Perturbation of phase normal in weakly anisotropic media - a preliminary study [1]

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Summary

The relation between phase normal and ray vector (unit vector in the direction of energy flux) and the expression for the ray velocity vector in an arbitrary weakly anisotropic medium is derived. The relation allows unique determination of the ray vector for a given phase normal and vice versa. In weakly anisotropic media, phase normal and ray vector differ by a first-order quantity. This means that in any expression containing a product of one of the mentioned vectors with a first-order quantity, both vectors are interchangable. The relation between phase normal and ray vector is linear. Therefore, the relation cannot describe, in principle, triplication of qS wavefront.

1 Introduction

Perturbation formulae derived by Jech and Pšenčík (1989) and specified for weakly anisotropic media yield the phase normal n_i, whose direction is in the weakly anisotropic medium the same as in the background isotropic medium. In this way, the ray vector in the anisotropic medium differs from the ray vector in the background isotropic medium. There are, however, many situations, in which, we would prefere to keep the ray vector fixed and change the phase normal accordingly. A simple example of such an situation are tomographic studies. In them, we wish to evaluate wave quantities along rays in the background medium, i.e. we want to evaluate the quantities for a given ray direction rather than for the given phase normal direction. Another example, in which the formula for a fixed phase normal cannot be used is the determination of quantities corresponding to waves generated by an incidence of a wave at an interface. The phase normals of the generated waves in the weakly anisotropic medium differ from the phase normals in the background isotropic medium. From these reasons, in the following we study the effects of the perturbation of the phase normal on the phase velocities and polarization vectors in weakly anisotropic media.

The used lowercase indices attain values 1, 2, 3, the uppercase indices only values 1

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and 2. Einstein summation convention is used over repeating indices.

2 Formulae for the phase velocity and polarization vectors

2.1 Formulae for a fixed phase normal

Let us consider a tensor of the density normalized elastic parameters a_{ijkl}, which describes a weakly anisotropic medium. Let us express the tensor a_{ijkl} in the following way

$$a_{ijkl} = a^0_{ijkl} + \Delta a_{ijkl}, \tag{1}$$

where a^0_{ijkl} is a tensor of density-normalized elastic parameters in a background isotropic medium and Δa_{ijkl} is its small perturbation. The tensor a^0_{ijkl} has the well-known form

$$a^0_{ijkl} = (\alpha^2 - 2\beta^2)\delta_{ij}\delta_{kl} + \beta^2(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}). \tag{2}$$

In (2), α and β denote P and S wave velocities in the background isotropic medium. The above perturbation of elastic properties of the medium produces perturbations in quantities describing wave propagation in the medium. For a fixed phase normal direction, i.e. for the wavefield whose phase normal is the same in the background isotropic and weakly anisotropic medium, the equations given by Jech & Pšenčík (1989) reduce to the following formulae. A perturbation Δc of the phase velocity from the background isotropic to an arbitrary weakly anisotropic medium is given by

$$\Delta c^{(3)} = \frac{1}{2\alpha} B_{33} \tag{3a}$$

for qP waves and by

$$\Delta c^{(1)} = \frac{1}{2\beta} B_{11}, \quad \Delta c^{(2)} = \frac{1}{2\beta} B_{22} \tag{3b}$$

for qS waves.

The perturbation $\Delta g_i^{(3)}$ of the P wave polarization vector is given by

$$\Delta g_i^{(3)} = \frac{B_{13}e_{1i} + B_{23}e_{2i}}{\alpha^2 - \beta^2}. \tag{4a}$$

The perturbations $\Delta g_i^{(K)}$ of the qS wave polarization vectors are given by

$$\Delta g_i^{(1)} = \frac{B_{13}n_i}{\beta^2 - \alpha^2}, \quad \Delta g_i^{(2)} = \frac{B_{23}n_i}{\beta^2 - \alpha^2}. \tag{4b}$$

Note that in contrast to Jech & Pšenčík (1989), we consider here only the components of the perturbations of the qS waves polarization vectors into the phase normal n_i. The components of the perturbations into the vectors e_{1i} and e_{2i} are of a higher order and, therefore, are not considered here. In Eqs.(3), (4), we used the following notation. The elements of the matrix B_{mn} are given as

$$B_{mn} = \Delta\Gamma_{jk} e_{mj} e_{nk} \quad . \tag{5}$$

The symbol $\Delta\Gamma_{jk}$ represents a perturbation of the Christoffel matrix

$$\Gamma_{jk} = a_{ijkl} n_i n_l. \tag{6}$$

The symbols e_{1i}, e_{2i} and e_{3i} denote three mutually perpendicular unit vectors in the background isotropic medium. The P wave polarization vector e_{3i} is parallel to the unit phase normal n_i, i.e., $e_{3i} = n_i$ since in an isotropic medium phase normal and polarization vector of a P wave are identical. In case of qP waves, the vectors e_{1i} and e_{2i} can be arbitrarily chosen in the plane perpendicular to n_i. In case of qS waves, we consider them chosen so that the element B_{12} of the matrix B_{mn} is zero. Only such vectors e_{1i} and e_{2i} can be used for the determination of the perturbed qS waves polarization vectors. For details, see Jech & Pšenčík (1989).

Similarly as Jech & Pšenčík (1989), we first assume that the perturbation of the Christoffel matrix is only caused by perturbations of elastic parameters a_{ijkl}. The phase normal appearing in the Christoffel matrix, see Eq.(6) remains fixed, the same in the background isotropic and anisotropic medium. In such a case, equation (5) defining the matrix B_{mn}, can be rewritten in the following way

$$B_{mn} = (a_{ijkl} - a^0_{ijkl}) n_i n_l e_{mj} e_{nk} = a_{ijkl} n_i n_l e_{mj} e_{nk} - c_0^2 \delta_{mn}. \tag{7}$$

Here c_0 denotes a phase velocity in the unperturbed medium. For $m = n = 3$, we have $c_0 = \alpha$ and for $m = n = 1$ or 2, we have $c_0 = \beta$. Note that in Eq.(7), the elements of the matrix B_{mn} are expressed in terms of elastic parameters of a weakly anisotropic medium instead of the perturbations Δa_{ijkl}.

2.2 Formulae for a perturbed phase normal

In many situations perturbations of phase velocities or of polarization vectors are sought for the phase normal, which in the anisotropic medium is different from the phase normal in the background medium. For example, we often seek quantities on a ray. Since in an anisotropic medium, the ray direction and the phase normal generally differ, such a phase normal must be sought which yields the required ray direction.

Let us assume that the deviation Δn_i of the phase normal from its value in the background isotropic medium is a first-order quantity. The perturbation $\Delta\Gamma_{jk}$ of the Christoffel matrix has, in this case, the form

$$\Delta\Gamma_{jk} = \Delta a_{ijkl} n_i n_l + a^0_{ijkl} \Delta n_i n_l + a^0_{ijkl} n_i \Delta n_l. \tag{8}$$

Let us present formulae for the quantities B_{mn} corresponding to such a perturbation of the Christoffel matrix. In order to distinguish quantities related to the perturbation of the Christoffel matrix (8) from the quantities specified for a fixed direction of n_i and thus related to Eq.(7), we denote the former quantities by a bar. From Eqs.(5) and (8), we simply get

$$\bar{B}_{11} = B_{11}, \quad \bar{B}_{22} = B_{22}, \quad \bar{B}_{33} = B_{33}, \quad \bar{B}_{12} = B_{12},$$
$$\bar{B}_{13} = B_{13} + (\alpha^2 - \beta^2)(e_{1i} \Delta n_i), \quad \bar{B}_{23} = B_{23} + (\alpha^2 - \beta^2)(e_{2i} \Delta n_i), \tag{9}$$

As we can see, the formulae (3) for the perturbation of phase velocity remain unchanged by the phase normal perturbation since the diagonal elements of the matrix B_{mn} are unchanged, see (9). Formulae (4) for the perturbation of the polarization vector, however, are affected by the perturbation of the phase normal. For the perturbation of the qP wave polarization vector, formulae (4a) and (9) yield

$$\Delta \bar{g}_i^{(3)} = \frac{B_{13}e_{1i} + B_{23}e_{2i}}{\alpha^2 - \beta^2} + (e_{1k}\Delta n_k)e_{1i} + (e_{2k}\Delta n_k)e_{2i} = \Delta g_i + \Delta n_i. \tag{10a}$$

For the perturbation of the K-th qS wave polarization vector we get from (4b) and (9)

$$\Delta \bar{g}_i^{(K)} = \frac{B_{K3}n_i}{\beta^2 - \alpha^2} - (e_{Kk}\Delta n_k)n_i = \Delta g_i + (\Delta e_{Kk}n_k)n_i = \Delta g_i + \Delta e_{Ki}. \tag{10b}$$

In deriving Eq.(10b), we used the obvious identity $\Delta e_{Ik}n_k = -e_{Ik}\Delta n_k$, following from the perpendicularity of perturbed vectors n_i and e_{Ii}, and the fact that the perturbation Δe_{Ii} has only nonzero projection into the phase normal, see Appendix A. For both qP and qS waves we arrived to an interesting result. If the phase normal is perturbed by a first-order vector Δn_i, then the perturbations of the corresponding polarization vectors consist of terms (4), corresponding to unperturbed phase normal, and of perturbed background polarization vectors $e_{mi} + \Delta e_{mi}$. In other words, in case of the perturbed phase normal, we can use the same formulae (4) as for the case of the fixed phase normal. We have only to substitute the corresponding background polarization vector by a perturbed one. This is very important finding since it allows an easy transformation of results obtained for a fixed phase normal into the results for a perturbed phase normal.

In the next section, we seek a special perturbation Δn_i of the phase normal, which yields the ray vector in the anisotropic medium coinciding with the ray vector in the background isotropic medium. To find it, we must first determine the relation between the phase normal and the ray vector in weakly anisotropic media. Under the ray vector, we understand a unit vector in the direction of the ray velocity.

3 Relation between phase normal and ray vector

We start from the well-known formula for the ray velocity vector v_i, see e.g. Červený et al.(1977), Gajewski and Pšenčík (1990):

$$v_i = vN_i = \frac{dx_i}{dt} = c^{-1}a_{ijkl}n_l g_j g_k. \tag{11}$$

Here, v and c are ray and phase velocities and g_i denotes a polarization vector of a considered wave. The ray vector N_i and the phase vector n_i generally differ. This can be simply shown if we consider the following expression for the phase velocity, see also the above references:

$$c^2 = a_{ijkl}n_i n_l g_j g_k. \tag{12}$$

Multiplying Eq.(11) by the phase normal n_i and taking into account Eq.(12), we find a useful and well-known identity

$$n_i v_i = c. \tag{13}$$

The determination of the ray vector N_i from the phase normal n_i can be performed with the use of relation (11). Determination of n_i from N_i, is however, much more complicated task and it is usually solved by an iterative use of the equation (11), see e.g. Jech (1991). As shown later, in the first-order approximation of the perturbation theory the relation between the vectors n_i and N_i is rather simple and can be written explicitely.

Let us consider the tensor a_{ijkl} given in Eq.(1) with isotropic background specified by Eq.(2). The ray velocity vector v_i can be also expressed as a sum of the ray velocity vector in the background isotropic medium and of its perturbation

$$v_i = v_{0i} + \Delta v_i = c_0 n_i + \Delta v_i. \tag{14}$$

In (14), the ray velocity vector v_{0i} is substituted by the term $c_0 n_i$ because in an isotropic medium the phase and ray velocities are equal in their size and direction. Indices 0 indicate that the corresponding quantities are related to the background isotropic medium. The phase normal n_i in (14) is written intentionally without such an index because we consider it fixed and thus the same in the isotropic as well as in anisotropic medium.

Before proceeding to the specification of Δv_i, let us estimate the size v of the ray velocity vector v_i given by Eq.(14). To the first-order, we get

$$v = v_0 + \Delta v = c_0 + N_i \Delta v_i. \tag{15}$$

The perturbation Δv_i of the group velocity vector v_i can be found by perturbing Eq.(11) from an isotropic background to a weakly anisotropic medium. For the perturbation of the ray velocity of the m-th wave, $\Delta v_i^{(m)}$, we get

$$\Delta v_i^{(m)} = -\Delta c^{(m)} n_i + (c_0^{(m)})^{-1} [a_{ijkl} n_l e_{mj} e_{mk} - \beta^2 n_i$$

$$+(\alpha^2 - \beta^2)\Big((n_k e_{mk})\Delta g_i^{(m)} - (n_k e_{mk}) g_i^{(m)} + (n_k \Delta g_k^{(m)}) e_{mi}\Big)]. \tag{16}$$

The projection of the perturbation of the ray velocity vector (16) into the vector n_i leads to

$$n_i \Delta v_i = \Delta c. \tag{17}$$

Where it is not necessary, we omit the superscripts (m). In deriving Eq.(17), we took into account Eqs.(12) and (13) specified for the background isotropic medium, Eq.(3) for perturbation of the corresponding phase velocity Δc and an obvious identity $e_{mj}\Delta g_j^{(m)} = 0$. Combining Eqs.(14) and (17), we can thus express the group velocity vector in the following form

$$v_i = (c_0 + \Delta c) n_i + A_I e_{Ii}. \tag{18}$$

The quantity A_I represents a projection of the ray velocity vector v_i into a vector e_{Ii} and is given by the formula simply following from (16)

$$A_I^{(m)} = (c_0^{(m)})^{-1} [a_{ijkl} n_l e_{mj} e_{mk} e_{Ii}$$

$$+(\alpha^2 - \beta^2)\Big((n_k e_{mk})(\Delta g_i^{(m)} e_{Ii}) - (n_k e_{mk})(e_{mi} e_{Ii}) + (n_k \Delta g_k^{(m)})(e_{mi} e_{Ii})\Big)]. \tag{19}$$

We can see that A_I is a first-order quantity.

Taking into account that $v_i = vN_i$, see Eq.(11), and keeping only terms upto the first order, we can find from (18)

$$N_i = n_i + \frac{n_i}{c_0}(\Delta c - N_k \Delta v_k) + c_o^{-1} A_i e_{Ii}. \tag{20}$$

Equation (20) clearly shows that the difference between the vectors n_i and N_i is of the first order. This means that everywhere, where a first-order quantity is multiplied by N_i or n_i, both vectors are interchangeable. We can use this property directly in Eq.(20) and rewrite it into the final form, which reads

$$N_i = n_i + c_o^{-1} A_I e_{Ii}. \tag{21}$$

Equation (21) can be used for the determination of each of the vectors n_i, N_i from the knowledge of the other one. It is a linear equation and as such it can yield only one vector n_i for a specified N_i. Thus, Eq.(21) cannot, in principle, give correct results in regions of triplication of a qS wave wavefront.

Equation (21) can be useful in many applications. For example, it can be useful in ray tracing in anisotropic media, where it can serve for the estimation of phase vector directions, for which rays at a source have prescribed directions. For such a case, Eq.(21) yields for the correction of the phase normal

$$\Delta n_i = -c_0^{-1} A_I e_{Ii}. \tag{22}$$

For the application of Eq.(21), only knowledge of the quantities A_I is necessary. For qP waves, Eq.(19) yields

$$A_I = 2\alpha^{-1} B_{I3}, \quad B_{I3} = a_{ijkl} n_i n_l n_j e_{Ik}. \tag{23}$$

For K-th of the qS waves, Eq.(19), gives

$$A_I^{(K)} = \beta^{-1}(a_{mjkl} n_l e_{Kj} e_{Kk} e_{Im} - B_{K3} e_{Ik} e_{Kk}), \quad B_{K3} = a_{ijkl} n_i n_l n_j e_{Kk}, \tag{24}$$

no summation over bracketted indices. It is very important to distinguish between the indices I and K in the equations (23) and (24). The index I is used to denote two mutually perpendicular unit vectors **arbitrarily** chosen in the plane perpendicular to n_i. The index K is used to denote two mutually perpendicular unit vectors **satisfying the condition** $B_{12} = 0$, see discussion after Eq.(6).

Equation (21) used in Eq.(15) gives, in the first-order approximation, and taking into account Eq.(17), a well-known relation, see e.g. Backus (1965)

$$v = v_0 + \Delta v = c_0 + \Delta c = c. \tag{25}$$

In weakly anisotropic media, the phase and ray velocities have equal size.

We can now write final expressions for the ray velocity vectors in a weakly anisotropic medium. For qP waves, the vector has a form

$$v_i = (\alpha + \Delta c)n_i + 2\alpha^{-1} B_{I3} e_{Ii}. \tag{26}$$

For K-th qS wave, the ray vector is

$$v_i^{(K)} = (\beta + \Delta c^{(K)})n_i + \beta^{-1}(a_{mjkl} n_l e_{Kj} e_{Kk} e_{Im} - B_{K3} e_{Ik} e_{Kk}) e_{Ii}, \tag{27}$$

The above Eqs.(26) and (27) could be used for finding a plane of symmetry of a studied weakly anisotropic material. According to Norris (1989), *necessary and sufficient conditions that the direction n_i be normal to a plane of symmetry are that it be a longitudinal direction and that the energy-flux vector be in the direction n_i for any direction of transverse waves with phase direction n_i.* The obvious conditions for the existence of longitudinal and transverse directions in weakly anisotropic media are

$$B_{13} = 0, \quad B_{23} = 0, \tag{28}$$

see formulae (4a) and (4b). The second condition of Norris' (1989) theorem is satisfied when the bracket on the RHS of Eq.(27) is zero, which implies

$$a_{mjkl} n_l e_{Kj} e_{Kk} e_{1m} = 0, \quad a_{mjkl} n_l e_{Kj} e_{Kk} e_{2m} = 0. \tag{29}$$

Note that the qP wave ray velocity is also parallel to n_i under the above conditions. Eqs.(28) and (29) are equations for the determination of the direction n_i, which is normal to a plane of symmetry.

4 Conclusions

It was shown that a first-order perturbation of a phase normal in a weakly anisotropic medium does not lead to changes in phase velocities determined for the unperturbed normal. The study also confirms the well-known fact that the phase velocity and the ray velocity in weakly anisotropic media are the same. This is, however, not true about phase normal and ray vector. An explicit formula which relates linearly these two vectors in weakly anisotropic media was presented. It can be used for a simple straightforward determination of the phase normal in a weakly anisotropic medium, which yields in this medium a ray vector coinciding with the ray vector in the background isotropic medium.

As to the polarization vectors, it was shown that for the determination of the polarization corresponding to a slightly perturbed phase normal, it is sufficient to use the polarization corresponding to the unperturbed phase normal and to substitute the corresponding background polarization vector by a perturbed one.

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Appendix A: Perturbation of vectors e_{1i} and e_{2i}

The vectors e_{1i} and e_{2i} can be expressed in terms of the phase normal n_i in the following way

$$e_{1i} = \epsilon_{ijk} e_{2j} n_k, \quad e_{2i} = \epsilon_{ijk} n_j e_{1k}, \tag{A.1}$$

where ϵ_{ijk} denotes the Levi-Civita's symbol. Perturbation of the phase normal Δn_i produces perturbations Δe_{1i} and Δe_{2i} of the vectors e_{1i} and e_{2i}. From Eqs.(A.1) we can write

$$\Delta e_{1i} = \epsilon_{ijk} e_{2j} \Delta n_k + \epsilon_{ijk} \Delta e_{2j} n_k, \quad \Delta e_{2i} = \epsilon_{ijk} \Delta n_j e_{1k} + \epsilon_{ijk} n_j \Delta e_{1k}. \tag{A.2}$$

Inserting the expression for Δe_{2i} to the RHS of the first equation and Δe_{1i} to the RHS of the second equation, we get, after some manipulations,

$$\epsilon_{ijk} e_{2j} \Delta n_k = n_i (\Delta e_{1k} n_k), \quad \epsilon_{ijk} \Delta n_j e_{1k} = n_i (\Delta e_{2k} n_k). \tag{A.3}$$

Using (A.3), we can rewrite (A.2) into the form

$$\Delta e_{1i} = n_i (\Delta e_{1k} n_k) + \epsilon_{ijk} \Delta e_{2j} n_k, \quad \Delta e_{2i} = n_i (\Delta e_{2k} n_k) + \epsilon_{ijk} n_j \Delta e_{1k}. \tag{A.4}$$

Multiplying both equations by n_i, we get

$$\epsilon_{ijk} \Delta e_{2j} n_k = 0, \quad \epsilon_{ijk} n_j \Delta e_{1k} = 0. \tag{A.5}$$

Thus the perturbations of the vectors e_{1i} and e_{2i} have only projections to the phase normal:

$$\Delta e_{1i} = n_i (\Delta e_{1k} n_k), \quad \Delta e_{2i} = n_i (\Delta e_{2k} n_k). \tag{A.6}$$

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