Fast Sweeping Methods for Factored TTI Eikonal Equation

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June 9, 2015

Traveltime computation

- **Key role in seismic data processing**
- **Applications : Near-surface modeling, microseismic monitoring, reservoir characterization, etc.**
- **Travel computations must honor anisotropy**
- **Ray tracing and FD methods are mostly used**

Why develop eikonal solver?

Advantages of eikonal solver

- **Computationally more efficient**
- **Interpolation on regular grids not required**
- **Rarely produce shadow zones**
- **Limitation?**

Which eikonal solving algorithm to use?

Eikonal Solvers

Different approaches proposed:

- **Embedding methods**
- **Iterative methods**
- **Single-pass methods**
- **Sweeping methods**

Single-pass methods

- **Isotropic eikonal equation (Sethian and Popovici, 1999)**
- **Works for VTI media (Alton and Mitchell, 2007)**
- **Ordered Upwind Method (Sethian and Vladimirsky, 2003)**
- **TEA eikonal solver (Konukoglu et al., 2007)**

Sweeping methods

- **Isotropic eikonal equation (Zhao, 2005)**
- **Modifications to incorporate anisotropy (Qian et al., 2007)**
- **Solves the TEA eikonal equation at most**

Advantages of FS over FM

- **Simple to implement**
- **Convergence achieved in finite number of iterations**
- **FS more flexible for general equation**
- **Cost:** $\mathcal{O}(N)$ **vs.** $\mathcal{O}(N\, log N_{NB})$

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We will use Fast sweeping!

Challenge

- **Isotropic/elliptic eikonal equation requires solving:**

$$a\,T_{i,j,k}^{2} + b\,T_{i,j,k} + c = 0$$

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$$\alpha_4 T_{i,j,k}^{4} + \alpha_3 T_{i,j,k}^{3} + \alpha_2 T_{i,j,k}^{2} + \alpha_1 T_{i,j,k} + \alpha_0 = 0$$

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- **Orthorhombic eikonal equation requires solving:**

$$\beta_6 T_{i,j,k}^6 + \beta_5 T_{i,j,k}^5 + \beta_4 T_{i,j,k}^4 + \beta_3 T_{i,j,k}^3 + \beta_2 T_{i,j,k}^2 + \beta_1 T_{i,j,k} + \beta_0 = 0$$

Iterative Eikonal Solver

(Waheed et al., 2015)

TTI Eikonal Equation

- **Eikonal equation for 2D TTI media:**

$$A\left(\frac{\partial T}{\partial x}\right)^2 + B\left(\frac{\partial T}{\partial z}\right)^2 + C\left(\frac{\partial T}{\partial x}\right)^2\left(\frac{\partial T}{\partial z}\right)^2 = 1$$

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- **At the first iteration, we solve:**

$$A\left(\frac{\partial T_1}{\partial x}\right)^2 + B\left(\frac{\partial T_1}{\partial z}\right)^2 = 1$$

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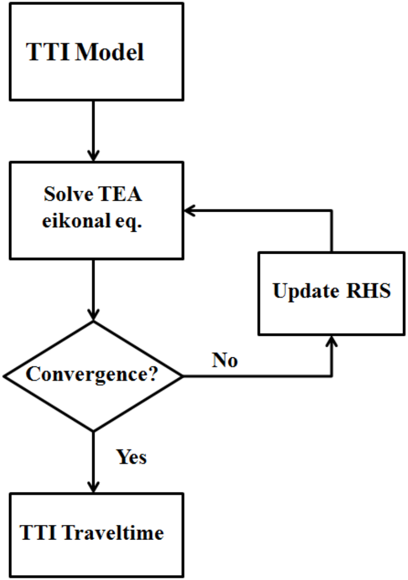
- **At the n-th iteration, we solve:**

$$A\left(\frac{\partial T_n}{\partial x}\right)^2 + B\left(\frac{\partial T_n}{\partial z}\right)^2 = 1 - C\left(\frac{\partial T_{n-1}}{\partial x}\right)^2\left(\frac{\partial T_{n-1}}{\partial z}\right)^2$$

Algorithm


TTI Model
Solve TEA
eikonal eq.
Update RHS
Convergence?
No
Yes
TTI Traveltime

Source Singularity

- **The problem of upwind source singularity**
- **At best we get polluted first-order convergence**
- **Prevents reliable computations of take-off angles and amplitudes**

Tackling the singularity problem

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- **Fixed local grid refinement method (Kim and Cook, 1999)**

- **Adaptive grid refinement method (Qian and Symes, 2002)**

Tackling the singularity problem

- **Factored eikonal equation for isotropic media (Fomel et al., 2009)**
- **Clean first-order convergence**
- **Traveltime factored into two multiplicative functions**

Tackling the singularity problem

- **Factored eikonal equation for isotropic media (Fomel et al., 2009)**

- **Clean first-order convergence**

- **Traveltime factored into two multiplicative functions**

- **Factored eikonal equation for TEA media (Luo and Qian, 2012)**

Factored TTI Eikonal Equation

TI Eikonal Equation

- **Eikonal equation for 2D VTI media (Alkhalifah, 2000):**

$$\boldsymbol{v_x^2 \left(\frac{\partial T}{\partial x}\right)^2 + v_t^2 \left(\frac{\partial T}{\partial z}\right)^2 \left(1 - \frac{2\eta v_x^2}{1+2\eta}\left(\frac{\partial T}{\partial x}\right)^2\right) = 1}$$

TI Eikonal Equation

- **Eikonal equation for 2D VTI media (Alkhalifah, 2000):**

$$v_x^2 \left(\frac{\partial T}{\partial x}\right)^2 + v_t^2 \left(\frac{\partial T}{\partial z}\right)^2 \left(1 - \frac{2\eta v_x^2}{1+2\eta}\left(\frac{\partial T}{\partial x}\right)^2\right) = 1$$

- **Re-write as:**

$$v_x^2 \left(\frac{\partial T}{\partial x}\right)^2 + v_t^2 \left(\frac{\partial T}{\partial z}\right)^2 = 1 + \frac{2\eta v_x^2 v_t^2}{1+2\eta}\left(\frac{\partial T}{\partial x}\right)^2 \left(\frac{\partial T}{\partial z}\right)^2$$

TI Eikonal Equation

- **Can be re-written as:**

$$\tilde{v}_x^2 \left(\frac{\partial T}{\partial x}\right)^2 + \tilde{v}_t^2 \left(\frac{\partial T}{\partial z}\right)^2 = 1$$

where

$$\tilde{v}_x = \frac{v_x}{\sqrt{f(T)}}, \qquad \tilde{v}_t = \frac{v_t}{\sqrt{f(T)}}$$

$$f(T) = 1 + \frac{2\eta v_x^2 v_t^2}{1+2\eta} \left(\frac{\partial T}{\partial x}\right)^2 \left(\frac{\partial T}{\partial z}\right)^2$$

Additive Factorization

- **Factorize the solution into two additive factors:**

$$\boldsymbol{T} = \boldsymbol{T_0} + \boldsymbol{\tau}$$

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Additive Factorization

- **Factorize the solution into two additive factors:**

$$T = T_0 + \tau$$

$$\Rightarrow \nabla T = \nabla T_0 + \nabla \tau$$

- **Plugging this, the TI eikonal equation becomes:**

$$\tilde{v}_x^2 \left(\frac{\partial T_0}{\partial x} + \frac{\partial \tau}{\partial x} \right)^2 + \tilde{v}_t^2 \left(\frac{\partial T_0}{\partial z} + \frac{\partial \tau}{\partial z} \right)^2 = 1$$

Additive Factorization

- T_0 solves the homogeneous eikonal equation:

$$\tilde{v}_{x0}^2 \left(\frac{\partial T_0}{\partial x}\right)^2 + \tilde{v}_{t0}^2 \left(\frac{\partial T_0}{\partial z}\right)^2 = 1$$

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$$\tilde{v}_{x0}^2 \left(\frac{\partial T_0}{\partial x}\right)^2 + \tilde{v}_{t0}^2 \left(\frac{\partial T_0}{\partial z}\right)^2 = 1$$

$$\tilde{v}_{x0} = \tilde{v}_x(x_0, z_0)$$

$$\tilde{v}_{t0} = \tilde{v}_t(x_0, z_0)$$

$$(x_0, z_0) : \text{source location}$$

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(x_0, z_0) : source location

$$T_0(x,z) = \sqrt{\frac{\tilde{v}_{t0}^2(x - x_0) + \tilde{v}_{x0}^2(z - z_0)}{\tilde{v}_{t0}^2\, \tilde{v}_{x0}^2}}$$

Algorithm

STEPS

1. **Compute $\tilde{v}_x$ and $\tilde{v}_t$**
2. **Solve the factored elliptic eikonal equation**
3. **Update the right hand term $f(T)$**
4. **Repeat the above steps till convergence**

$$|(\tilde{v}_x)_n - (\tilde{v}_x)_{n-1}|_\infty < \epsilon, \quad |(\tilde{v}_t)_n - (\tilde{v}_t)_{n-1}|_\infty < \epsilon$$

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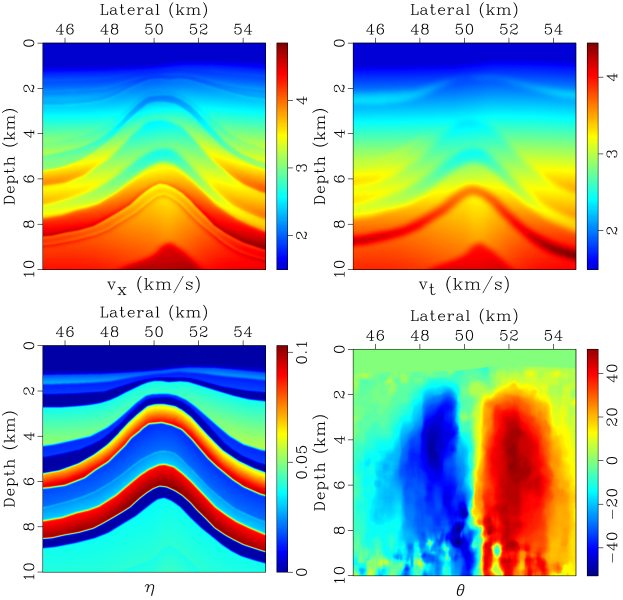
- **Plugging this, the TI eikonal equation becomes:**

$$\tilde{v}_x^2 \left(\frac{\partial T_0}{\partial x} \tau + \frac{\partial \tau}{\partial x} T_0 \right)^2 + \tilde{v}_t^2 \left(\frac{\partial T_0}{\partial z} \tau + \frac{\partial \tau}{\partial z} T_0 \right)^2 = 1$$

Numerical Tests

BP TTI Model

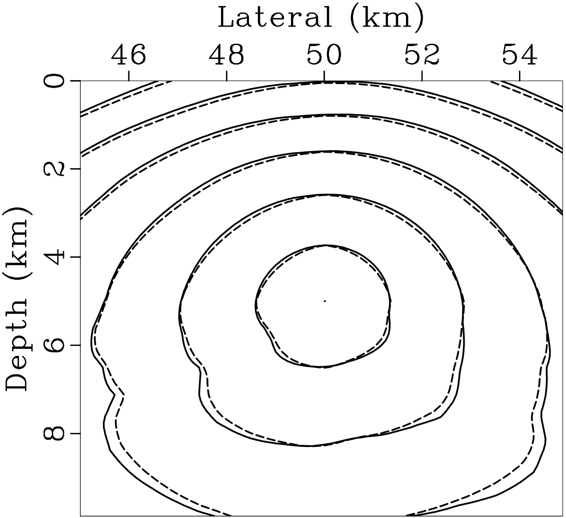


Lateral (km)
46 48 50 52 54
Depth (km)
0 2 4 6 8 10
2 3 4
v_x (km/s)
Lateral (km)
46 48 50 52 54
Depth (km)
0 2 4 6 8 10
2 3 4
v_t (km/s)
Lateral (km)
46 48 50 52 54
Depth (km)
0 2 4 6 8 10
0 0.05 0.1
η
Lateral (km)
46 48 50 52 54
Depth (km)
0 2 4 6 8 10
−40 −20 0 20 40
θ

BP TTI Model Test

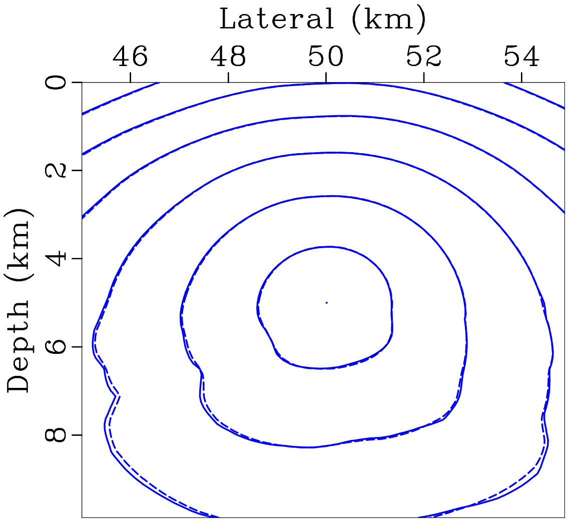
- **Source: (50 km, 5 km)**
- **Fine grid model: $\Delta x = \Delta z = 12.5$ m**
- **Coarse grid model: $\Delta x = \Delta z = 125$ m**

BP TTI Model Test



No factorization

BP TTI Model Test



Multiplicative factorization

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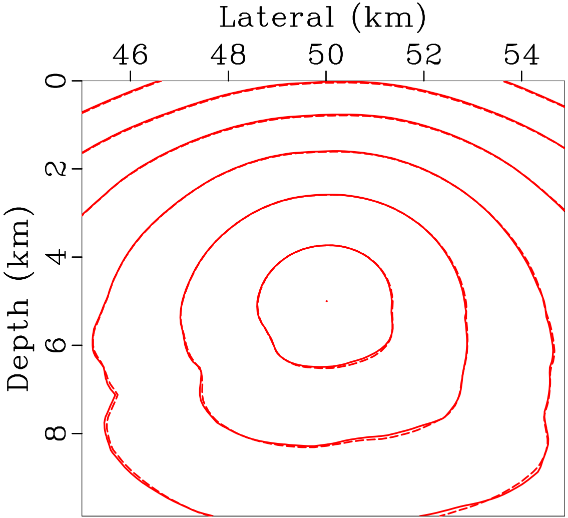


Additive factorization

Convergence Order

No Factorization

Mesh	Error (s)	Convergence order
800×800	0.0178	-
400×400	0.0501	0.711
200×200	0.1451	0.691
100×100	0.3687	0.787
50×50	1.0433	0.707

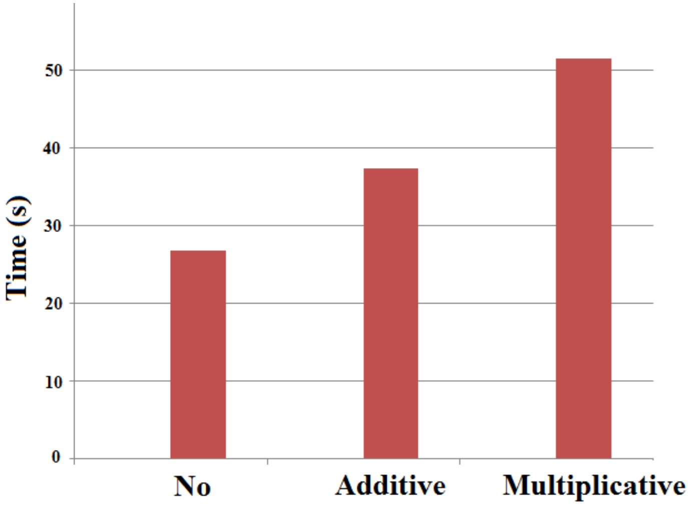
Multiplicative Factorization

Mesh	Error (s)	Convergence order
800 × 800	**0.00479**	**-**
400 × 400	**0.00960**	**0.998**
200 × 200	**0.01878**	**1.022**
100 × 100	**0.03480**	**1.079**
50 × 50	**0.06543**	**1.06**

Additive Factorization

Mesh	Error (s)	Convergence order
800×800	0.00477	-
400×400	0.00971	0.982
200×200	0.01923	1.009
100×100	0.03852	0.998
50×50	0.07205	1.069

Cost comparison



Summary

- **The idea of factorization for TTI eikonal equation**
- **Additive and multiplicative factorizations presented**
- **Which factorization is better?**
- **Easily extendable to lower anisotropic symmetries**

Acknowledgments

- **KAUST for financial support**

- **Members of Seismic Wave Analysis Group**

Convergence Test

- $v_t = 2$ km/s
- $v_x = 2.83$ km/s
- $\eta = 0.21$
- $\theta = 40^\circ$ km/s