Phase shift of the Green function due to caustics in anisotropic media

Luděk Klimeš

Department of Geophysics, Charles University, Ke Karlovu 3, 121 16 Praha 2, Czech Republic, Fax: +420-2-2191 1214, E-mail: klimes@seis.karlov.mff.cuni.cz

Summary

Equations destined to determine the phase shift of the amplitude of the elastic Green function due to caustics in anisotropic media are presented.

Keywords

Anisotropy, Green function, caustics, KMAH index, paraxial rays.

1 Introduction

Equations for ray tracing and dynamic ray tracing are required to calculate first–order, anisotropic ray theory, high–frequency asymptotic approximations of elastic wavefields or Green functions. The equations should also be supplemented with rules determining which ray to follow through singularities in the ray tracing equations, and which of the square roots of the determinant of the matrix of geometrical spreading to select when touching a caustic. The Green function also requires the equations of the initial conditions corresponding to a unit force at the source point, including the initial value of the KMAH index. In layered and block models, equations for the reflection/transmission coefficients and for the transformation of the dynamic ray tracing at structural interfaces are also required.

The initial conditions for the Green function are the same as in the anisotropic medium, locally homogeneous in a small vicinity of the point source. Refer, e.g., to Buchwald (1959).

For the reflection/transmission coefficients refer, e.g., to Fedorov (1968).

The dynamic ray tracing equations in Cartesian coordinates, supplementing the ray tracing equations in anisotropic media (Babich 1961), were derived by Červený (1972). For the transformation of the dynamic ray tracing at curved interfaces between anisotropic materials refer, e.g., to Gajewski & Pšenčík (1990).

The rules determining the sign of the phase shift of the amplitude of the elastic Green function due to caustics in anisotropic media are presented in this paper. The equations for the phase shift corresponding to a general wavefield have been derived by Bakker (1997). Unlike in isotropic media, where the increment of the KMAH index determining the phase shift due to caustics is always positive, the increment of the KMAH index of S waves in anisotropic media may be either positive, or negative. A simple example is given to illustrate this.

A KMAH index of +1 indicates a phase shift of the complex–valued amplitude by $\frac{\pi}{2}$ in the direction corresponding to increasing time (or decreasing travel time) of the time–harmonic wave. Hereinafter the time–harmonic wavefield, dependent on time t through

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the multiplication factor of $\exp(-\mathrm{i}\omega t)$, is chosen for positive circular frequencies ω. The imaginary part of the matrix of the second travel–time derivatives of a Gaussian beam then has to be positive–definite, and the KMAH index of +1 corresponds to the amplitude multiplication factor of $\exp(-\mathrm{i}\frac{\pi}{2})$.

2 Simple (line) caustic

Assume a simple (line) caustic at the ray bundle corresponding to a point source. At the simple caustic, submatrix $\mathbf{Q}_2$ of the paraxial–ray propagator matrix

$$\begin{pmatrix} \mathbf{Q}_1 & \mathbf{Q}_2 \\ \mathbf{P}_1 & \mathbf{P}_2 \end{pmatrix} \tag{1}$$

has just one zero eigenvalue. The ray–theory wavefield is the limiting case of the Gaussian beam, infinitely narrow at the point source. The real part of the matrix of the second travel–time derivatives of the Gaussian beam at the point source is taken to be zero, and the imaginary part is denoted $\mathbf{Y}_0$. Inverse matrix $\mathbf{Y}_0^{-1}$ is thus a positive–definite symmetrical matrix limiting to the zero matrix.

The matrix of geometrical spreading of the Gaussian beam is

$$\mathbf{Q} = [\mathbf{Q}_1 + \mathrm{i}\mathbf{Q}_2\mathbf{Y}_0]\mathbf{Q}_0 \quad , \tag{2}$$

where $\mathbf{Q}_0$ is the value of the matrix of geometrical spreading of the Gaussian beam at the point source.

In the vicinity of the caustic we may approximate $\mathbf{Q}$ by the linear Taylor expansion with respect to the increasing monotonic parameter γ_3 along the ray,

$$\mathbf{Q}(\Delta\gamma_3) \approx [\mathbf{Q}_1 + \mathrm{i}\mathbf{Q}_2\mathbf{Y}_0 + \mathbf{Q}_1'\Delta\gamma_3 + \mathrm{i}\mathbf{Q}_2'\mathbf{Y}_0\Delta\gamma_3]\mathbf{Q}_0 \quad , \tag{3}$$

centred at the caustic where $\Delta\gamma_3 = 0$. Here

$$\mathbf{Q}_1' = \frac{\mathrm{d}\mathbf{Q}_1}{\mathrm{d}\gamma_3} \quad , \qquad \mathbf{Q}_2' = \frac{\mathrm{d}\mathbf{Q}_2}{\mathrm{d}\gamma_3} \quad . \tag{4}$$

For very small $\mathbf{Y}_0^{-1}$ and $\Delta\gamma_3$ we may further approximate (3) by

$$\mathbf{Q}(\Delta\gamma_3) \approx [\mathbf{Q}_2 - \mathrm{i}\mathbf{Q}_1\mathbf{Y}_0^{-1} + \mathbf{Q}_2'\Delta\gamma_3]\ \mathrm{i}\mathbf{Y}_0\mathbf{Q}_0 \quad . \tag{5}$$

Introducing matrix

$$\mathbf{K} = \mathbf{Q}_2^{-1}\det\mathbf{Q}_2 \tag{6}$$

of rank 1 composed of the subdeterminants of $\mathbf{Q}_2$, the determinant of (5) may be approximated by

$$\det\mathbf{Q}(\Delta\gamma_3) \approx [\mathrm{tr}(\mathbf{K}\mathbf{Q}_2')\Delta\gamma_3 - \mathrm{i}\ \mathrm{tr}(\mathbf{K}\mathbf{Q}_1\mathbf{Y}_0^{-1})]\ \det(\mathrm{i}\mathbf{Y}_0\mathbf{Q}_0) \quad . \tag{7}$$

If numbers

$$\overline{\overline{K}}_1 = \mathrm{tr}(\mathbf{K}\mathbf{Q}_1\mathbf{Y}_0^{-1}) \quad , \qquad K_2 = \mathrm{tr}(\mathbf{K}\mathbf{Q}_2') \tag{8}$$

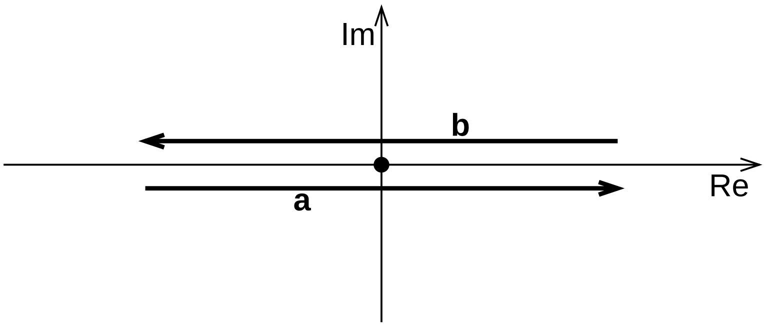

Figure 1. Values of function $-\mathrm{i}K_2 + \overline{\overline{K}}_1 \Delta\gamma_3$ in the vicinity of the origin of the complex plane: **(a)** for $\overline{\overline{K}}_1 > 0$ and $K_2 > 0$, **(b)** for $\overline{\overline{K}}_1 < 0$ and $K_2 < 0$. The arrows correspond to increasing $\Delta\gamma_3$. The constant multiplication factor of $\det(\mathrm{i}\mathbf{Y}_0\mathbf{Q}_0)$ in (7) just rotates the diagram of the functional values around the origin by the corresponding angle.

have equal signs, determinant (7) passes for increasing $\Delta\gamma_3$ the origin of the complex plane counterclockwise (see Figure 1), and the increment of the KMAH index is $+1$, otherwise the increment of the KMAH index is -1.

Since $\mathbf{KQ}_1$ is a symmetrical matrix of rank 1, and $\mathbf{Y}_0^{-1}$ is positive–definite, the sign of $\overline{\overline{K}}_1$ is the same as the sign of

$$\overline{K}_1 = \mathrm{tr}(\mathbf{KQ}_1) \quad . \tag{9}$$

Because of the symplectic property of the paraxial–ray propagator matrix (1),

$$\mathrm{tr}(\mathbf{KQ}_1)\ \mathrm{tr}(\mathbf{KP}_2) = \mathrm{tr}(\mathbf{KK}^{\mathrm{T}}) \quad , \tag{10}$$

see Appendix A, and the sign of $\overline{K}_1$ is the same as the sign of

$$K_1 = \mathrm{tr}(\mathbf{KP}_2) \quad . \tag{11}$$

The sign of the increment of the KMAH index is thus the same as the sign of $K_1 K_2$, where K_1 and K_2 are given by equations (11) and (8) with $\mathbf{K}$ defined by (6), regardless of whether we consider 2×2 or 3×3 matrices in the ray–centred coordinates, or 3×3 matrices in general coordinates. Severe caustics with $K_2 = 0$ are not considered in this paper.

Note that **(a)** K_2 always has the sign of $\det\mathbf{Q}_2$ beyond the caustic; **(b)** in an isotropic medium, both K_1 and K_2 have the same sign, and the increment of the KMAH index is thus always positive there; **(c)** the increment of the KMAH index is negative if the sign of the wavefront curvature perpendicular to the caustic is opposite to that the curvature always has in isotropic media.

3 Point caustic

Assume a point caustic at the ray bundle corresponding to a point source. At the caustic, submatrix $\mathbf{Q}_2$ of the paraxial–ray propagator matrix (1) has two zero eigenvalues. If it is a 2×2 matrix expressed in ray–centred coordinates, it becomes a zero matrix.

3.1 Matrices 2×2 in ray–centred coordinates

For 2×2 matrices expressed in ray–centred coordinates, the determinant of (5) may be approximated by

$$\det\mathbf{Q}(\Delta\gamma_3) \approx \det[-\mathrm{i}\mathbf{Q}_1\mathbf{Y}_0^{-1}\mathbf{P}_2^{-1} + \mathbf{Q}_2'\mathbf{P}_2^{-1}\Delta\gamma_3]\ \det(\mathrm{i}\mathbf{P}_2\mathbf{Y}_0\mathbf{Q}_0) \quad . \tag{12}$$

Since

$$\mathbf{Q}_1\mathbf{Y}_0^{-1}\mathbf{P}_2^{-1} = (\mathbf{P}_2^{\mathrm{T}})^{-1}\mathbf{Y}_0^{-1}\mathbf{P}_2^{-1} \quad , \tag{13}$$

see ($A2$), is a positive–definite symmetrical 2×2 matrix, the phase shift due to caustics depends on the signs of the two eigenvalues of the symmetrical 2×2 matrix $\mathbf{Q}_2'\mathbf{P}_2^{-1}$. If

$$\det(\mathbf{Q}_2'\mathbf{P}_2^{-1}) > 0 \quad , \qquad \mathrm{tr}(\mathbf{Q}_2'\mathbf{P}_2^{-1}) > 0 \quad , \tag{14a}$$

both the eigenvalues of $-\mathrm{i}\mathbf{Q}_1\mathbf{Y}_0^{-1}\mathbf{P}_2^{-1} + \mathbf{Q}_2'\mathbf{P}_2^{-1}\Delta\gamma_3$ pass for increasing $\Delta\gamma_3$ the origin of the complex plane counterclockwise, and the increment of the KMAH index is +2. If

$$\det(\mathbf{Q}_2'\mathbf{P}_2^{-1}) > 0 \quad , \qquad \mathrm{tr}(\mathbf{Q}_2'\mathbf{P}_2^{-1}) < 0 \quad , \tag{14b}$$

both the eigenvalues of $-\mathrm{i}\mathbf{Q}_1\mathbf{Y}_0^{-1}\mathbf{P}_2^{-1} + \mathbf{Q}_2'\mathbf{P}_2^{-1}\Delta\gamma_3$ pass for increasing $\Delta\gamma_3$ the origin of the complex plane clockwise, and the increment of the KMAH index is −2. If

$$\det(\mathbf{Q}_2'\mathbf{P}_2^{-1}) < 0 \quad , \tag{14c}$$

one eigenvalue of $-\mathrm{i}\mathbf{Q}_1\mathbf{Y}_0^{-1}\mathbf{P}_2^{-1} + \mathbf{Q}_2'\mathbf{P}_2^{-1}\Delta\gamma_3$ passes for increasing $\Delta\gamma_3$ the origin of the complex plane counterclockwise, whereas the other one passes the origin clockwise, and the increment of the KMAH index is 0. Severe caustics with $\det(\mathbf{Q}_2'\mathbf{P}_2^{-1}) = 0$ are not considered in this paper.

3.2 Matrices 3×3

Determinant $\det(\mathbf{Q}_2'\mathbf{P}_2^{-1})$ in conditions ($14a, b, c$) may be replaced by an analogous determinant

$$\det(\widehat{\mathbf{Q}}_2'\widehat{\mathbf{P}}_2^{-1}) \tag{15}$$

of the corresponding 3×3 matrices in general coordinates because the third eigenvalue of the 3×3 matrix $\widehat{\mathbf{Q}}_2'\widehat{\mathbf{P}}_2^{-1}$ is always positive (Klimeš 1994).

Trace $\mathrm{tr}(\mathbf{Q}_2'\mathbf{P}_2^{-1})$ in conditions ($14a, b, c$) should be replaced by

$$\mathrm{tr}(\widehat{\mathbf{Q}}_2'\widehat{\mathbf{P}}_2^{-1}) - \mathbf{p}^{\mathrm{T}}\widehat{\mathbf{Q}}_2'\widehat{\mathbf{P}}_2^{-1}\mathbf{p}\ (\mathbf{p}^{\mathrm{T}}\mathbf{p})^{-1} \quad , \tag{16}$$

where p is the slowness vector. Expression (16) is the trace of the projection of matrix $\widehat{\mathbf{Q}}_2'\widehat{\mathbf{P}}_2^{-1}$ onto the wavefront tangent plane.

4 Reciprocity of the phase shift

Kendall, Guest & Thomson (1992) show the reciprocity of the first–order anisotropic ray theory Green function except for the discussion of the reciprocity of the phase shift of the Green function due to caustics. The phase–shift reciprocity is thus demonstrated in this section.

The phase shift of the Green function due to caustics is obviously reciprocal for two very close points A and B. Now fix point A and move point B along a ray away from point A.

Assuming that we have hereinbefore considered initial conditions $\mathbf{Y}_0$, $\mathbf{Q}_0$ at point A and have studied the determinant of matrix $\mathbf{Q}$ at point B, we shall now consider initial conditions $\mathbf{Y}_0$, $\mathbf{Q}_0$ at point B and study the determinant of matrix $\mathbf{Q}$ at point A.

The matrix inverse to the paraxial–ray propagator matrix (1) from point A to point B is

$$\begin{pmatrix} \mathbf{P}_2^{\mathrm{T}} & -\mathbf{Q}_2^{\mathrm{T}} \\ -\mathbf{P}_1^{\mathrm{T}} & \mathbf{Q}_1^{\mathrm{T}} \end{pmatrix} \quad , \tag{17}$$

which follows from the symplectic property of the paraxial–ray propagator matrix (Thomson & Chapman 1985). Matrix (17) is the paraxial–ray propagator matrix from point B to point A, but still corresponds to the wave propagation from point A to point B. The opposite wave propagation implies the opposite orientation of the slowness vectors. Keeping the same spatial coordinates and changing the signs of all slowness vector components in (17), we thus arrive at the paraxial–ray propagator matrix

$$\begin{pmatrix} \mathbf{P}_2^{\mathrm{T}} & \mathbf{Q}_2^{\mathrm{T}} \\ \mathbf{P}_1^{\mathrm{T}} & \mathbf{Q}_1^{\mathrm{T}} \end{pmatrix} \tag{18}$$

from point B to point A, consistent with the direction of propagation.

Equations (1) to (16) may thus be modified for the propagation from point B to point A by the substitution

$$\mathbf{Q}_1 \longrightarrow \mathbf{P}_2^{\mathrm{T}} \ , \qquad \mathbf{P}_1 \longrightarrow \mathbf{P}_1^{\mathrm{T}} \ , \qquad \mathbf{Q}_2 \longrightarrow \mathbf{Q}_2^{\mathrm{T}} \ , \qquad \mathbf{P}_2 \longrightarrow \mathbf{Q}_1^{\mathrm{T}} \ , \tag{19}$$

including the derivatives of these matrices. Since matrices $\mathbf{Q}_2$ and $\mathbf{Q}_2^{\mathrm{T}}$ have the same rank, the caustics arise for the same distances between points A and B for both the directions of propagation.

With a simple (line) caustic, substitution (16) implies the substitution

$$K_2 \longrightarrow K_2 \ , \qquad \overline{K}_1 \longrightarrow K_1 \ , \qquad K_1 \longrightarrow \overline{K}_1 \ , \tag{20}$$

and, in consequence, the same phase shift.

With a point caustic, substitution (16) implies the substitution

$$\mathbf{Q}_2' \, \mathbf{P}_2^{-1} \longrightarrow [\mathbf{Q}_2^{\mathrm{T}}]' \, [\mathbf{Q}_1^{\mathrm{T}}]^{-1} \tag{20}$$

in $(14a, b, c)$. Since, at a point caustic,

$$[\mathbf{Q}_2^{\mathrm{T}}]' \, [\mathbf{Q}_1^{\mathrm{T}}]^{-1} = [\mathbf{Q}_2^{\mathrm{T}}]' \, \mathbf{P}_2 = \mathbf{P}_2^{\mathrm{T}} \, [\mathbf{Q}_2' P_2^{-1}]^{\mathrm{T}} \, \mathbf{P}_2 \quad , \tag{21}$$

see $(A2)$, conditions $(14a, b, c)$ determine the same phase shift for both the directions of propagation.

5 Example

Assume a 2-D homogeneous anisotropic medium with triplication on the S–wave group–velocity surface. Select a ray perpendicular to the reverse branch of the group–velocity surface as the axial ray. Assume that the homogeneous anisotropic medium is separated from the homogeneous isotropic medium by a planar interface perpendicular to the axial ray. For the sake of simplicity, we may assume that the (phase and group) velocities in the direction of the axial ray are similar (or equal) in both media.

The slowness vectors of the paraxial rays, propagating in the anisotropic medium to one side of the axial ray, point to the other side of the axial ray, see Figure 2. On entering the isotropic medium, the paraxial rays are refracted to the direction of the slowness vector and cross the axial ray at the caustic.

Consider 3 points on the axial ray: Point A in the anisotropic medium, point B situated in the isotropic medium between the interface and the caustic corresponding to the point source at A, and point C situated in the isotropic medium beyond the caustic corresponding to the point source at A. The paraxial rays from all 3 points are depicted in Figure 2. The paraxial rays of the same perpendicular component of the slowness vector are parallel in each of the halfspaces.

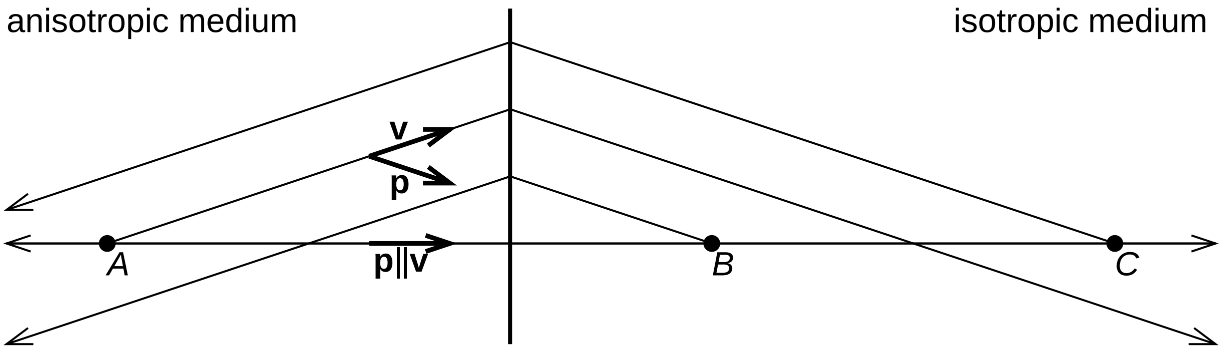

Figure 2. The axial ray and paraxial rays from points A, B, C in the discussed example. In the anisotropic halfspace, paraxial group–velocity vector **v** and slowness vector **p** are deflected in opposite directions.

The phase shift along the four rays between points A, B and C is accumulated in the following way:

$C \rightarrow A$: There is no caustic along the ray from point C to point A. If the KMAH index is defined zero at the point source in the isotropic medium, the resulting KMAH index at point A is 0.

$A \rightarrow C$: The initial KMAH index at point A on the reverse wavefront branch must be smaller by 1 than on the forward branch, i.e. -1 in our 2-D case with zero initial value of the KMAH index in the isotropic medium. There is a single caustic along the ray from point A to point C, located in the isotropic medium, causing the KMAH index to increase by $+1$. The resulting KMAH index at point C is then 0.

$A \rightarrow B$: The initial KMAH index at point A is again -1, but there is no caustic along the ray from point A to point B. The resulting KMAH index at point C is now -1.

$B \rightarrow A$: The initial KMAH index at point B is 0, as for $C \rightarrow A$. There is a single caustic along the ray from point B to point A, located in the anisotropic medium on the reverse branch of the group–velocity surface, causing the KMAH index to decrease by 1. The resulting KMAH index at point A is then -1.

Appendix A

The symplectic property of the paraxial–ray propagator matrix (1) may be expressed in the form of

$$\begin{pmatrix} \mathbf{Q}_1^{\mathrm{T}} & \mathbf{P}_1^{\mathrm{T}} \\ \mathbf{Q}_2^{\mathrm{T}} & \mathbf{P}_2^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{Q}_1 & \mathbf{Q}_2 \\ \mathbf{P}_1 & \mathbf{P}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{pmatrix} \quad . \tag{A1}$$

It implies, inter alia, the symmetry of matrices $\mathbf{Q}_1\mathbf{Q}_2^{\mathrm{T}}$ and $\mathbf{Q}_2^{\mathrm{T}}\mathbf{P}_2$, and consequently of matrices $\mathbf{K}\mathbf{Q}_1$ and $\mathbf{P}_2\mathbf{K}$, and the identity

$$\mathbf{P}_2^{\mathrm{T}}\mathbf{Q}_1 - \mathbf{P}_1^{\mathrm{T}}\mathbf{Q}_2 = \mathbf{1} \quad . \tag{A2}$$

Multiplying ($A2$) by $\mathbf{K}^{\mathrm{T}}$ and $\mathbf{K}$ from left and right, respectively, we arrive at

$$\mathbf{K}^{\mathrm{T}}\mathbf{P}_2^{\mathrm{T}}\ \mathbf{Q}_1\mathbf{K} = \mathbf{K}^{\mathrm{T}}\mathbf{K} \quad . \tag{A3}$$

Since $\mathbf{P}_2\mathbf{K}$ is a symmetrical matrix and $\mathbf{K}$ is of rank 1,

$$\mathbf{K} = \mathbf{a}\ \mathbf{b}^{\mathrm{T}} \tag{A4}$$

$\mathbf{a}$ and $\mathbf{b}$ being vectors,

$$\mathrm{tr}(\mathbf{K}^{\mathrm{T}}\mathbf{P}_2^{\mathrm{T}}\mathbf{Q}_1\mathbf{K}) = \mathrm{tr}(\mathbf{P}_2\mathbf{K}\mathbf{Q}_1\mathbf{K}) = \mathbf{b}^{\mathrm{T}}\mathbf{P}_2\mathbf{a}\ \mathbf{b}^{\mathrm{T}}\mathbf{Q}_1\mathbf{a} = \mathrm{tr}(\mathbf{K}\mathbf{P}_2)\ \mathrm{tr}(\mathbf{K}\mathbf{Q}_1) \tag{A5}$$

and equation (10) is the direct consequence of ($A3$).

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