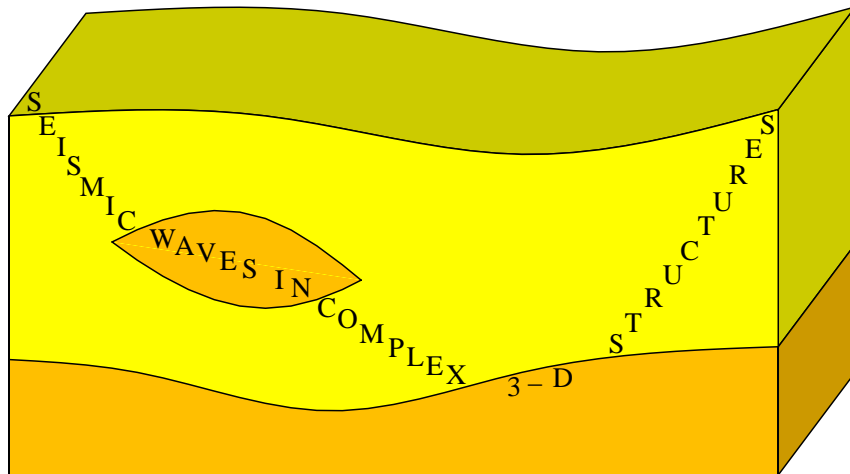


# Phase shift of a general wavefield due to caustics in anisotropic media

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## Transport equation for the scalar amplitude

Multivalued zero-order ray-theory scalar amplitude  $A$  of a general elastic wavefield satisfies **transport equation**

$$\frac{\partial}{\partial x_i} (A^2 \varrho V^i) = 0 \quad ,$$

where  $V^i$  is the ray velocity vector. Function  $\varrho$  is a function parametrizing the transport equation. If  $A$  is the amplitude of the displacement of an elastic wavefield,  $\varrho$  is the density.

The transport equation is a partial differential equation for the square  $A^2$  of the amplitude, not for the amplitude itself. Even if the solution  $A^2$  of the transport equation is real-valued, amplitude  $A$  becomes complex-valued if its square  $A^2$  becomes negative. Amplitude  $A$  is thus complex-valued. Since the complex-valued square root has two branches, it is difficult to determine amplitude  $A$  from its square  $A^2$ . We must determine which branch of the amplitude is correct.

## Complex modulus of the scalar amplitude and phase shift due to caustics

We thus separate square root  $A = \sqrt{A^2}$  into its **complex modulus**  $|A|$  and complex argument  $\varphi$ :

$$A = |A| \exp(i\varphi) \quad ,$$

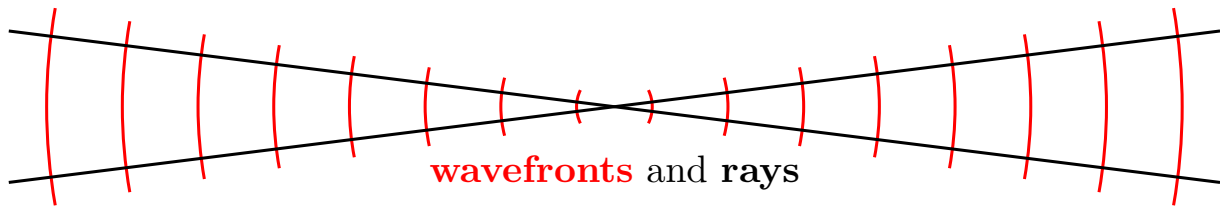
where  $\varphi$  is the **phase shift due to caustics** often expressed in terms of the **KMAH index**. A KMAH index of +1 indicates a phase shift of the complex-valued amplitude by  $\frac{\pi}{2}$  in the direction corresponding to increasing time (or decreasing travel time) of the time-harmonic wave.

For the expressions for the complex modulus of the scalar amplitude of a wavefield with general initial conditions, and for the complex modulus of the scalar amplitude of the elastic Green tensor, refer to Klimeš (2014b).

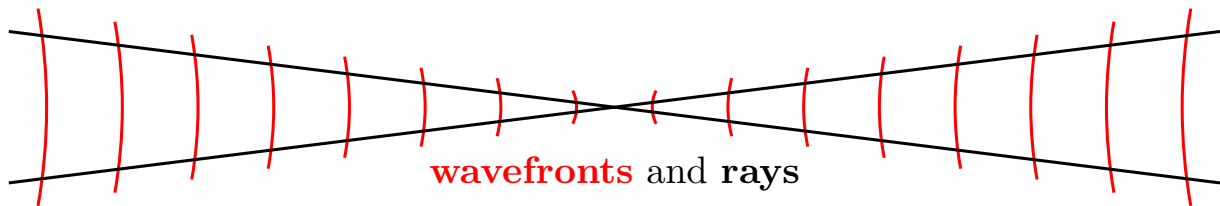
For the rules of determining the phase shift of the Green tensor due to caustics refer to Klimeš (2010).

In this contribution, we summarize the rules of determining the phase shift of a wavefield with general initial conditions due to caustics by Klimeš (2014a).

**Positive increment of the KMAH index at a caustic:**



**Negative increment of the KMAH index at a caustic:**



## Ray coordinates

In 3-D space, the orthonomic system of rays corresponding to the given initial conditions consists of a two-parametric system of rays which are parametrized by two ray parameters  $\gamma^A$ ,  $A = 1, 2$ . The ray parameters together with the independent parameter  $\gamma^3$  along the rays form ray coordinates  $\gamma^a$ ,  $a = 1, 2, 3$ .

## Ray-centred coordinates

Along a particular ray, we define ray-centred coordinates  $q^a$  (Klimeš, 2006).

We parametrize the points along the ray by an arbitrary monotonic variable  $q^3$ . At each point  $x^i(q^3)$  of the ray, we choose two contravariant basis vectors  $h_1^i(q^3)$  and  $h_2^i(q^3)$  perpendicular to slowness vector  $p_i$ ,

$$h_A^i(q^3) p_i = 0 \quad .$$

Contravariant basis vectors  $h_A^i$  should vary smoothly along the ray.

The transformation from the ray-centred coordinates  $q^a$  to Cartesian coordinates  $x^i$  is defined by relation

$$x^i = x^i(q^3) + h_A^i(q^3) q^A \quad .$$

We also define slowness vector

$$p_i^{(q)} = \frac{\partial \tau}{\partial q^i}$$

in ray-centred coordinates.

**Simple (line) caustic.**

**Paraxial matrices  $2 \times 2$  in ray-centred coordinates**

$$(Q_R)^I_A = \frac{\partial q^I}{\partial \gamma^A} \quad , \quad (P_R)_{IA} = \frac{\partial p_I^{(q)}}{\partial \gamma^A} \quad , \quad \mathbf{Q}'_R = \frac{d\mathbf{Q}_R}{d\gamma^3} \quad .$$

Matrix

$$\mathbf{K} = \mathbf{Q}_R^{-1} \det \mathbf{Q}_R$$

of rank 1 is the matrix of cofactors of matrix  $\mathbf{Q}_R$ .

The increment of the KMAH index is equal to

$$\text{sgn}[\text{Tr}(\mathbf{K}\mathbf{P}_R) \text{Tr}(\mathbf{K}\mathbf{Q}'_R)] \quad .$$

**Simple (line) caustic.**

**Paraxial matrices  $3 \times 3$  in Cartesian coordinates**

$$(\widehat{\mathbf{Q}}_{\mathbf{R}})^i_a = \frac{\partial x^i}{\partial \gamma^a} \quad , \quad (\widehat{\mathbf{P}}_{\mathbf{R}})_{ia} = \frac{\partial p_i}{\partial \gamma^a} \quad , \quad \widehat{\mathbf{Q}}'_{\mathbf{R}} = \frac{d\widehat{\mathbf{Q}}_{\mathbf{R}}}{d\gamma^3} \quad .$$

Matrix

$$\widehat{\mathbf{K}} = \widehat{\mathbf{Q}}_{\mathbf{R}}^{-1} \det \widehat{\mathbf{Q}}_{\mathbf{R}}$$

of rank 1 is the matrix of cofactors of matrix  $\widehat{\mathbf{Q}}_{\mathbf{R}}$ .

We assume that rays are parametrized by travel time  $\gamma^3 = \tau$ .

The increment of the KMAH index is equal to

$$\text{sgn}[\text{Tr}(\widehat{\mathbf{K}}\widehat{\mathbf{P}}_{\mathbf{R}}) \text{Tr}(\widehat{\mathbf{K}}\widehat{\mathbf{Q}}'_{\mathbf{R}})] \quad .$$



## Point caustic. Paraxial matrices $2 \times 2$ in ray-centred coordinates

$$(Q_R)^I_A = \frac{\partial q^I}{\partial \gamma^A} \quad , \quad (P_R)_{IA} = \frac{\partial p_I^{(q)}}{\partial \gamma^A} \quad , \quad \mathbf{Q}'_R = \frac{d\mathbf{Q}_R}{d\gamma^3} \quad .$$

If

$$\det(\mathbf{Q}'_R \mathbf{P}_R^{-1}) > 0 \quad , \quad \text{Tr}(\mathbf{Q}'_R \mathbf{P}_R^{-1}) > 0 \quad ,$$

the increment of the KMAH index is  $+2$ .

If

$$\det(\mathbf{Q}'_R \mathbf{P}_R^{-1}) > 0 \quad , \quad \text{Tr}(\mathbf{Q}'_R \mathbf{P}_R^{-1}) < 0 \quad ,$$

the increment of the KMAH index is  $-2$ .

If

$$\det(\mathbf{Q}'_R \mathbf{P}_R^{-1}) < 0 \quad ,$$

the increment of the KMAH index is  $0$ .

## Point caustic. Paraxial matrices $3 \times 2$ in Cartesian coordinates

$$(\tilde{Q}_R)^i{}_A = \frac{\partial x^i}{\partial \gamma^A} \quad , \quad (\tilde{P}_R)_{iA} = \frac{\partial p_i}{\partial \gamma^A} \quad , \quad \tilde{Q}'_R = \frac{d\tilde{Q}_R}{d\gamma^3} \quad .$$

We assume that rays are parametrized by travel time  $\gamma^3 = \tau$ .

If

$$\det(\tilde{\mathbf{P}}_R^T \tilde{\mathbf{Q}}'_R) > 0 \quad , \quad \text{Tr}(\tilde{\mathbf{P}}_R^T \tilde{\mathbf{Q}}'_R) > 0 \quad ,$$

the increment of the KMAH index is  $+2$ .

If

$$\det(\tilde{\mathbf{P}}_R^T \tilde{\mathbf{Q}}'_R) > 0 \quad , \quad \text{Tr}(\tilde{\mathbf{P}}_R^T \tilde{\mathbf{Q}}'_R) < 0 \quad ,$$

the increment of the KMAH index is  $-2$ .

If

$$\det(\tilde{\mathbf{P}}_R^T \tilde{\mathbf{Q}}'_R) < 0 \quad ,$$

the increment of the KMAH index is  $0$ .

## References

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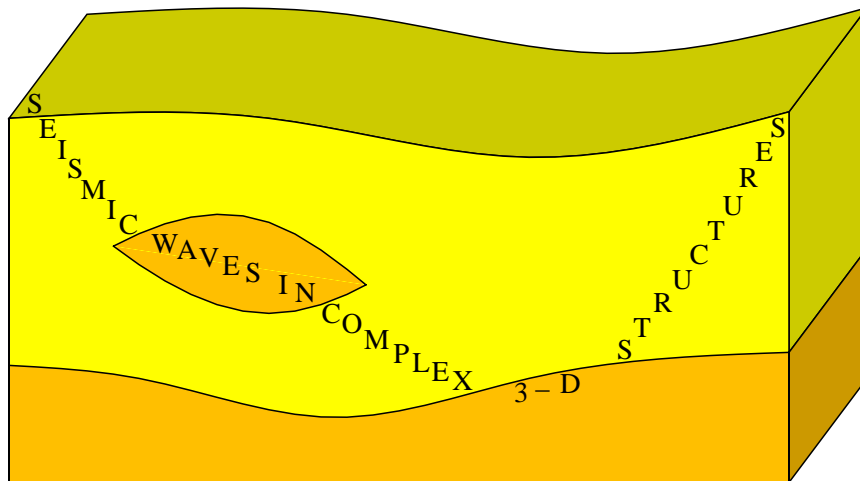
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