ON RICHARDS' PARADOX

Bjørn Ursin, NTNU

Lasse Amundsen, NTNU & Equinor

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The problem:

- Richards (1984) showed that there may be plane-wave solutions to the wave equation which violate the radiation condition.
- Krebes and Dayley (2007) showed that this also creates erroneous R/T coefficients.
- Ursin et al. (2017) gave a stationary-phase solution which bypasses these problems.

Outline

- ❑ Plane waves
- ❑ Dispersion relation
- ❑ The radiation condition
- ❑ Reflection plane-wave integral
    - Complex ray solution
- ❑ Stationary-phase approximation
    - Real ray solution
- ❑ Numerical examples
- ❑ Conclusions

The wave equation

$$\nabla^2\phi + \frac{\omega^2}{c^2}\phi = 0$$

where c is the complex wave speed with

$$\frac{1}{c} = \mathrm{Re}\frac{1}{c} + i\,\mathrm{Im}\frac{1}{c}$$

and where

$$\mathrm{Re}\frac{1}{c} > 0 \text{ and } \mathrm{Im}\frac{1}{c} \geq 0$$

A plane-wave solution is

$$\phi \sim \mathrm{e}^{i\omega(px+qz-t)}$$

The horizontal slowness p and the vertical slowness q are complex

$$p = p_R + ip_I \text{ and } q = q_R + iq_I$$

The dispersion relation

$$p^2 + q^2 = \frac{1}{c^2}$$

The real and imaginary part of the dispersion relation are

$$p_R^2 - p_I^2 + q_R^2 - q_I^2 = \left(\mathrm{Re}\frac{1}{c}\right)^2 - \left(\mathrm{Im}\frac{1}{c}\right)^2$$

and

$$q_R q_I + p_R p_I = \mathrm{Re}\frac{1}{c} \cdot \mathrm{Im}\frac{1}{c}$$

The radiation condition

No disturbance may be radiated from infinity into the finite source region, and the source field must remain finite or go to zero at infinity.

For a complex plane wave this requires that

$$p_R p_I \geq 0 \text{ and } q_R q_I \geq 0$$

The dispersion relation gives

$$q_R q_I = \mathrm{Re}\frac{1}{c} \cdot \mathrm{Im}\frac{1}{c} - p_R p_I \geq 0$$

This is always satisfied if p is real.

The reflected wavefield

For a point source at a vertical distance z_1 above a horizontal reflector and a receiver at a horizontal distance x from the source and a vertical distance z_2 above the reflector:

$$\phi = \frac{A}{4\pi\mu}\int_{-\infty}^{\infty}\frac{R(p)}{q(p)}\, e^{i\omega\tau(p)}dp$$

where all variables may depend on ω; A is the source spectrum,

$$\tau = px + qz, \qquad z = z_1 + z_2$$

is the phase function, $\mu = \rho c^2$ is the shear modulus, and the reflection coefficient is

$$R = \frac{\mu q - \mu' q'}{\mu q + \mu' q'}$$

where μ' and q' denote the variables in the lower medium; p is real and q is complex.

The steepest-descent approximation

The stationary point satisfies

$$\frac{d\tau}{dp} = x + z\frac{dq}{dp} = 0$$

From the dispersion equation

$$p + q\frac{dq}{dp} = 0$$

and then

$$\bar{p} = \frac{x}{rc} = \frac{\sin\theta}{c}\ , \qquad \bar{q} = \frac{z}{rc} = \frac{\cos\theta}{c}$$

Complex rays, homogeneous waves

In the lower medium

$$q'_R q'_I = \mathrm{Re}\frac{1}{c'} \cdot \mathrm{Im}\frac{1}{c'} - \bar{p}_R \bar{p}_I \,, = \mathrm{Re}\frac{1}{c'} \cdot \mathrm{Im}\frac{1}{c'} - \sin^2\theta \, \mathrm{Re}\frac{1}{c} \cdot \mathrm{Im}\frac{1}{c} \geq 0$$

where Snell′s law has been used.

For specific combinations of wave speeds and recording geometries this quantity will be negative corresponding to a non-physical plane wave which propagates away from the interface with increasing amplitude.

The critical angle is

$$\theta_c = \arcsin\left(\frac{\mathrm{Re}\frac{1}{c'} \cdot \mathrm{Im}\frac{1}{c'}}{\mathrm{Re}\frac{1}{c} \cdot \mathrm{Im}\frac{1}{c}}\right)^{\frac{1}{2}}$$

Simple equation

Quite often a very simple inverse velocity function

$$\frac{1}{c} = \frac{1}{c_R}\left(1 + \frac{i}{2Q}\right)$$

is used, and similarly for the lower medium. Then the critical angle is

$$\theta_c = \arcsin\left[\frac{c_R}{c_R'}\left(\frac{Q}{Q'}\right)^{\frac{1}{2}}\right]$$

When $Q = Q'$ this is the standard critical angle for reflection.

The stationary-phase approximation

Here p is real, and the stationary-phase condition is

$$\frac{\partial \tau_R}{\partial p} = x + z\frac{\partial q_R}{\partial p} = 0$$

Approximate solution

$$p = \sin\theta\left(\mathrm{Re}\frac{1}{c^2}\right)^{\frac{1}{2}}$$

$$q_R = \cos\theta\left(\mathrm{Re}\frac{1}{c^2}\right)^{\frac{1}{2}}$$

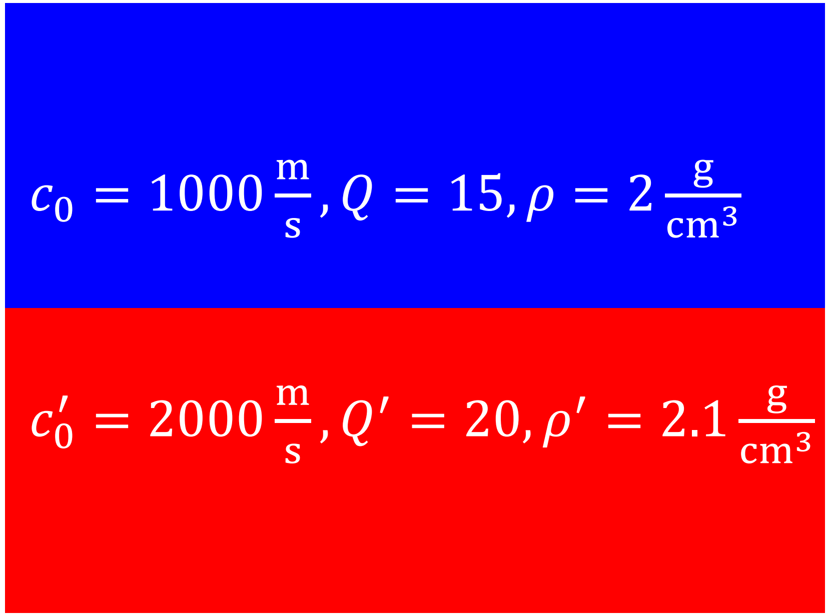
$$q_I = \frac{\mathrm{Im}\frac{1}{c^2}}{2q_R} = \frac{\mathrm{Im}\frac{1}{c^2}}{2\cos\theta\left(\mathrm{Re}\frac{1}{c^2}\right)^{\frac{1}{2}}}$$

Real ray with attenuation computed along the ray

Numerical example (Ursin et al. 2017)

Complex velocity

$$\frac{1}{c} = \frac{1}{c_0}\left(1 + \frac{i}{2Q}\right)$$

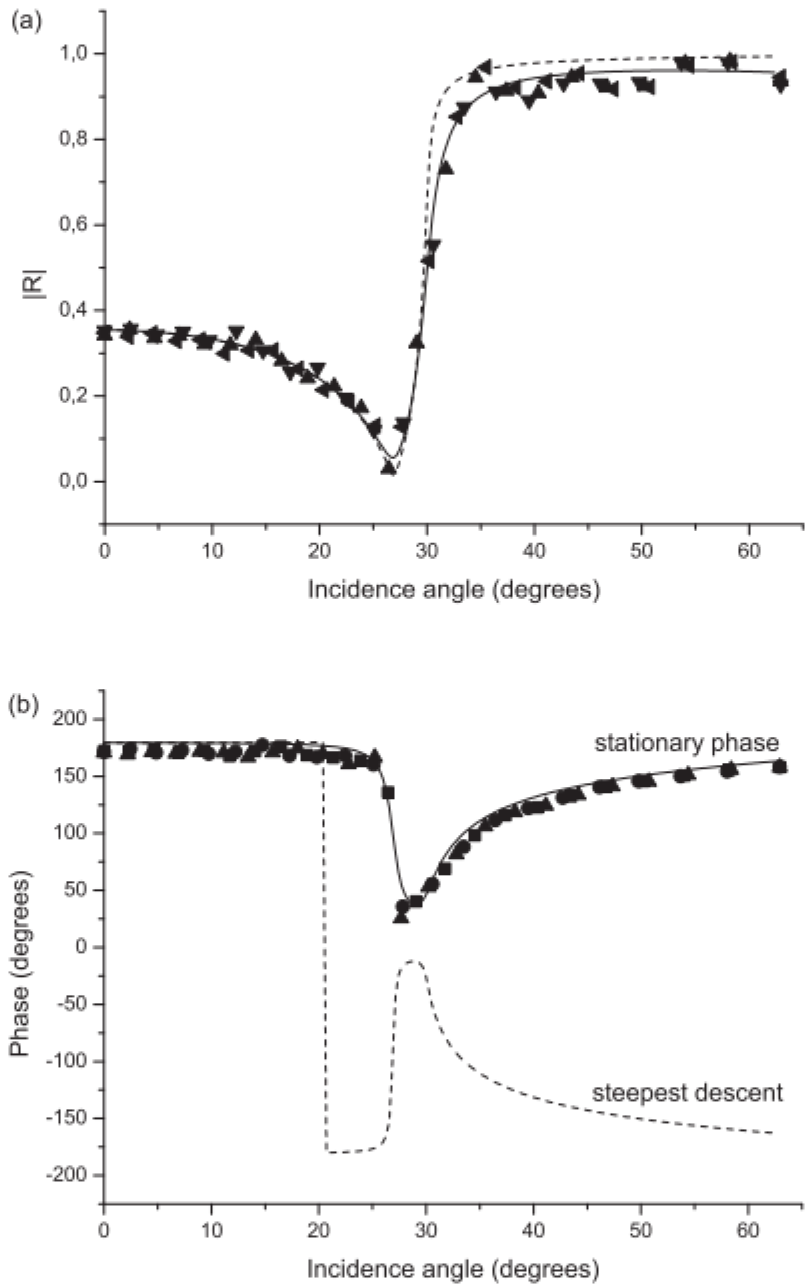


Elastic critical angle

$$\sin\theta_e = \frac{1}{2} \quad , \qquad \theta_e = 30^{\mathrm{o}}$$

New critical angle

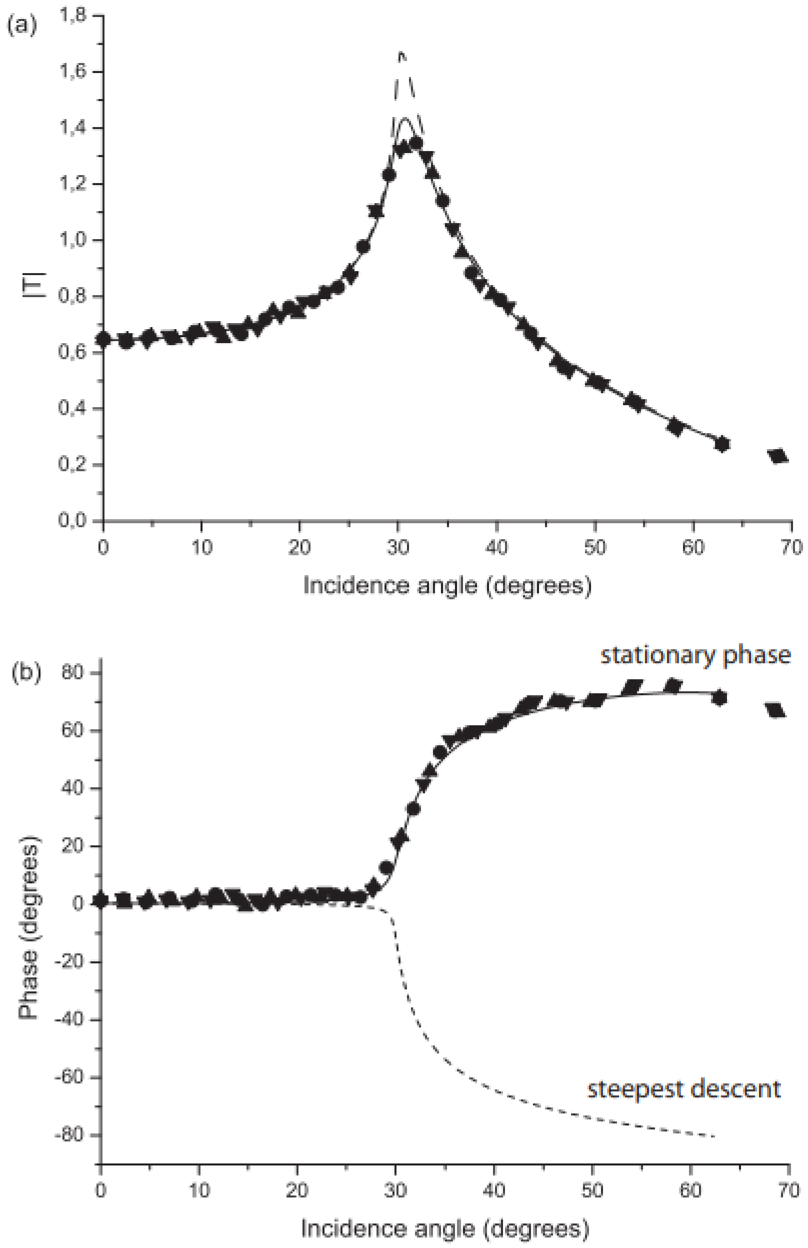
$$\sin\theta_c = \frac{\sqrt{3}}{4} \quad , \qquad \theta_c = 25.7^{\mathrm{o}}$$



Reflection coefficient (a) and phase angle (b) as a function of the angle of incidence.

The solid and dashed lines correspond to the stationary phase and steepest-descent solutions, respectively.

The symbols indicate the numerical evaluation for different frequencies (9 (triangles), 10 (circles) and 11 (squares) Hz).



Transmission coefficient (a) and phase angle (b) as a function of the angle of incidence.

The solid and dashed lines correspond to the stationary-phase and steepest-descent solutions, respectively.

The symbols indicate the numerical evaluation for different frequencies (9 (triangles up), 10 (circles) and 11 (triangles down) Hz).

Conclusions

- The steepest-desent approximation gives complex rays.

  It may lead to non-physical wave solutions.

- The stationary-phase approximation is always stable.

  It gives a good match to the numerical solution.

- Complex rays may lead to non-physical waves.

References

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