine the initial values of $\mathbf{x}_\tau$ and $\mathbf{p}_\tau$ from (25).

We can also express $\mathbf{p}$ in the form:

$$\mathbf{p} = \sigma \mathbf{n}^{\Sigma} + T_s^0 \mathbf{t} \ , \quad \mathbf{t} = \mathbf{x}_1 / ||\mathbf{x}_1|| \ . \tag{69}$$

Here, $\mathbf{n}^{\Sigma}$ and $\mathbf{t}$ are two mutually perpendicular unit vectors. The unit vector $\mathbf{t}$ is tangential to the initial line C^0, and the unit vector $\mathbf{n}^{\Sigma}$ is perpendicular to C^0. The quantity T_s^0 is the slowness corresponding to the initial travel time T^0 along the initial line C^0, $T_s^0 = T_1^0/||\mathbf{x}_1|| = \partial T^0/\partial s$, where s is the arclength along C^0. Finally, σ should be determined from the eikonal equation $\mathcal{H}(\mathbf{x}, \mathbf{p}) = 0$, see (22) and the text following it.

Consequently, for every $\mathbf{n}^{\Sigma}$, the initial slowness vector $\mathbf{p}$ is situated in a plane specified by unit vectors $\mathbf{t}$ and $\mathbf{n}^{\Sigma}$, and the one-parametric system of initial slowness vectors at any fixed point $S \in C^0$, may take two forms:

a) If the initial travel time T^0 is constant close to $S \in C^0$, the quantity T_s^0 is zero, and $\mathbf{p} = \sigma \mathbf{n}^{\Sigma}$. In this case we can speak of an *exploding initial line* C^0 (or *exploding initial wire* in electromagnetism). Then all initial slowness vectors at S are situated in the plane perpendicular to C^0 at S. In an isotropic medium, the loci of the end points of the initial slowness vectors form a circle, with its centre at S. In an anisotropic medium, the initial slowness vectors at S are again situated in the plane perpendicular to C^0 at S, but the loci of their end points do not form a circle, but may be more complicated (similar to the phase velocity polar diagram). The dynamic ray equations for the case of an exploding line are given in Section 6.3.2.

b) If the travel time T^0 is a variable function of u_1 in the vicinity of S, the quantity T_s^0 is generally non-zero. Then the one-parametric system of initial slowness vectors at S is not situated in the plane perpendicular to C^0 at S, but the initial slowness vectors at S are *inclined* in the direction of $\mathbf{t}$, see (69). Consequently, the one-parametric system of initial slowness vectors at S forms a conical surface, with its apex at S. The opening or apex angle may be different for each $\mathbf{n}^{\Sigma}$ and $\mathbf{t}$, see (69). The loci of the one-parametric system of initial slowness vectors at S form again a circle in isotropic media, but this circle is generally not situated in the plane perpendicular to C^0 at S. In anisotropic media, they form a more complicated curve.

Once we know the initial slowness vector for given u_1 and u_2, we can also compute the ray-velocity vector $\mathbf{x}_\tau$ at the initial point $S \in C^0$, see (25). The behaviour of the one-parametric system of ray-velocity vectors at S is not as simple as the behaviour of the one-parametric system of initial slowness vectors. The reason is that $\mathbf{x}_\tau$ and $\mathbf{p}$ are not parallel in anisotropic media. Only in isotropic media is $\mathbf{x}_\tau$ parallel to $\mathbf{p}$, and the behaviour of $\mathbf{x}_\tau$ is the same as the above described behaviour of the initial slowness vectors. In anisotropic media, the one-parametric system of $\mathbf{x}_\tau$ is situated strictly in the plane perpendicular to C^0 at S for $T_s^0 = 0$ only in exceptional cases. Moreover, the loci of the end-points of $\mathbf{x}_\tau$ form a curve closer to the ray-velocity polar diagram. For fixed $\mathbf{n}^{\Sigma}$ and $\mathbf{t}$, we can also obtain several $\mathbf{x}_\tau$'s. Thus, the loci of end points of $\mathbf{x}_\tau$ may form loops and cuspoidal points. For $T_s^0 \neq 0$, the one-parametric system of $\mathbf{x}_\tau$ at S again forms a conical surface with the apex at S, which may be rather complicated, possibly multivalued, forming loops and cuspoidal points.

6.2 Dynamic ray equations

For $\mathbf{h}_i$, we use (35), and obtain:

$$\mathbf{h}_1 = (\mathbf{f}_2 \times \mathbf{x}_\tau)/F \ , \quad \mathbf{h}_2 = (\mathbf{x}_\tau \times \mathbf{x}_1)/aF \ , \quad \mathbf{h}_3 = (\mathbf{x}_1 \times \mathbf{f}_2)/F \ , \tag{70}$$

where $F = \mathbf{x}_\tau^T(\mathbf{x}_1 \times \mathbf{f}_2)$. Taking $a \to 0$, we obtain from (39) the final results for $\mathbf{Q}$ and $\mathbf{P}$:

$$\mathbf{x}_{\gamma_1} = \mathbf{x}_1 - T_1^0\mathbf{x}_\tau \ , \quad \mathbf{x}_{\gamma_2} = \mathbf{0} \ , \quad \mathbf{x}_{\gamma_3} = \mathbf{x}_\tau \ , \tag{71}$$

and

$$\begin{aligned}
\mathbf{p}_{\gamma_1} &= [(\mathbf{f}_2 \times \mathbf{x}_\tau)(T_{11}^0 - \mathbf{x}_{11}^T\mathbf{p}) - (\mathbf{x}_\tau \times \mathbf{x}_1)(\mathbf{f}_{12}^T\mathbf{p}) + (\mathbf{x}_1 \times \mathbf{f}_2)(\mathbf{x}_1^T\mathbf{p}_\tau)]/F - T_1^0\mathbf{p}_\tau \ , \\
\mathbf{p}_{\gamma_2} &= -(\mathbf{x}_\tau \times \mathbf{x}_1)(\mathbf{f}_{22}^T\mathbf{p})/F \ , \\
\mathbf{p}_{\gamma_3} &= \mathbf{p}_\tau \ .
\end{aligned} \tag{72}$$

These are the final expressions for the initial conditions of $\mathbf{Q}$ and $\mathbf{P}$ for dynamic ray tracing from the line C^0. The choice of $\mathbf{f}$ is quite arbitrary; we only require $\mathbf{f}(u_1, u_2)$ to be a unit vector perpendicular to $\mathbf{x}_1(u_1)$ for any given u_1 and u_2.

We shall now consider an important choice of $\mathbf{f}(u_1, u_2)$, suitable in applications:

$$\mathbf{f}(u_1, u_2) = \mathbf{v}(u_1)\cos u_2 + \mathbf{w}(u_1)\sin u_2 \ , \tag{73}$$

where $\mathbf{v}(u_1)$ and $\mathbf{w}(u_1)$ are two mutually perpendicular unit vectors, perpendicular to $\mathbf{x}_1(u_1)$. We obtain then

$$\mathbf{f}_2 = -\mathbf{v}\sin u_2 + \mathbf{w}\cos u_2 \ , \quad \mathbf{f}_{22} = -\mathbf{f} \ , \quad \mathbf{f}_{12} = -\mathbf{v}_1 \sin u_2 + \mathbf{w}_1 \cos u_2 \ . \tag{74}$$

The unit vectors $\mathbf{v}$ and $\mathbf{w}$ can be chosen in different ways. Once one of them, for example $\mathbf{v}(u_1)$, is specified, the second one is determined by the relation $\mathbf{w}(u_1) = \mathbf{t}(u_1) \times \mathbf{v}(u_1)$, where $\mathbf{t}(u_1) = \mathbf{x}_1(u_1)/||\mathbf{x}_1(u_1)||$ is the unit tangent vector to Σ^0. We shall briefly mention two possibilities of specifying $\mathbf{v}(u_1)$.

1) Consider the problem of edge waves, where the line C^0 is an intersection of two known surfaces Σ^A and Σ^B. We can determine the normal vectors $\mathbf{n}^A(u_1)$ and $\mathbf{n}^B(u_1)$ to the surfaces Σ^A and Σ^B at any point u_1 of C^0. We can then determine $\mathbf{v}(u_1)$ in the plane specified by $\mathbf{n}^A$ and $\mathbf{n}^B$ (perpendicular to C^0), arbitrarily related to $\mathbf{n}^A$ and/or $\mathbf{n}^B$. Numerical procedures should be used to determine the derivatives $\mathbf{v}_1 = \partial\mathbf{v}/\partial u_1$ and $\mathbf{w}_1 = \partial\mathbf{w}/\partial u_1$.

2) Consider a line C^0 situated in a smooth medium. We can then use $\mathbf{v} = \mathbf{n}$ and $\mathbf{w} = \mathbf{b}$, where $\mathbf{n}$ is the main normal and $\mathbf{b}$ the binormal to C^0 (defined by $\mathbf{n} = (\partial\mathbf{t}/\partial u_1)/||\partial\mathbf{t}/\partial u_1||$ and $\mathbf{b} = \mathbf{t} \times \mathbf{n}$). If $\mathbf{x}(u_1)$ is known analytically, $\mathbf{f}_{12}$ can be also determined analytically. The price of this, however, is that we have to compute the main curvature $\mathcal{K}$ and torsion $\mathcal{T}$ along C^0 (following Frenet's formulae):

$$\mathbf{v}_1 = ||\mathbf{x}_1||(\mathcal{T}\mathbf{b} - \mathcal{K}\mathbf{t}) \ , \quad \mathbf{w}_1 = -||\mathbf{x}_1||\mathcal{T}\mathbf{n} \ , \quad \mathbf{n}^\Sigma = -\mathbf{f} \ . \tag{75}$$

This yields

$$\mathbf{f}_{12} = \mathcal{K}\mathbf{x}_1 \sin u_2 - \mathcal{T}||\mathbf{x}_1||\mathbf{f} \ . \tag{76}$$

3) Take a fixed vector $\mathbf{e}$, such that $\mathbf{x}_1 \times \mathbf{e} \neq \mathbf{0}$ for the region of interest. Define $\mathbf{v}(u_1) = \mathbf{x}_1(u_1) \times \mathbf{e}/||\mathbf{x}_1(u_1) \times \mathbf{e}||$ and $\mathbf{w}(u_1) = \mathbf{t}(u_1) \times \mathbf{v}(u_1)$. Then $\mathbf{f}$, as given by (73) depends smoothly on u_1 and u_2, with $\mathbf{f}^T\mathbf{x}_1 = \mathbf{g}_1^T\mathbf{g}_2 = 0$, and $\mathbf{f}_2$, $\mathbf{f}_{12}$, and $\mathbf{f}_{22}$ are obtained by differentiation.

6.3 Special cases

The expressions for the initial conditions for dynamic ray equations for an initial line C^0, given in Section 6.2 are very general. They may be applied to a three-dimensional smooth initial line with non-vanishing torsion and curvature, parametrized by any monotonic parameter u_1 along C^0. The medium in the vicinity of the initial line may be spatially varying, isotropic or anisotropic. The initial line, forming an intersection of two surfaces (edge) is also considered.

In this section, we treat the initial line C^0 situated in a smooth medium, and give compact solutions for several special cases. In all cases, we only give the expressions for $\mathbf{p}_{\gamma_1}$ and $\mathbf{p}_{\gamma_2}$ as the remaining expressions (for $\mathbf{p}_{\gamma_3}$, $\mathbf{x}_{\gamma_1}$, $\mathbf{x}_{\gamma_2}$ and $\mathbf{x}_{\gamma_3}$) are straightforward, see (71) and (72).

6.3.1 Parametrization of the initial line by arclength

Here we consider the natural parametrization u_1 along C^0, namely by arclength s, $u_1 = s$. Then $\mathbf{x}_1 = \mathbf{t}$, where $\mathbf{t}$ is the unit vector tangent to C^0. We further obtain $||\mathbf{x}_1|| = 1$ and the following relations:

$$\begin{array}{ll} \mathbf{f}_{12} = \mathcal{K}\mathbf{t}\sin u_2 - \mathcal{T}\mathbf{f} \quad , & \mathbf{x}_{11} = \mathcal{K}\mathbf{n} \ , \\ \mathbf{t} \times \mathbf{f}_2 = -\mathbf{f} \quad , & F = -\mathbf{x}_\tau^T\mathbf{f} \ . \end{array} \tag{77}$$

We also take into account the expression (69) for the slowness vector $\mathbf{p}$ (where $T_s^0 = \partial T^0/\partial s$), and obtain:

$$\mathbf{f}^T\mathbf{p} = -\sigma \ , \quad \mathbf{t}^T\mathbf{p} = T_s^0 \ , \quad \mathbf{n}^T\mathbf{p} = -\sigma\cos u_2 \ . \tag{78}$$

Inserting (77) and (78) into (72), we obtain the final expressions for $\mathbf{p}_{\gamma_1}$ and $\mathbf{p}_{\gamma_2}$:

$$\begin{array}{rcl} \mathbf{p}_{\gamma_1} & = & [(\mathbf{x}_\tau \times \mathbf{f}_2)(T_{11}^0 + \mathcal{K}\sigma\cos u_2) + (\mathbf{x}_\tau \times \mathbf{t})(\mathcal{K}T_s^0 \sin u_2 + \mathcal{T}\sigma) + \mathbf{f}(\mathbf{t}^T\mathbf{p}_\tau)]/\mathbf{x}_\tau^T\mathbf{f} - T_s^0\mathbf{p}_\tau \ , \\ \mathbf{p}_{\gamma_2} & = & \sigma(\mathbf{x}_\tau \times \mathbf{t})/\mathbf{x}_\tau^T\mathbf{f} \ . \end{array} \tag{79}$$

For completeness, the expressions for $\mathbf{f}$ and $\mathbf{f}_2$ are:

$$\mathbf{f} = \mathbf{n}\cos u_2 + \mathbf{b}\sin u_2 \ , \quad \mathbf{f}_2 = -\mathbf{n}\sin u_2 + \mathbf{b}\cos u_2 \ . \tag{80}$$

We remind the reader that $\mathbf{n}$ is a unit normal, $\mathbf{b}$ the unit binormal, $\mathcal{K}$ the curvature and $\mathcal{T}$ the torsion of C^0, and $T_s^0 = \partial T^0/\partial s$. Note that σ depends on elastic moduli, on T_s^0, and on two unit vectors $\mathbf{f}$ and $\mathbf{t}$.

6.3.2 Exploding initial line

In this case, $T_{11}^0 = 0$ and $T_s^0 = 0$. The general expression (79) for $\mathbf{p}_{\gamma_1}$ then yields:

$$\mathbf{p}_{\gamma_1} = [(\mathbf{x}_\tau \times \mathbf{f}_2)\mathcal{K}\sigma \cos u_2 + (\mathbf{x}_\tau \times \mathbf{t})\mathcal{T}\sigma + \mathbf{f}(\mathbf{t}^T \mathbf{p}_\tau)]/\mathbf{x}_\tau^T \mathbf{f} \ . \tag{81}$$

Here, $\mathbf{f}$ and $\mathbf{f}_2$ are given by (80); the equation for $\mathbf{p}_{\gamma_2}$ remains the same as in (79). In this case, the quantity σ depends only on elastic moduli.

6.3.3 Planar two-dimensional line

In this case the torsion $\mathcal{T}$ is zero, $\mathcal{T} = 0$. The simplification of (79) and (81) (for variable and constant initial travel time, respectively) in this case is straightforward.

6.3.4 Straight initial line

In this case, the curvature $\mathcal{K}$ and torsion $\mathcal{T}$ are both zero, $\mathcal{K} = \mathcal{T} = 0$. The expression for $\mathbf{p}_{\gamma_1}$ now simplifies considerably:

$$\mathbf{p}_{\gamma_1} = [(\mathbf{x}_\tau \times \mathbf{f}_2)T_{11}^0 + \mathbf{f}(\mathbf{t}^T \mathbf{p}_\tau)]/\mathbf{x}_\tau^T \mathbf{f} - T_s^0 \mathbf{p}_\tau \ . \tag{82}$$

The expression for $\mathbf{p}_{\gamma_2}$ remains the same as in (79). If also $T_{11}^0 = 0$ and $T_s^0 = 0$, $\mathbf{p}_{\gamma_1}$ becomes very simple.

6.3.5 Homogeneous media

In a homogeneous medium, $\mathbf{p}_\tau = \mathbf{0}$. This slightly simplifies equation (79) for $\mathbf{p}_{\gamma_1}$. The initial conditions (79) for the dynamic ray equations (8) are, however, very important in this case, as the dynamic ray equations can be solved analytically, see (9).

6.3.6 Isotropic media

In isotropic media, we have

$$\begin{array}{lll} \mathbf{p} = -\mathbf{f}\sigma + T_s^0 \mathbf{t} \ , & \mathbf{p}^T \mathbf{p} = 1/V^2 \ , & \sigma = \sqrt{1/V^2 - (T_s^0)^2} \ , \\ \mathbf{p}_\tau = -V^{-1}\nabla V \ , & \mathbf{x}_\tau = (-\mathbf{f}\sigma + T_s^0 \mathbf{t})V^2 \ . & \end{array} \tag{83}$$

Also

$$\mathbf{x}_\tau \times \mathbf{t} = V^2 \sigma \mathbf{f}_2 \ , \quad \mathbf{x}_\tau \times \mathbf{f}_2 = -V^2(\sigma \mathbf{t} + T_s^0 \mathbf{f}) \ , \quad \mathbf{x}_\tau^T \mathbf{f} = -\sigma V^2 \ . \tag{84}$$

Inserting these expressions into (79), we obtain the final equations for $\mathbf{p}_{\gamma_1}$ and $\mathbf{p}_{\gamma_2}$:

$$\begin{aligned} \mathbf{p}_{\gamma_1} &= \mathbf{f}\sigma^{-1}(T_s^0 T_{ss}^0 + V^{-3}\mathbf{t}^T \nabla V) + \sigma(\mathcal{K}\cos u_2 \mathbf{t} - \mathcal{T}\mathbf{f}_2) + \mathcal{K}T_s^0 \mathbf{n} + T_{ss}^0 \mathbf{t} + T_s^0 V^{-1}\nabla V \ , \\ \mathbf{p}_{\gamma_2} &= -\sigma \mathbf{f}_2 \ . \end{aligned} \tag{85}$$

Here, $\mathbf{f}$ and $\mathbf{f}_2$ are given by (80).

We assume here that the function $\sigma = \sqrt{1/V^2 - (T_s^0)^2}$ is real-valued. In other words, we require $V < 1/T_s^0$. The physical explanation of this requirement is that the medium velocity V must be smaller than the velocity $1/T_s^0$ along the line C^0. If this requirement is not satisfied, the generated waves would be inhomogeneous, which we do not consider in this paper. For $V = 1/T_s^0$, we obtain $\sigma = 0$, and the equation (85) breaks down.

Initial conditions for dynamic ray equations in isotropic media at an initial line were also derived by Červený (2001, sec. 4.5.5), using ray-centred coordinates, instead of the Cartesian coordinates used in this paper. Actually, Červený (2001) also treated dynamic ray equations in Cartesian coordinates as a prerequisite for dynamic ray equations in ray-centred coordinates; see p. 321, unnumbered equations for $\mathbf{p}_{,1}$ and $\mathbf{p}_{,2}$. These results fully correspond to our equations (85) for $\mathbf{p}_{\gamma_1}$ and $\mathbf{p}_{\gamma_2}$. Some differences in the signs of certain terms are caused by the fact that Červený (2001) uses $\mathbf{n}^\Sigma$ (denoted by $\mathbf{n}^0$) instead of $\mathbf{f}$, and that $\mathbf{n}^\Sigma = -\mathbf{f}$, see (75). Thus, our equations (85) for isotropic media are confirmed by independent computations of Červený (2001), and equations (72) represent their generalization to anisotropic media.

7 Concluding remarks

The expressions for the initial conditions for dynamic ray equations, derived in this paper, are very general. They can be applied to rays with initial points situated at a smooth initial surface, at a smooth initial line, or in a smooth medium. They are valid for isotropic or anisotropic, homogeneous or inhomogeneous media. The initial surface may be planar or curved; the initial line may be three-dimensional, with non-vanishing curvature and torsion. A non-vanishing initial travel time, which may vary with position, may be considered along the initial surface and initial line. As a special case, exploding initial surface and exploding initial line can be used. Along the initial surface, general non-orthogonal (Gaussian) coordinates or local Cartesian coordinates at the initial point of the ray may be introduced. The coordinates along the initial line may also be introduced in several ways.

The dynamic ray equations and, consequently, also the initial conditions for them, play a basic role in the paraxial ray methods. They are very important in computing geometrical spreading and, consequently, ray amplitudes. They are also required in computing approximately paraxial rays, paraxial travel times, paraxial slowness vectors, paraxial ray-velocity vectors, etc., in the vicinity of the reference ray. They can be used to compute the curvature of the wavefront, the ray Jacobian, the phase shift due to caustics, higher-order derivatives of travel times, etc. They have found important applications in two-point ray tracing, in the solution of various boundary-value problems, in ray-perturbation methods, in the investigation of chaotic rays (Lyapunov exponents), in the Gaussian beam and Gaussian packet summation methods, in the Maslov seismogram method, in the solution of various diffraction problems, etc.

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