Extension of ray theory to anisotropic viscoelastic media

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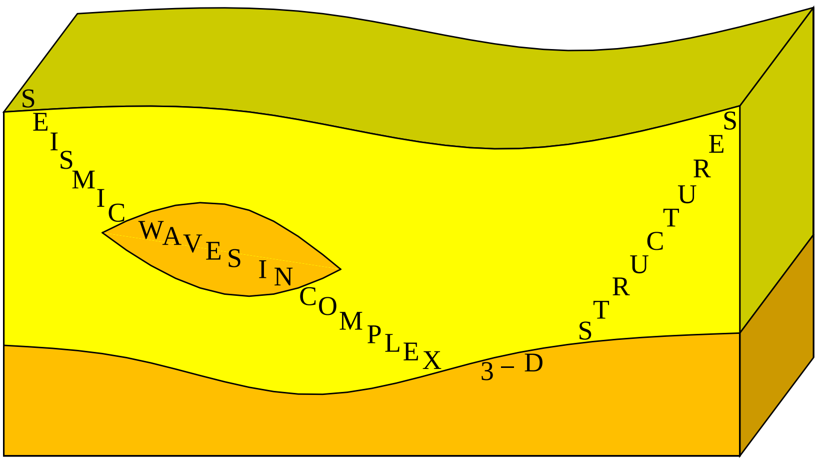

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Attenuation is a very important phenomenon in wave propagation, and is essential whenever the intensity of waves matters.

It is thus fundamental to extend the ray theory from anisotropic elastic media to anisotropic viscoelastic media.

Real-valued ray tracing in viscoelastic media and perturbation towards the complex-valued velocity model

The idea of the extension looks simple: The eikonal equation in an attenuating medium has the form of a complex-valued Hamilton-Jacobi equation, which would generate complex-valued rays. Since we know the velocity model in real space rather than in complex space, we have to trace the real-valued reference rays using the reference Hamiltonian function, and calculate the complex-valued travel time right in real space by the perturbation from the reference travel time calculated along real-valued reference rays to the complex-valued travel time defined by the complex-valued Hamilton-Jacobi equation (Klimeš, 2002; 2016). Analogously for the corresponding amplitudes (Klimeš, 2006a) and other quantities.

For this purpose, Klimeš & Klimeš (2011) designed the construction of the optimum reference Hamiltonian function $\widetilde{H}(x^m, p_n)$ corresponding to a given complex-valued Hamiltonian function $H(x^m, p_n)$:

$$\widetilde{H}(x^m, p_n) = \sum_{\Omega=0}^{+\infty} \frac{\mathrm{i}^\Omega}{\Omega!} \,\mathrm{Re}[H^{,k_1 k_2 \ldots k_\Omega}(x^m, \mathrm{Re}\, p_n)]\, \mathrm{Im}(p_{k_1})\, \mathrm{Im}(p_{k_2}) \ldots \mathrm{Im}(p_{k_\Omega})$$

where

$$H^{,k_1 k_2 \ldots k_\Omega}(x^m, p_n) = \frac{\partial}{\partial p_{k_1}} \frac{\partial}{\partial p_{k_2}} \ldots \frac{\partial}{\partial p_{k_\Omega}} H(x^m, p_n) \quad .$$

We have encountered various more or less expected problems:

Missing S-wave eigenvector of the complex-valued Christoffel matrix at an S-wave singularity.

Divergence of the S-wave eigenvectors of the Christoffel matrix when approaching an S-wave singularity.

Divergence of the phase-space derivatives of the anisotropic-ray-theory S-wave Hamiltonian function at S-wave singularities.

Divergence of the anisotropic-ray-theory vectorial amplitudes at S-wave singularities.

Complex-valued S-wave eigenvectors in the coupling ray theory.

Diverging S-wave eigenvectors in the coupling ray theory.

Let us discuss these problems and their solutions.

Problems with missing eigenvector of the Christoffel matrix

The ray tracing equations and the corresponding equations of geodesic deviation (Červený, 1972) are often formulated in terms of the eigenvectors of the Christoffel matrix (Klimeš, 2006b). Unfortunately, a complex-valued Christoffel matrix need not have all three eigenvectors at an S-wave singularity (Klimeš, 2021).

We thus have to formulate the ray tracing equations and the corresponding equations of geodesic deviation using the characteristic values of the complex-valued Christoffel matrix, without the eigenvectors of the Christoffel matrix (Klimeš, 2020).

The resulting equations for the real-valued reference P-wave rays and real-valued reference common S-wave rays are applicable everywhere, including S-wave singularities.

Divergence of the S-wave eigenvectors of the Christoffel matrix

The eigenvectors of the complex-valued Christoffel matrix are normalized to unit complex-valued pseudonorm with respect to pseudoscalar product $a_i b_i$ of vectors a_i and b_i rather than to unit real-valued norm with respect to scalar product $a_i^* b_i$, because the eigenvectors are pseudoorthogonal with respect to the pseudoscalar product (Klimeš, 2018). Note that

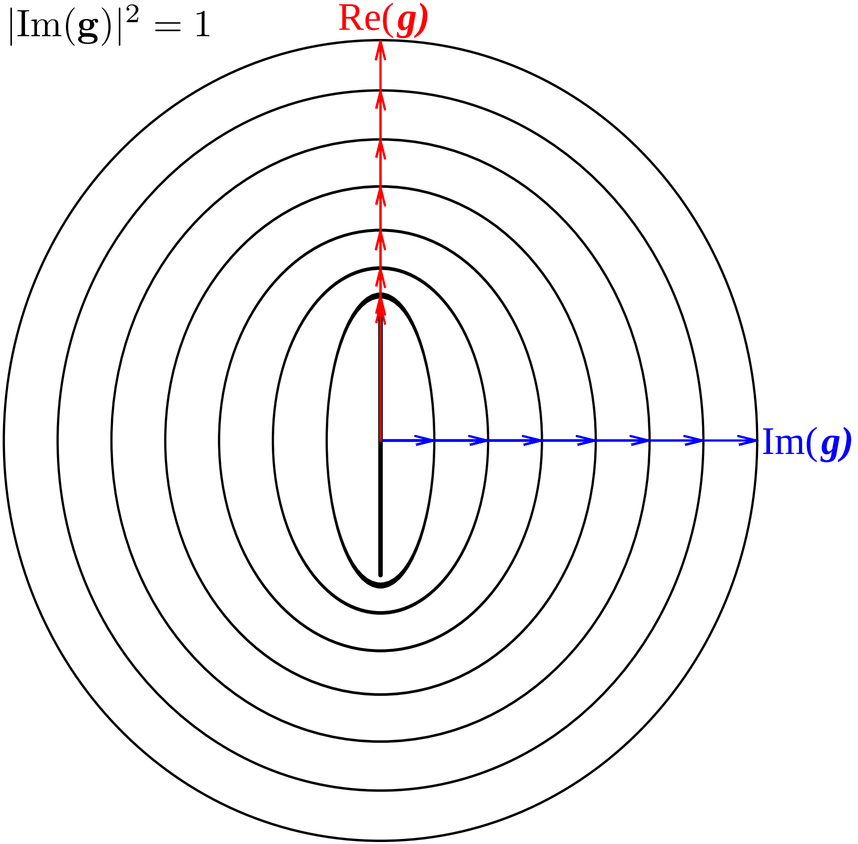
$$G = g_i \Gamma_{ij} g_j \quad \Longleftrightarrow \quad g_i g_i = 1$$

where G is the characteristic value corresponding to the eigenvector g_i of the complex-valued Christoffel matrix.

Consequently, the pseudonormal S-wave eigenvectors frequently diverge when approaching an S-wave singularity (Klimeš, 2022). This divergence does not occur in elastic media.

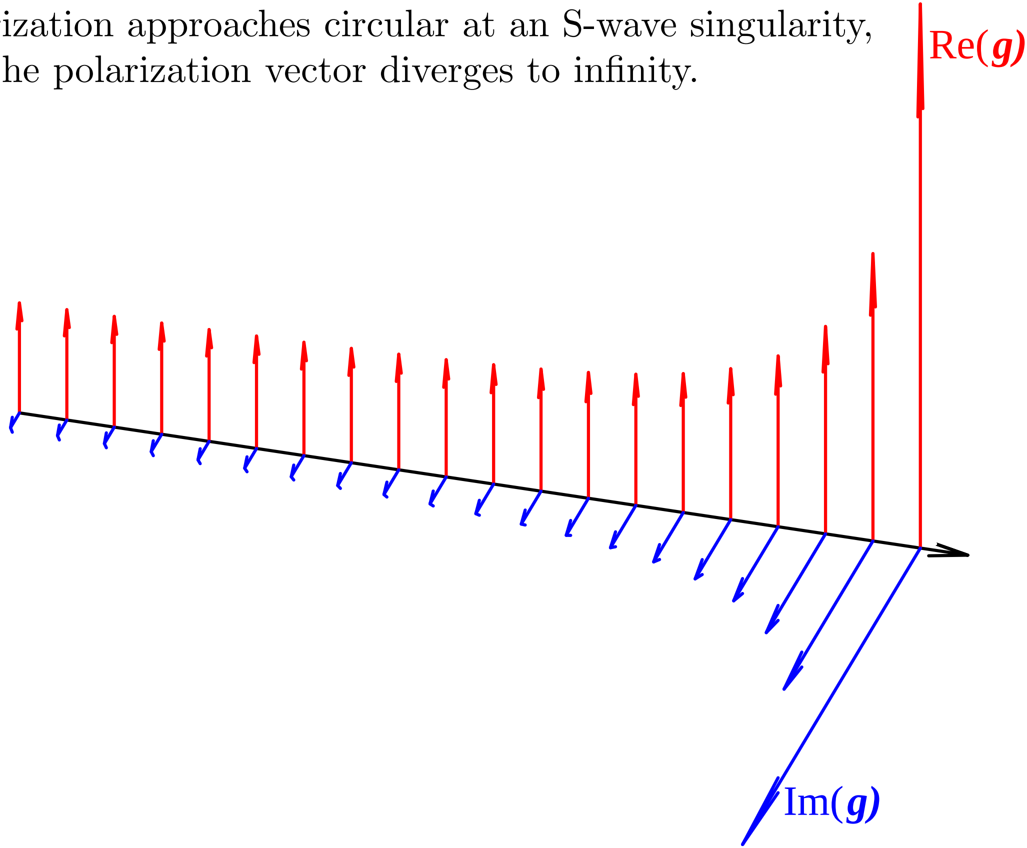
Varying ellipticity of a pseudonormal S-wave eigenvector

$|\mathrm{Re}(\mathbf{g})|^2 - |\mathrm{Im}(\mathbf{g})|^2 = 1$



Varying ellipticity of a pseudonormal S-wave eigenvector in 3-D

Polarization approaches circular at an S-wave singularity, and the polarization vector diverges to infinity.



Divergence of the phase-space derivatives of the anisotropic-ray-theory S-wave Hamiltonian function

As a result of diverging S-wave eigenvectors, the phase-space derivatives of the anisotropic-ray-theory Hamiltonian function used to trace the anisotropic-ray-theory rays may also diverge when approaching an S-wave singularity. Note that the phase-space derivatives

$$G' = g_i \Gamma'_{ij} g_j$$

of characteristic value G often diverge with diverging g_i because the structure and characteristic values of Γ'_{ij} differ from those of Γ_{ij}.

Fortunately, the phase-space derivatives of the reference common S-wave Hamiltonian function used to trace the reference common S-wave rays are smooth at S-wave singularities (Klimeš, 2020).

S-wave characteristic values G_1 and G_2 and P-wave characteristic value G_3 satisfy relation

$$\tfrac{1}{2}(G_1 + G_2) = \tfrac{1}{2}(\Gamma_{ii} - G_3)$$

which implies that complex-valued common S-wave Hamiltonian function $H = \frac{1}{2}(H_1 + H_2)$ is smooth. Then the corresponding real-valued reference Hamiltonian function $\widetilde{H}$ of a real-valued slowness vector is smooth as well.

Divergence of the anisotropic-ray-theory vectorial amplitudes

If the S-wave eigenvector g_i of the complex-valued Christoffel matrix diverges at an S-wave singularity, the corresponding anisotropic-ray-theory vectorial amplitude $A_i = A\, g_i$ diverges at the S-wave singularity as well. This divergence does not occur in elastic media where the anisotropic-ray-theory vectorial amplitudes diverge at caustics only.

Coupling ray theory for S waves in viscoelastic media

Since the variation of the S-wave eigenvectors along the reference ray plays a decisive role in the coupling equations (Bulant & Klimeš, 2002; Klimeš, 2013), the divergence of S-wave eigenvectors in a vicinity of an S-wave singularity represents a considerably challenging problem in the coupling ray theory for S waves in viscoelastic media.

The coupling equations for elastic media contain the real-valued angle φ of rotation of the S-wave eigenvectors about the P-wave eigenvector. The coupling equations for viscoelastic media contain the complex-valued angle $\varphi+\mathrm{i}\psi$ of rotation:

$$\begin{pmatrix} \cos(\varphi+\mathrm{i}\psi) & -\sin(\varphi+\mathrm{i}\psi) \\ \sin(\varphi+\mathrm{i}\psi) & \cos(\varphi+\mathrm{i}\psi) \end{pmatrix} = \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{pmatrix} \begin{pmatrix} \cos(\mathrm{i}\psi) & -\sin(\mathrm{i}\psi) \\ \sin(\mathrm{i}\psi) & \cos(\mathrm{i}\psi) \end{pmatrix}$$

where

$$\begin{pmatrix} \cos(\mathrm{i}\psi) & -\sin(\mathrm{i}\psi) \\ \sin(\mathrm{i}\psi) & \cos(\mathrm{i}\psi) \end{pmatrix} = \begin{pmatrix} \cosh(\psi) & -\mathrm{i}\sinh(\psi) \\ \mathrm{i}\sinh(\psi) & \cosh(\psi) \end{pmatrix} \quad .$$

Importance of the coupling ray theory in viscoelastic media

Fortunately, the coupling equations for viscoelastic media compensate the divergence of S-wave eigenvectors and yield vectorial amplitudes smoothly varying through S-wave singularities.

Whereas the coupling ray theory corrects just the S-wave polarization in elastic media, it corrects also the S-wave amplitudes at S-wave singularities in viscoelastic media which may diverge in the anisotropic ray theory.

To overcome infinite ray-theory amplitudes, we thus need coupling-ray-theory Gaussian beams or packets.

References:

Bulant, P. & Klimeš, L. (2002): Numerical algorithm of the coupling ray theory in weakly anisotropic media. *Pure appl. Geophys.*, **159**, 1419–1435.

Červený, V. (1972): Seismic rays and ray intensities in inhomogeneous anisotropic media. *Geophys. J. R. astr. Soc.*, **29**, 1–13.

Klimeš, L. (2002): Second-order and higher-order perturbations of travel time in isotropic and anisotropic media. *Stud. geophys. geod.*, **46**, 213–248.

Klimeš, L. (2006a): Spatial derivatives and perturbation derivatives of amplitude in isotropic and anisotropic media. *Stud. geophys. geod.*, **50**, 417–430.

Klimeš, L. (2006b): Common-ray tracing and dynamic ray tracing for S waves in a smooth elastic anisotropic medium. *Stud. geophys. geod.*, **50**, 449–461.

Klimeš, L. (2013): Coupling ray series. *Stud. geophys. geod.*, **57**, 253–266.

Klimeš, L. (2016): Transformation of spatial and perturbation derivatives of travel time at a curved interface between two arbitrary media. *Stud. geophys. geod.*, **60**, 451–470.

Klimeš, L. (2018): Frequency-domain ray series for viscoelastic waves with a non-symmetric stiffness matrix. *Stud. geophys. geod.*, **62**, 261–271.

Klimeš, L. (2020): Tracing real-valued reference rays in anisotropic viscoelastic media. *Seismic Waves in Complex 3-D Structures*, **30**, 77–94.

Klimeš, L. (2021): Two S-wave eigenvectors of the Christoffel matrix need not exist in anisotropic viscoelastic media. *Stud. geophys. geod.*, **65**, 291–295.

Klimeš, L. (2022): S-wave polarization vectors in anisotropic viscoelastic media. *Seismic Waves in Complex 3-D Structures*, **31**, 69–87.

Klimeš, M. & Klimeš, L. (2011): Perturbation expansions of complex-valued traveltime along real-valued reference rays. *Geophys. J. int.*, **186**, 751–759.

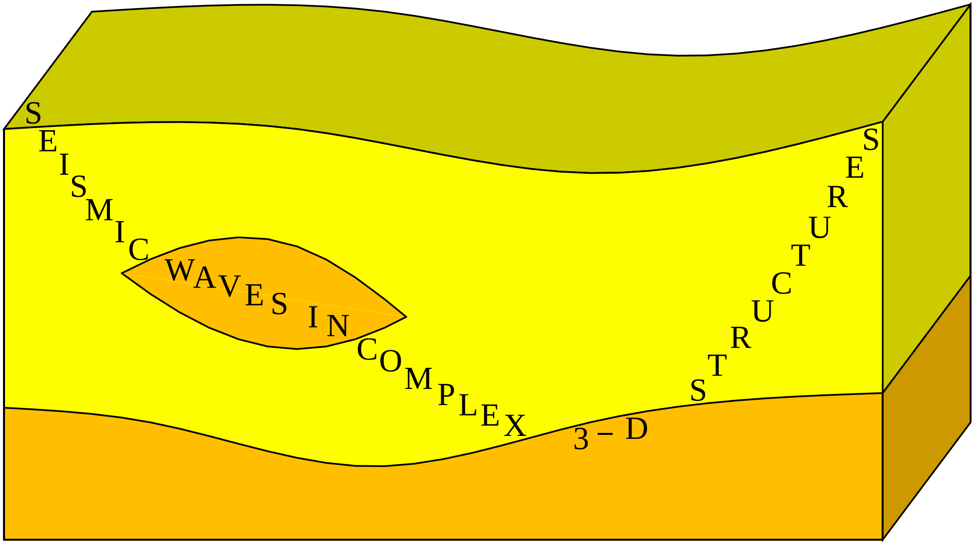
The references are available online at “http://sw3d.cz”.

Acknowledgements

The research has been supported:

by the Czech science foundation under contract 20-06887S,

and by the consortium "Seismic Waves in Complex 3-D Structures"



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