Note on chaos in ray propagation

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Abstract: The aim of this note is to mention some potential tools for studies of chaotical behaviour of rays. First a local exponential divergence of rays in a phase-space is demonstrated. Next the possibilities of a study of ray chaos by means of the Lyapunov exponents for both continuos time-evolution and iterative maps on Poincaré sections are discussed. In the continuos case the Lyapunov exponents can be obtained from the analyses of the propagator matrices and in the latter problems their discrete analogies have to be constructed. The correlation technique is then shortly pointed out and the employment of wavelet multiresolution analysis is proposed in the last part of the note.

Introduction

The propagation of rays can be described by means of the Hamilton canonical equations in six-dimensional phase-space, where the Hamiltonian is equal to half of the eigenvalue of the Christoffel matrix (e.g., Klimeš, 1994). The phase-space coordinates are formed by generalized coordinates and the slowness vector components, which play the role of conjugate momenta.

From the formal point of view, the ray tracing equations represent a (nonlinear) dynamical system, which can exhibit a chaotical behaviour. In general, there is no simple and clear measure of chaoticity, which could just be routinely applied to any model. On the other side, there are several quantities—namely the Lyapunov exponents, Poincare's sections, correlations, Fourier and wavelet power spectra— which could be tested for studied models and here we present their concise review.

We will neither deal with the qualitative theory of Hamiltonian systems in ray propagation, which can be found, e.g., in Abdullaev (1993), nor with special quantifications of chaos by means of the number of eigenrays connecting the source with the receiver, which can be suitable for infinite waveguides (Tappert and Tang, 1996). Instead we will concentrate our attention to geometrical aspects of ray propagation, which can be studied by means of the analyses of the propagator matrix. These geometrical methods can be developed well for closed or periodic models, where rays can be constructed for very long periods of time. However, in many models the rays are constructed for only limited time periods. This is the reason why we propose to pay attention also to statistical and spectral characteristics of constructed ray time series.

Local exponential behaviour of rays

Consider a general Hamiltonian ray tracing system (e.g., Klimeš, 1994)

$$\frac{dw_i}{dt} = F_i(\mathbf{w}) \ , \quad i = 1, 2, \ldots, N \ , \tag{1}$$

where $N = 6$ ($N = 4$) for 3-D (2-D) problems and w_i are the phase-space coordinates. This system is called *autonomous* as the vector function $\mathbf{F}$ does not depend explicitly on time. Let $\mathbf{w}(t; \mathbf{w}^0)$ is the ray satisfying (1) and the initial condition $\mathbf{w} = \mathbf{w}^0$ at $t = 0$. We are interested in the behaviour of rays in the neighbourhood of a selected ray $\mathbf{w}(t; \mathbf{w}^0)$. To demonstrate that their behaviour in the phase-space E_N is, in principle, exponential, study now the propagation of rays in a region, where F_i are continuously differentiable. Employing the Taylor expansion and Einstein's summation rule, we may rewrite the ray tracing system (1) into

$$\frac{d}{dt}(w_i + \delta w_i) \doteq F_i(\mathbf{w}) + \left.\frac{\partial F_i}{\partial w_j}\right|_{\mathbf{w}} \delta w_j \ . \tag{2}$$

Eqn. (1) then immediately yields

$$\frac{d}{dt}\delta w_i \doteq \left.\frac{\partial F_i}{\partial w_j}\right|_{\mathbf{w}} \delta w_j \ . \tag{3}$$

As F_i are smooth enough, we can write for a selected ray $\mathbf{w}(t; \mathbf{w}^0)$ and a fixed time τ

$$\left.\frac{\partial F_i}{\partial w_j}\right|_{\mathbf{w}(t;\mathbf{w}^0)} = \left.\frac{\partial F_i}{\partial w_j}\right|_{\mathbf{w}(\tau;\mathbf{w}^0)} \quad \text{for} \quad t \to \tau \ , \tag{4}$$

and the solution of (3) for $t \to \tau$ is

$$\delta\mathbf{w}(t) \doteq \exp[(t - \tau)\boldsymbol{D}_F(\mathbf{w}(\tau; \mathbf{w}^0))] \, \delta\mathbf{w}(\tau) \ , \tag{5}$$

where the *Jacobi matrix* $\boldsymbol{D}_F$ and $\exp(\boldsymbol{D}_F)$ are evaluated as follows,

$$(D_F)_{ij}(\mathbf{w}) = \left.\frac{\partial F_i}{\partial w_j}\right|_{\mathbf{w}} \ , \quad \exp(t\boldsymbol{D}_F) = \sum_{n=0}^{\infty} \frac{1}{n!}(t\boldsymbol{D}_F)^n \ . \tag{6}$$

To construct N independent solutions of the studied linearized system we determine eigenvalues λ and eigenvectors $\mathbf{e}$ of the matrix $\boldsymbol{D}_F$. Eigenvalues and eigenvectors are introduced by the relation

$$(\boldsymbol{D}_F - \lambda\boldsymbol{I})\mathbf{e} = \mathbf{0} \ , \tag{7}$$

where $\boldsymbol{I}$ is the identity matrix. In general, the Jacobi matrix is not symmetric and thus both eigenvalues and eigenvectors can be complex. If $\boldsymbol{D}_F$ has N linearly independent eigenvectors $\mathbf{e}_j$, $j = 1, 2, \ldots, N$, then the independent solutions of (3) are simply

$\exp(\alpha_j t)\mathbf{e}_j$, where α_j is an eigenvalue corresponding to $\mathbf{e}_j$. If we have a pair of complex conjugate eigenvalues α_j, α_{j+1} with the corresponding complex eigenvectors, a pair $\exp(\Re\alpha_j t)[\cos(\Im\alpha_j t)\Re\mathbf{e}_j - \sin(\Im\alpha_j t)\Im\mathbf{e}_j]$, $\exp(\Re\alpha_j t)[\sin(\Im\alpha_j t)\Re\mathbf{e}_j + \cos(\Im\alpha_j t)\Im\mathbf{e}_j]$ represents linearly independent real solutions. Here $\Re$ denotes real and $\Im$ imaginary part of the complex numbers.

The time dependence of rays in the neighbourhood of a selected ray at a particular time $t = \tau$ can thus be exponential. This local behaviour is controlled by the real parts of the eigenvalues of the Jacobi matrix. If there are no multiple eigenvalues, the phase-space E_N is divided into the three linearly independent subspaces, E^s, E^u and E^c, where the *stable subspace* E^s is spanned by the eigenvectors whose eigenvalues have negative real parts, the *unstable subspace* E^u is spanned by the eigenvectors whose eigenvalues have positive real parts and the *center subspace* E^c is spanned by the eigenvectors whose eigenvalues have zero real parts. Each of the subspaces is invariant under the time evolution of the linearized system (3), (4), i.e., the solutions lying in E^s are characterized by exponential decay, those lying in E^u by exponential growth, and those lying in E^c by neither (see also Guckenheimer and Holmes, 1983). Remind, however, that this analysis is valid only for a short-time evolution of rays near a selected phase-space point.

Note here that the dimension of $\Re\alpha_j$ is given by the inverse of a time unit, which means that it does not depened on a spatial unit chosen to describe a studied ray tracing model. The ray tracing system (1) is *Hamiltonian* (e.g., Klimeš, 1994) and thus it preserves phase-space volume during time evolution. This implies that either $E^c = E_N$ or $\dim E^u > 0$. Let us number the eigenvalues α_j so that α_N is the eigenvalue with the largest real part. Then $(\Re\alpha_N)^{-1}$ represents the *local characteristic time of predictability* of ray behaviour.

The exponential amplification of small perturbations δw_i can result in a *sensitive dependence on initial conditions* for long times. However, the Jacobi matrix $\boldsymbol{D}_F$ can change along rays, i.e., $\boldsymbol{D}_F$ is a function of time for a selected ray $\mathbf{w}(t;\mathbf{w}^0)$ and thus such a global instability need not occur under the presence of local instabilities. The quantitative description of instabilities for a long time propagation can be characterized by means of the Lyapunov exponents, which is the topic of the next sections.

Global chaotic behaviour of rays

Suppose now that we deal with a model with both geometrically smooth interfaces and smooth material properties outside the interfaces so that $\mathbf{w}(t;\mathbf{w}^0)$ smoothly depends on $\mathbf{w}^0$. Employing again the Taylor expansion and Einstein's summation rule, we may write

$$w_i(t;\mathbf{w}^0 + \delta\mathbf{w}^0) \doteq w_i(t;\mathbf{w}^0) + \frac{\partial w_i}{\partial w_j^0}(t;\mathbf{w}^0)\delta w^0{}_j \ , \tag{8}$$

where

$$\Pi_{ij}(t;\mathbf{w}^0) \equiv \frac{\partial w_i}{\partial w_j^0}(t;\mathbf{w}^0) \tag{9}$$

is the *propagator matrix* of a selected ray (see, e.g., Klimeš, 1994). The expansion (8) can be rewritten into the form

$$\delta\mathbf{w}(t;\mathbf{w}^0;\delta\mathbf{w}^0) \doteq \boldsymbol{\Pi}(t;\mathbf{w}^0)\,\delta\mathbf{w}^0 \ . \tag{10}$$

The propagator matrix can thus be used to analyze the chaoticity of rays due to both a spatial-dependence of material properties outside the interfaces and undulations of the interfaces. Both these effects can lead to an exponential divergences of rays in the phase-space, which can be globally described by means of the exponents formally corresponding to the eigenvalues of the Jacobi matrix $\boldsymbol{D}_F$ in the cases with constant $\boldsymbol{D}_F$ along a ray.

Consider below that the Jacobi matrix has N linearly independent eigenvectors and corresponding eigenvalues have distinct real parts. In other words, we can number α_i in such an order that $\Re\alpha_1 < \Re\alpha_2 < \ldots < \Re\alpha_N$. Let E_k be the k-dimensional subspace of E_N spanned by the eigenvectors corresponding to α_j, $j = 1, 2, \ldots, k$. Then E_k may be characterized as (Lanford, 1981)

$$E_k = \left\{ \mathbf{u} \in E_N ; \limsup_{t\to\infty} \frac{1}{t} \ln ||\exp(t\boldsymbol{D}_F)\,\mathbf{u}|| \le \Re\alpha_k \right\} \ , \tag{11}$$

where the norm $||\ ||$ is generated by means of a scalar product $(\cdot,\cdot)$. Moreover, if $\mathbf{u} \in E_k$ but not in E_{k-1}, then

$$\limsup_{t\to\infty} \frac{1}{t} \ln ||\exp(t\boldsymbol{D}_F)\,\mathbf{u}|| = \Re\alpha_k \ . \tag{12}$$

Definition (see also Lanford, 1981): The point $\mathbf{w}^0$ has the *Lyapunov* (or *characteristic*) *exponents* $\lambda_1 < \lambda_2 < \ldots < \lambda_N$ if there exist subspaces $E_1 \subset E_2 \subset \ldots \subset E_N$, with each E_k of dimension k, such that

$$E_k = \left\{ \mathbf{u} \in E_N ; \limsup_{t\to\infty} \frac{1}{t} \ln ||\boldsymbol{\Pi}(t;\mathbf{w}^0)\,\mathbf{u}|| \le \lambda_k \right\} \tag{13}$$

and

$$\limsup_{t\to\infty} \frac{1}{t} \ln ||\boldsymbol{\Pi}(t;\mathbf{w}^0)\,\mathbf{u}|| = \lambda_k \quad \text{if } \mathbf{u} \in E_k \text{ but } \mathbf{u} \notin E_{k-1} \ . \tag{14}$$

The Lyapunov exponents represent the rate of exponential growth or damping of infinitesimal displacements for $t \to \infty$, i.e., the inverse of the largest positive Lyapunov exponent gives the *global characteristic time of predictability* of ray behaviour near a selected ray. However, I would like to emphasize that the Lyapunov exponents are the quantities associated with a particular ray $\mathbf{w} = \mathbf{w}^0$ at $t = t^0$. This is the reason why a simple quantification of a whole model by, e.g., the largest Lyapunov exponent found, may result in a deep misunderstanding of the basic physics of the model because we thus obtain the characteristic time of predictability of ray propagation near only the ray corresponding to this exponent.

Numerical evaluation of Lyapunov exponents

There is a problem in numerical computations that the components of the phase-space represent different physical quantities (generalized coordinates and generalized slowness vector components). Although the above definition of the Lyapunov exponents does not depend on the scaling of w_i, we are not able to operate with infinitely long time-series in such problems, where the ray paths is not closed in a studied domain. Finite-time guesses of the Lyapunov exponents are then dependent on a choice of the scalar product generating the employed norm.

The largest Lyapunov exponent can be numerically computed by a random choice of $\mathbf{u}$ in the expression $t^{-1} \ln ||\boldsymbol{\Pi}(t; \mathbf{w}^0)\, \mathbf{u}||$ (see Keers et al., 1997). Since the subspace (13) corresponding to λ_N is the basic space E_N, there is zero probability that randomly chosen $\mathbf{u}$ is from the $(N-1)$-dimensional space E_{N-1}. Hence, (14) is satisfied for $k = N$. However, in practical applications we can only estimate λ_N on a finite-time interval. It is, therefore, useful to estimate $t^{-1} \ln ||\boldsymbol{\Pi}(t; \mathbf{w}^0)\, \mathbf{u}||$ for several randomly chosen $\mathbf{u}$ with unit modulus under the selected norm. A generalization of this procedure for computations of other Lyapunov exponents (together with the bibliography of original papers) can be found in Marek and Schreiber (1991), Section 4.2.

We may also try to find the Lyapunov exponents λ_k as the limits

$$\lambda_k = \limsup_{t \to \infty} \frac{1}{t} \ln |\sigma_k(t)| \ , \tag{15}$$

where $\sigma_k(t)$ are the eigenvalues of the propagator matrix. For example, Marek and Schreiber (1991) referenced a paper by Farmer et al. (1983) in defining the Lyapunov exponents even in a simplified version of (15), where $\limsup_{t\to\infty}$ was replaced by only $\lim_{t\to\infty}$. However, in dynamic ray tracing models $(1/t) \ln |\sigma_k(t)|$ can be a function oscillating over a finite interval (Klimeš, 1998). Hence we deal with the limiting behaviour of an upper envelope of $(1/t) \ln |\sigma_k(t)|$.

Iterative maps and Poincaré sections

As mentioned in the previous section, an additional effect to that caused by smoothly heterogeneous media is the effect of undulating interfaces. In such a case it is possible to restrict the attention to the values of phase-space coordinates of a selected ray at instants of its incidence on chosen interfaces. In other words, these incidences define a discrete map

$$\mathbf{w}^{(i+1)} = \mathbf{G}(\mathbf{w}^{(i)}) \ . \tag{16}$$

To simplify the description of ray interactions with the interfaces, let us confine our further considerations only to the special problems when we study the repeated incidence of a ray on the same interface. Then we can simply redefine the phase-space coordinates since the location of the selected interface in the real space can be parameterized by means of some parameters s_j, $j = 1, N/2-1$ and the incidence of a ray by the angles s_j, $j = N/2, N-2$ and

$$\mathbf{w}^{(i)} = \mathbf{w}^{(i)}(\mathbf{s}). \tag{17}$$

This relation descibes a parametrization of a cross-section in the original phase-space. On this section we have a map

$$s_j^{(i+1)} = G_j(s_1^{(i)}, \dots, s_{N-2}^{(i)}). \tag{18}$$

A sequence of points $\{\mathbf{s}^{(i)}\}_{i=-\infty}^{\infty}$ in E_{N-2} is then called an *orbit.* Each interface with infinitely repeated incidences of the ray represents the special case of *Poincaré sections. Chaotic behaviour of rays can be visualized* by drawing orbits in such a cross-section of the original phase-space.

The *Lyapunov exponents* associated with a selected orbit can be defined by only a slight change of the previous definition for continuous time: the time t is replaced by the number of iteration i and the propagator matrix $\boldsymbol{\Pi}(t;\mathbf{w}^0)$ is replaced by the Jacobi matrix $\boldsymbol{D}_G^i(\mathbf{s}^{(0)})$ of the i-th iteration of the procedure starting at $\mathbf{s} = \mathbf{s}^{(0)}$. It is clear that $\boldsymbol{D}_G^i(\mathbf{s}^{(0)}) = \boldsymbol{D}_G(\mathbf{s}^{(i-1)})\boldsymbol{D}_G(\mathbf{s}^{(i-2)})\dots\boldsymbol{D}_G(\mathbf{s}^{(0)})$, where $\boldsymbol{D}_G(\mathbf{s}^j)_{kl} = \partial G_k(\mathbf{s}^j)/\partial s_l$.

The discrete definition of the Lyapunov coefficient was employed by Keers et al. (1997) for the description of chaotic propagation of rays in the Earth's crust, which is caused by undulations of the Moho topography. The authors considered a 2-D model with the flat surface and the Moho, which was undulated periodically. The periodicity of their model was the reason why they identified the phase-space coordinate s_1 with the horizontal distance from the interval $[0, \text{period}]$ (they projected all rays with epicentral distances larger than the period back to this basic interval). The second phase-space coordinate s_2 was the incidence angle on the surface. They computed largest Lyapunov exponent by means of both definitions (14) and (15) and they obtained the same results. The visualization of orbits on the Poincaré section corresponding to the above mentioned parameters s_1, s_2 distinctly separated regions with chaotic and regular behaviour. This kind of a waveguide was also studied by Abdullaev (1993), who similarly demonstrated the route to chaos for increasing incidence angle at the source. The Lyapunov exponents and Poincare's section were, e.g., succesfully employed in the problems of chaotic ray trajectories in ocean acoustics (Smith et al., 1992a).

Correlations

Definition: Correlation matrix of phase-space velocities $\boldsymbol{C}(\tau;\mathbf{w}^0)$ has the elements

$$C_{ij}(\tau;\mathbf{w}^0) =$$

$$\lim_{T\to\infty}\frac{1}{T}\left[\int_0^T F_i(\mathbf{w}(t;\mathbf{w}^0))F_j(\mathbf{w}(t+\tau;\mathbf{w}^0))\,dt-\frac{1}{T}\int_0^T F_i(\mathbf{w}(t;\mathbf{w}^0))\,dt\int_0^T F_j(\mathbf{w}(t+\tau;\mathbf{w}^0))\,dt\ .\right] \tag{19}$$

In many chaotic systems $|C_{ij}|$ decay exponentially with the time-shift τ. The decay coefficient can be then viewed similarly as the Lyapunov exponent—its inverse is the characteristic time of predictability of the ray behaviour near the selected ray $\mathbf{w}(t;\mathbf{w}^0)$.

Of course, this is not the only possibility of correlating. We can define other correlation coefficients associated separately with the time-evolution of generalized coordinates w_i, $i=1,\ldots,N/2$ and with their conjugate momenta w_i, $i=N/2+1,\ldots,N$. Define the *coordinate correlation coefficient* $C_q(t;\mathbf{w}^0)$ by means of the corresponding scalar products of coordinate vector of small perturbations in time t with its original position,

$$C_q(t;\mathbf{w}^0)=$$

$$\frac{\int_M\sum_{i=1}^{N/2}\sum_{j=1}^{N}\delta w_i^0\Pi_{ij}(t;\mathbf{w}^0)\delta w_j^0\,d\mu(\delta\mathbf{w}^0)}{\sqrt{\int_M\sum_{i=1}^{N/2}\delta w_i^0\delta w_i^0\,d\mu(\delta\mathbf{w}^0)\int_M\sum_{i=1}^{N/2}\sum_{j=1}^{N}\sum_{k=1}^{N}\Pi_{ij}(t;\mathbf{w}^0)\delta w_j^0\Pi_{ik}(t;\mathbf{w}^0)\delta w_k^0\,d\mu(\delta\mathbf{w}^0)}}\ , \tag{20}$$

where $M\subset E_N$ is a local neighbourhood of $\mathbf{0}$ and μ is a probability measure defined on M. Analogically, we can define the *momentum correlation coefficient* $C_p(t;\mathbf{w}^0)$,

$$C_p(t;\mathbf{w}^0)=$$

$$\frac{\int_M\sum_{i=N/2+1}^{N}\sum_{j=1}^{N}\delta w_i^0\Pi_{ij}(t;\mathbf{w}^0)\delta w_j^0\,d\mu(\delta\mathbf{w}^0)}{\sqrt{\int_M\sum_{i=N/2+1}^{N}\delta w_i^0\delta w_i^0\,d\mu(\delta\mathbf{w}^0)\int_M\sum_{i=N/2+1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}\Pi_{ij}(t;\mathbf{w}^0)\delta w_j^0\Pi_{ik}(t;\mathbf{w}^0)\delta w_k^0\,d\mu(\delta\mathbf{w}^0)}}\ . \tag{21}$$

Probability measure enables to introduce guesses of initial-value uncertainities and thus relate spatial and momentum coordinates which have different physical dimensions. For example, if we have an exact initial spatial location of a ray, μ consists of delta function over spatial coordinates. C_q is not defined in such a way but we may write for C_p

$$C_p(t;\mathbf{w}^0)=$$

$$\frac{\int_{M_p}\sum_{i=N/2+1}^{N}\sum_{j=N/2+1}^{N}\delta w_i^0\Pi_{ij}(t;\mathbf{w}^0)\delta w_j^0\,d\mu_p(\delta\mathbf{w}^0)}{\sqrt{\int_{M_p}\sum_{i=N/2+1}^{N}\delta w_i^0\delta w_i^0\,d\mu_p(\delta\mathbf{w}^0)\int_{M_p}\sum_{i=N/2+1}^{N}\sum_{j=N/2+1}^{N}\sum_{k=N/2+1}^{N}\Pi_{ij}(t;\mathbf{w}^0)\delta w_j^0\Pi_{ik}(t;\mathbf{w}^0)\delta w_k^0\,d\mu_p(\delta\mathbf{w}^0)}}\ , \tag{22}$$

where M_p is orthogonal projection of M onto the subspace of momentum coordinates and μ_p is corresponding probability measure on M_p.

Since $\Pi_{ij} w_j \sim \exp \lambda_N t$, we normalize the correlation coefficients C_q and C_p to one in the definitions (20) and (21). The time-behaviour of $\ln |C_q|$ or $\ln |C_p|$ can again yield the guess of time-predictability of rays near the selected ray.

Wavelet spectra: A tool for analysis of local chaoticity

The traditional analysis of time series is executed by means of the Fourier transform. The Fourier power spectra enables to differentiate between regular and irregular parts of signals considering both characteristic frequencies revealed as peaks in spectra and broadband "noisy" background corresponding to the chaoticity in the signals. This distinction is, however, rather subjective. The Fourier analysis is mainly the tool for analysis of measured time series, where the level of broadband "noise" is the main practical criterion for the determination of chaos, but it can be applied also to synthetic time-series, which can improve the insight into the studied problem (Smith et al., 1992a,b). If we numerically integrate (1), it is natural to deal with the frequency contents of phase-space velocities.

$$S_i(f; \mathbf{w}^0) = \lim_{T \to \infty} \frac{1}{T} \left| \int_0^T F_i(\mathbf{w}(t; \mathbf{w}^0)) e^{-2\pi i f t} \, dt \right|^2 . \tag{23}$$

The Lyapunov exponents, the correlation matrix as well as the Fourier spectra yield the characteristics of global behaviour ($t \to \infty$) of a selected ray. In various models the chaotic behaviour of rays can be limited to only a finite time interval, which can hardly be revealed by the global methods. Moreover, synthetic models are usually finite, which brings problems with evaluation of $\lim_{t \to \infty}$. This is the reason why a method, which is able to reveal a multiscale contents in analyzed signal on a neighbourhood of a fixed time t^0 can be highly desirable.

In the last decade the multiscale analysis by means of *wavelets* became very popular in various physical and technical applications. There are many textbooks and papers on wavelets, I can recommend to a potential reader the book by Kaiser (1994), where only a "friendly level of mathematics" is employed, or the classical monograph by Daubechies (1992). A review of other geophysical applications of wavelets can be found, e.g., in Kumar and Foufoula-Georgiou (1997). We will remind here how wavelets yield the multiresolution analysis of the space $L_2(E_1)$ to make it clear that the decomposition of a time series into a wavelet basis yield the information about the power of the signal on different scales in different times.

Let $\psi(t)$ be a real function from the Hilbert space $L_2(E_1)$ with the scalar product $< f | g >_{L^2} = \int_{E_1} f(t) g(t) \, dt$ and let m and n be numbers from the set of integers Z. The *translation operator* T^n is defined by the relation

$$T^n(\psi) = \psi(t - n) \tag{24}$$

and the *dilatation operator* D^m is

$$D^m(\psi) = 2^{-\frac{m}{2}}\psi(2^{-m}t) \ . \tag{25}$$

Both operators are the isometries, i.e., $||T^n\psi||_{L^2} = ||D^m\psi||_{L^2} = ||\psi||_{L^2}$, but they do not commute as $D^mT^n = T^{2^m n}D^m$.

A function $\phi(t)$ is called *scaling function*, if the functions

$$\phi_{m,n}(t) \equiv D^mT^n\phi(t) \tag{26}$$

are orthonormal for each fixed m, if

$$\int_{E_1} \phi(t)\, dt = 1 \ . \tag{27}$$

and if the sequence of spaces $\{V_m \subset L_2(E_1)\}_{m=-\infty}^{\infty}$

$$V_m = \text{span}\,\{\phi_{m,k}(t), k \in Z\} \tag{28}$$

satisfies

1. $V_{m+1} \subset V_m$,
2. $P_m(f) = 0 \ \ \forall f \in L^2(E_1)$, $\ m \to \infty$,
3. $P_m(f) = f \ \ \forall f \in L^2(E_1)$, $\ m \to -\infty$,

where P_m is the projection operator onto the space V_m. The sequence of spaces $\{V_m\}_{m=-\infty}^{\infty}$ is called the *multiresolution analysis* of $L_2(E_1)$.

Define the spaces W_m by the relation

$$V_{m-1} = V_m \oplus W_m \tag{29}$$

and denote by Q_m the orthogonal projection onto W_m. Moreover, define the operators

$$H = D^{-1}P_1 \ , \quad G = D^{-1}Q_1 \ , \tag{30}$$

and their adjoints

$$H^* = DP^0 \ , \quad G^* = DQ^0 \ . \tag{31}$$

If we consider them like the operators between the spaces

$$H : V^0 \to V^0 \ , \quad H^* : V^0 \to V^0 \ , \tag{32}$$

$$G : V^0 \to W^0 \ , \quad G^* : W^0 \to V^0 \ , \tag{33}$$

we may write

$$H^*(\phi) = \sum_n h_n T^n(\phi) \ , \tag{34}$$

$$G^*(\psi) = \sum_n g_n T^n(\phi) \ , \tag{35}$$

where $\{h_n\}$ ($\{g_n\}$) are called *low-pass (high-pass) filter coefficients* and

$$\psi = D^{-1} \sum_n g_n T^n(\phi) \tag{36}$$

is the *mother wavelet.* The functions

$$\psi_{m,n}(t) \equiv D^m T^n(\psi) \tag{37}$$

are called *wavelets* and it holds that

$$W_m = \text{span}\,\{\psi_{m,n}, n \in Z\} \ . \tag{38}$$

Since

$$L_2(E_1) = \bigoplus_{m=-\infty}^{\infty} W_m \ , \tag{39}$$

the decomposition of time series into wavelets represents the natural way of time-scale analysis.

Concluding remark

The chaotic behaviour of rays due to reflecting and/or transmitting interfaces represents the special problem which is the extension of the so-called "chaotic billiards", i.e., the propagation of rays in a homogeneous medium bounded by curved reflecting boundaries. Although the billiard problems could seem rather simple in comparison with general ray propagation problems, they are still studied even for elementary geometries of the walls. For example, Artuso et al. (1995) investigated decay properties of the velocity-velocity correlation function for "diamond billiards" and demonstrated its exponential decay in their numerical experiments.

If we concentrate on chaotic propagation due to the volume heterogeneities, the fundamental question is, which perturbations of a homogeneous model yield the exponential divergence of rays. The ray tracing equations are just the canonical equations of a Hamiltonian system. Hence the Kolmogorov-Arnold-Moser (KAM) theory (e.g., Guckenheimer and Holmes, 1983) may be applied to bounded or periodic models (see also Arnold, 1989). Remind here several points of the classical theory of Hamiltonian systems.

We suppose that the total Hamiltonian H can be decoupled into the unperturbed and perturbed parts,

$$H = H^0 + \epsilon H_1 \ , \quad \epsilon > 0 \ , \tag{40}$$

where $\epsilon \ll 1$ is the perturbing parameter. If there exists a generating function $S = S(q_i, I_i)$ satisfying the Hamilton-Jacobi equation for the unperturbed Hamiltonian $H^0(q_i, p_i)$ (q_i are the generalized coordinates and p_i corresponding conjugate momenta), i.e.,

$$H^0(q_i, \frac{\partial S}{\partial I_i}) = 0 \ , \tag{41}$$

then we can introduce new generalized coordinates

$$\theta_i = \frac{\partial S}{\partial I_i} \ . \tag{42}$$

If, moreover, H^0 does not depend on θ_i, $H^0 = H^0(I_i)$, then the Hamilton equations are simply

$$\frac{dI_i}{dt} = 0 \quad \Rightarrow \ I_i = \text{const.} \ \ \forall i = 1, 2, \ldots, N/2 \ , \tag{43}$$

$$\frac{d\theta_i}{dt} = \omega_i = \text{const.} \quad \Rightarrow \ \theta_i = \omega_i t + {\theta^0}_i \ \ \forall i = 1, 2, \ldots, N/2 \ . \tag{44}$$

Because this formalism was originally developed in celestial mechanics, I_i are called the *actions*, θ_i are the *angles* and ω_i, which can be considered as the functions of actions, are the *frequencies* of motion. The actions are the independent integrals of motion and the $N/2$-dimensional manifold in the N-dimensional phase-space corresponding to a choice of values of these integrals can be described like a $N/2$-dimensional torus.

If the frequencies are algebraically independent, i.e.,

$$\sum_{i=1}^{N} m_i \omega_i \neq 0 \tag{45}$$

for all integers m_i and, moreover, far from the resonances, i.e.,

$$\left| \sum_{i=1}^{N} m_i \omega_i \right| > C \left(\sum_{i=1}^{N} |m_i| \right)^{-\nu} \qquad C > 0, \ \nu > 0 \ , \tag{46}$$

then, according to the KAM theory, for sufficiently small values of ϵ and sufficiently smooth perturbed Hamiltonian ($H_1 \in C^{2+\delta}$), most invariant tori are preserved and they differ from the unperturbed ones only by a weak deformation. Phase trajectories wind densely around the deformed tori everywhere and describe a quasi-periodic motion with $N/2$ frequencies.

In other words, homogeneous models do not directly change to chaotic models just by adding any smooth three-dimensional heterogeneity. In general cases, the situation may be very complicated. Route to chaos in ray tracing should thus be the goal of further intensive research.

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