Estimating the correlation function of a self–affine random medium

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Summary

Unknown slowness distribution describing the geological structure is assumed to be composed of a smoothly heterogeneous mean value and a realization of a stationary self–affine random medium. The power–law correlation function of a stationary self–affine random medium is described by the Hurst exponent and "standard deviation" expressed at a chosen reference length. The geometrical travel–time variances are then proportional to the power of ray lengths. A method designed to estimate the two parameters of the power–law medium correlation function using field travel times is proposed, and applied to data from the Western Bohemia region.

1 Introduction

In order to estimate the relation of a seismic model to the geological medium and, in consequence, to estimate the relation of the synthetic quantities, calculated in the seismic model, to reality, it is important to have an estimate of the medium covariance function. The medium covariance function is of principal importance in refraction travel–time tomographic inversion (Tarantola 1987, Klimeš 1997a), especially when estimating the accuracy of the seismic model, its relation to the geological medium, or the covariance matrix describing the statistics of synthetic travel times.

In a self–affine random medium, the material parameters may be scaled simultaneously with scaling the spatial dimensions in such a way that the statistical properties remain unchanged by the scaling. Since a geological medium contains heterogeneities of all sizes, very similar on various scales, a self–affine random medium is a mathematical model very suitable for approximating the statistics of a geological medium. We thus assume the geological medium to be a particular representation of the self–affine random medium in this paper. For an overview and brief discussion of other types of random media used in geophysics, refer to Klimeš (1997b).

In a stationary (i.e., statistically homogeneous) random medium, the medium covariance function may be expressed in terms of the medium correlation function. Here, a particular class of self–affine random media, composed of a heterogeneous mean value and a stationary covariance function, is considered.

Section 3 is devoted to the dependence of the a priori geometrical covariance matrix of field travel times (Tarantola 1987) on the medium covariance function. The a priori geometrical covariance matrix of field travel times describes the deviations of travel times from the mean travel–time curve. The deviations are caused by the heterogeneities, especially the lateral ones.

The medium correlation function is dependent on two parameters: the Hurst exponent and the corresponding "standard deviation". Section 4 is devoted to the estimation of these two parameters, essential for travel–time tomography, from field travel times.

The proposed method is applied to field data from the Western Bohemia region in Section 5 to demonstrate the possibilities of estimating the medium correlation function.

The reader should be aware that the Einstein summation does not apply to the equations anywhere in this paper.

2 Correlation function of a self–affine random medium

Random medium $u(\mathbf{x})$ is *stationary* if mean value $\langle u(\mathbf{x})\rangle$ is constant and the *medium covariance function*

$$C(\mathbf{x}_1,\mathbf{x}_2) = \langle u(\mathbf{x}_1)\, u(\mathbf{x}_2)\rangle - \langle u(\mathbf{x}_1)\rangle \langle u(\mathbf{x}_2)\rangle \qquad (1)$$

depends only on distance $\mathbf{x}_1-\mathbf{x}_2$, $C(\mathbf{x}_1,\mathbf{x}_2) = c(\mathbf{x}_1-\mathbf{x}_2)$, where $c(\mathbf{x})$ is the *medium correlation function* (Tarantola 1987).

Realizations of a stationary *self–affine random medium*, uniformly scalable over all lengths, may be obtained by multiplying the Fourier transform of the realizations of a stationary *white noise* by spectral filter

$$\widehat{f}(\mathbf{k}) = \kappa k^{-\frac{d}{2}-N} \ , \quad k = (\mathbf{k}^{\mathrm{T}}\mathbf{k})^{\frac{1}{2}} \ , \qquad (2)$$

and inversely Fourier transforming the products back into the space domain. Here κ is a constant proportional to the standard deviation of the resulting self–affine random functions, and N is the *Hurst exponent.*

Assuming that the white noise has a unit standard deviation, the medium covariance function for $-\frac{d}{2} < N < 0$ is (Klimeš 1997b)

$$C(\mathbf{x}_1,\mathbf{x}_2) = \sigma^2 \ \left(\frac{|\mathbf{x}_1-\mathbf{x}_2|}{L}\right)^{2N} \quad , \qquad (3)$$

where

$$\sigma^2 = \kappa^2 \ 2^{-d} \ \pi^{-\frac{d}{2}} \ \Gamma(|N|) \left[\Gamma(\tfrac{d}{2}+N)\right]^{-1} \left[\tfrac{1}{2}L\right]^{2N} \ . \qquad (4)$$

Here L is some reference distance supplemented to preserve the correct physical units in the expressions.

Hereinafter, we shall assume a stationary covariance function but allow for a spatially variable mean value $\langle u(\mathbf{x})\rangle$.

3 A priori geometrical covariance matrix of travel times

Assuming the ray–theory linearization approach, field travel times may be expressed in the form of

$$T_I = \tau_I + \delta T_I \qquad (5)$$

where

$$\tau_I = \int_0^{s_I} \mathrm{d}s \ u(\mathbf{x}(s)) \qquad (6)$$

is the integral of the slowness $u(\mathbf{x})$ along the corresponding ray of length s_I, and δT_I is the error in determining field travel time T_I.

The *a priori geometrical covariance* (Tarantola 1987)

$$S_{KL} = \left\langle (\tau_K - \tau_K^0)(\tau_L - \tau_L^0)\right\rangle \quad , \qquad (7)$$

of the $K^{\rm th}$ and $L^{\rm th}$ travel times is then given by

$$S_{KL} = \int_0^{s_K} \mathrm{d}s'_K \int_0^{s_L} \mathrm{d}s'_L \; C\big(\mathbf{x}(s'_K), \mathbf{x}(s'_L)\big) \quad , \tag{8}$$

where the integration is performed along the corresponding rays of lengths s_K and s_L. Here τ_K^0 are the reference travel times corresponding to mean value $u^0(\mathbf{x}) = \langle u(\mathbf{x}) \rangle$ of the random slowness. Notice that, for medium covariance function (3), $\sigma^{-2} S_{KL}$ is independent of σ and is determined by a single medium parameter, Hurst exponent N.

The derivative of geometrical covariance S_{KL} with respect to Hurst exponent N is then

$$\frac{\partial S_{KL}}{\partial N} = \int_0^{s_K} \mathrm{d}s'_K \int_0^{s_L} \mathrm{d}s'_L \, C\big(\mathbf{x}(s'_K), \mathbf{x}(s'_L)\big) \; 2 \ln\left(\frac{|\mathbf{x}(s'_K) - \mathbf{x}(s'_L)|}{L}\right) \tag{9}$$

and $\sigma^{-2} \frac{\partial S_{KL}}{\partial N}$ is again independent of σ.

Unlike geometrical crossvariances, variances S_{KK} and their derivatives with respect to N may be estimated analytically (Klimeš 1996).

4 Determination of the medium correlation function from the field travel times

4.1 Differences of the relative field travel times and their statistical moments

Let us study the mutual differences of the field travel times. Since the travel times are strongly dependent on hypocentral distances, it is possible to compare only the travel times T_K and T_L along the rays of similar lengths s_K and s_L. This restriction may, to some extent, be reduced if we relate the travel times to the reference travel–time curve $\tau^0 = \tau^0(s)$. We may than compare the relative travel times T_K/τ_K^0 and T_L/τ_L^0, where $\tau_I^0 = \tau^0(s_I)$. The differences of the relative travel times then depend on the local value of the reference travel–time curve in terms of a multiplication factor which has practically no influence on the statistics. The differences of the relative travel times are distorted especially by the error in the derivative of the reference travel–time curve multiplied by $|s_K - s_L|$. We assume that the local error in the derivative of the reference travel–time curve compared to the exact mean travel–time curve is locally negligible in intervals defined by

$$q\, T_L < T_K < T_L \tag{10}$$

with given parameter q, $0 \le q < 1$. We now define the variances of the relative travel–time differences

$$\vartheta_{KL,KL} = \left\langle (T_K/\tau_K^0 - T_L/\tau_L^0)^2 \right\rangle \quad , \tag{11}$$

and the fourth–order variances

$$\vartheta_{KL,KL,KL,KL} = \left\langle \left(\frac{T_K}{\tau_K^0} - \frac{T_L}{\tau_L^0} \right)^4 \right\rangle - [\vartheta_{KL,KL}]^2 \quad , \tag{12}$$

describing the standard deviations of the squared differences of the relative field travel times from variances (11).

We assume that picking errors δT_I are statistically independent of travel times τ_I, that the mean values of picking errors are zero, and that the *data covariance matrix* is known, at least approximately,

$$\langle \delta T_K \rangle = 0 \quad , \qquad \langle \delta T_K \; \delta T_L \rangle = T_{KL} \quad . \tag{13}$$

Than variances (11) depend on parameters σ and N of the medium covariance function (3) through (Klimeš 1996)

$$\vartheta_{KL,KL} = \vartheta^0_{KL,KL} + \sigma^2 \vartheta^1_{KL,KL}(N) \tag{14}$$

with

$$\vartheta^0_{KL,KL} = \frac{T_{KK}}{\tau_K^0 \tau_K^0} - 2 \frac{T_{KL}}{\tau_K^0 \tau_L^0} + \frac{T_{LL}}{\tau_L^0 \tau_L^0} \tag{15}$$

and

$$\vartheta^1_{KL,KL}(N) = \frac{\sigma^{-2} S_{KK}}{\tau_K^0 \tau_K^0} - 2 \frac{\sigma^{-2} S_{KL}}{\tau_K^0 \tau_L^0} + \frac{\sigma^{-2} S_{LL}}{\tau_L^0 \tau_L^0} \quad . \tag{16}$$

Values (15) are constants with respect to N and σ. Functions (16) of N are independent of σ.

To be able to approximate the fourth–order moments in (12) using the second–order moments, we assume Gaussian probability distributions for both the the self–affine random medium and the picking errors. If all probability distributions in the problem are Gaussian, the marginal probability distribution describing the relative travel–time differences is Gaussian too. If the probability distribution is Gaussian,

$$\left\langle (T_K/\tau_K^0 - T_L/\tau_L^0)^4 \right\rangle = 3 \left\langle (T_K/\tau_K^0 - T_L/\tau_L^0)^2 \right\rangle^2 \quad , \tag{17}$$

and equation (12) reads

$$\vartheta_{KL,KL,KL,KL} = 2\, [\vartheta_{KL,KL}]^2 \quad . \tag{18}$$

4.2 Objective function

We select the objective function in the form of

$$y = \left[\sum_{K,L} 1 \right]^{-1} \sum_{K,L} \frac{[(T_K/\tau_K^0 - T_L/\tau_L^0)^2 - \vartheta_{KL,KL}]^2}{\vartheta_{KL,KL,KL,KL}} \; , \tag{19}$$

and minimize it with respect to parameters σ and N of the medium covariance function (3). Let us emphasize that the minimum has to be sought for constant fourth–order variances $\vartheta_{KL,KL,KL,KL}$.

Inserting (14), and (18) with constant $\sigma = \sigma_0$ and $N = N_0$, objective function (19) reads

$$y(\sigma, N) = \tfrac{1}{2} \left[\sum_{K,L} 1 \right]^{-1} \sum_{K,L} \left[\frac{A_{KL,KL} - \sigma^2 \vartheta^1_{KL,KL}(N)}{B_{KL,KL}} \right]^2 , \tag{20}$$

where

$$A_{KL,KL} = (T_K/\tau_K^0 - T_L/\tau_L^0)^2 - \vartheta^0_{KL,KL} \; , \tag{21}$$

$$B_{KL,KL} = \vartheta^0_{KL,KL} + (\sigma_0)^2 \vartheta^1_{KL,KL}(N_0) \; . \tag{22}$$

Parameters σ_0 and N_0 are fixed during the minimization, but should be selected close to the final solution,

$$\sigma_0 \approx \sigma \quad , \qquad N_0 \approx N \quad . \tag{23}$$

Objective function (20) has its minimum

$$y(N) = \tfrac{1}{2} \left[\sum_{K,L} 1 \right]^{-1} \left[F_0(N) - \frac{[F_1(N)]^2}{F_2(N)} \right] \tag{24}$$

with respect to σ for

$$\sigma^2(N) = \frac{F_1(N)}{F_2(N)} \quad , \tag{25}$$

where

$$F_0(N) = \sum_{K,L} [A_{KL,KL}]^2 \, [B_{KL,KL}]^{-2} \quad , \tag{26}$$

$$F_1(N) = \sum_{K,L} A_{KL,KL} \vartheta^1_{KL,KL}(N) \, [B_{KL,KL}]^{-2} \; , \tag{27}$$

$$F_2(N) = \sum_{K,L} \left[\vartheta^1_{KL,KL}(N) \right]^2 [B_{KL,KL}]^{-2} \quad . \tag{28}$$

Since inaccurate field travel times may severely distort the estimated statistics of the geological medium, it is reasonable to restrict the summation only to travel times satisfying inequality

$$T_{KK} \leq (\sigma_{\rm err})^2 \; \sigma^{-2} S_{KK} \quad , \tag{29}$$

where $\sigma_{\rm err}$ is a given constant. The right–hand side of (29) has to be calculated at fixed $N = N_0$ in order not to alter the data set during the minimization of the objective function.

4.3 Minimization of the objective function

First we select reasonable values of constants q and $\sigma_{\rm err}$. Then we select the value of N_0. The corresponding value of σ_0 may be found iteratively: for an initial estimate of σ_0 we calculate new $\sigma_0 = \sigma(N_0)$ using (25) to get a better estimate, rapidly approaching the value of σ_0 consistent with the chosen value of N_0. For fixed N_0 and σ_0 we calculate the values of the objective function (24) at different values of N in order to find the minimum. If the values of N depart from N_0, we should select new N_0 and determine new σ_0.

The minimum of the objective function with respect to N is not very pronounced and is very sensitive to bad mistakes in the data. It is also influenced by artificial numerical parameters such as q or $\sigma_{\rm err}$. This behaviour is due to the sensitivity of N to the fourth statistical moment of the field travel times. That is why N cannot be determined very accurately. An accuracy of the order of ± 0.05 in N may be thought to be an excellent result, difficult to achieve in practice. However, the author hopes that some small uncertainty in N cannot influence the travel–time inversion considerably.

On the other hand, for given Hurst exponent N, "standard deviation" σ is dependent on the second statistical moment of the field travel times and may be determined very accurately.

For the Gaussian probability distribution, the resulting minimum objective function should be close to 1.

5 Example: Western Bohemia

An attempt is made to estimate the medium correlation function for the region of Western Bohemia and the surrounding part of Germany, using the travel times from the refraction measurements performed during the years 1989 to 1991 (Bucha et al. 1992).

5.1 Reference travel–time curve

The mean dependence of the travel times on the hypocentral distance has been roughly approximated by a rational function of the form

$$\tau^0(s) = (a\; s + b\; s^2)/(c + s) \quad . \tag{30}$$

At large distances s, the reference travel time (30) approaches the asymptotic line given by slowness b and the travel–time delay of a–bc. Constants a, b, and c have been fitted using least squares,

$$a = 0.50\,{\rm s} \;, \quad b = 0.17\,{\rm s\,km}^{-1} \;, \quad c = 1.25\,{\rm km} \;. \tag{31}$$

The deviations $T_K - \tau(s_K)$ of field travel times T_K with respect to reference travel–time curve (30) are shown in Figure 1. The area of the dots in Figure 1 is proportional to the weights w'_K used in the least squares. Weights

$$w_K = \left(1 + \frac{T_{KK}}{(\delta_{\rm err} + \rho_{\rm err}\; T_K)^2} \right)^{-1} \tag{32}$$

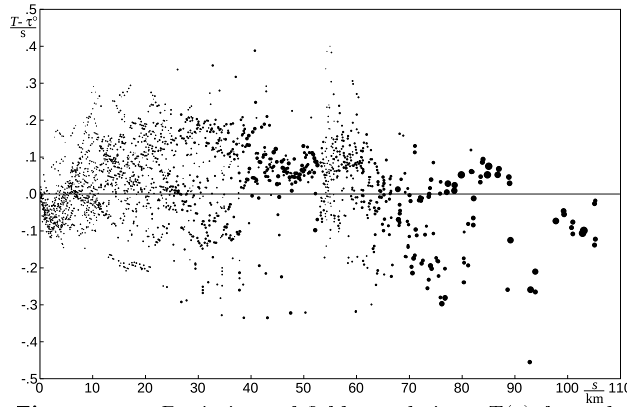

Figure 1. Deviations of field travel times $T(s)$ from the reference travel–time curve $\tau^0(s)$.

with

$$\delta_{\rm err} = 0.010\,{\rm s} \quad , \qquad \rho_{\rm err} = 0.005 \tag{33}$$

have been normalized separately in each interval of length

$$\Delta s = 1{\rm km} \tag{34}$$

using formula

$$w'_K = w_K \left[1 + \sum_L w_L \right]^{-1} \tag{35}$$

to get a relatively even coverage of all hypocentral distances s. The squared travel–time deviations have then been weighted with the least–squares weights

$$w''_K = w'_K \; s^{-p} \qquad \text{where} \qquad p = 0.5 \quad . \tag{36}$$

5.2 Calculation of covariances between travel times

For the estimation of the parameters of the medium correlation function, we consider straight rays, as in a homogeneous medium. For the used refraction travel times whose rays do not penetrate very deep into the Earth compared with the epicentral distance, it should be a reasonable approximation. Especially if the rays of considerably different lengths are not compared, see condition (10).

Geometrical covariance matrix (8) of travel times has been calculated numerically, dividing the rectangular integration area of dimensions $s_K \times s_L$ into small rectangular cells and replacing the integrand by a bilinear function in each cell. Since the integrand may reach infinity at some points, the integrand has been limited from above at each grid point in such a way as to get exact values of the integral in all square cells touched by the diagonal for the special case of variances S_{KK}.

5.3 Medium correlation function

First we attempted to find a good value of numerical parameter $\sigma_{\rm err}$ selecting the set of field travel times measured with sufficient accuracy, see (29). Table 1 shows the dependence of objective function $y(N)$ and number $\sum_K 1$ of the field travel times used on the selection of $\sigma_{\rm err}$, for $q = 0.90$, $N_0 = -0.10$, and $N = -0.11$. The reference length unit of $L = 1$km is used.

Here the value of $\sigma_0 = 0.011330\,{\rm s\,km}^{-1}$ has been determined for $\sigma_{\rm err} = 0.001558\,{\rm s\,km}^{-1}$ and has been kept fixed

$\sigma_{\rm err}$	$\sum_K 1$	$y(N)$	$\sigma_{\rm err}$	$\sum_K 1$	$y(N)$	$\sigma_{\rm err}$	$\sum_K 1$	$y(N)$
0.0008	1091	0.750	0.0013	1602	1.061	0.0018	1792	1.138
0.0009	1266	0.849	0.0014	1657	1.078	0.0019	1854	1.222
0.0010	1406	0.999	0.0015	1698	1.086	0.0020	1881	1.274
0.0011	1489	1.037	0.0016 •	1728	1.084	0.0025	1980	1.719
0.0012	1549	1.054	0.0017	1759	1.095	0.0030	2054	2.007

Table 1. The dependence of number $\sum_K 1$ of the field travel times used and of the value of objective function $y(N)$ on the selection of $\sigma_{\rm err}$, for $q = 0.90$, $N_0 = -0.10$, and $N = -0.11$.

in calculating $y(N)$ for different $\sigma_{\rm err}$. The value of the objective function is relatively stable for $0.0010\,{\rm s\,km^{-1}} \le \sigma_{\rm err} \le 0.0017\,{\rm s\,km^{-1}}$ and increases considerably for larger $\sigma_{\rm err}$. Such an increase probably indicates the influence of bad travel–time data. The author has chosen $\sigma_{\rm err} = 0.001558\,{\rm s\,km^{-1}}$ for the subsequent calculations.

The next task is to choose a reasonable value of the other numerical parameter q, which selects the pairs of field travel times according to (10). Unfortunately, the position $N = N_{\rm min}$ of the minimum of objective function $y(N)$ is influenced considerably by the choice of numerical parameter q. Table 2 shows the dependence of the minimum of objective function $y(N)$ on the selection of q, for $\sigma_{\rm err} = 0.001558\,{\rm s\,km^{-1}}$ and σ_0 corresponding to our choice of N_0.

There are at least 3 drawbacks of small values of q:

• For decreasing q, inaccurate reference travel–time curve $\tau^0(s)$ may begin to influence the results considerably.

• The number of differences $T_K/\tau^0_K - T_L/\tau^0_L$ of the relative travel times is much greater than the number of field travel times T_K, whereas we treat the differences as independent data in objective function (19). This processing need not be correct from the statistical point of view and may get worse for smaller values of q.

• Here we substituted curved rays with straight ones. It is probably a reasonable approximation for rays of similar lengths, but for rays of different lengths the straight approximations of rays may be much closer together than the correct rays, separated in depth, are. For small q, some geometrical covariances (8) may thus be calculated greater than correct, which may result in compensation by smaller (more negative) values of Hurst exponent N.

On the other hand, the drawback of q approaching 1 consists in exclusion of field travel times not surrounded by other travel times, and consequently in considerable limitation of the amount of available information. This may be the case of the results obtained for values of $q = 0.95$ and $q = 0.98$.

5.4 Results for Western Bohemia

Taking into account the above considerations, the medium correlation function (3) described by parameters obtained for q from $q = 0.75$,

$$N = -0.20 \quad , \qquad \sigma = 0.0094\,{\rm s\,km^{-1}} \quad , \tag{37}$$

to $q = 0.90$

$$N = -0.12 \quad , \qquad \sigma = 0.0106\,{\rm s\,km^{-1}} \quad , \tag{38}$$

look acceptable. For these parameters, related to the reference length unit of $L = 1$km, geometrical standard deviations

$$\sqrt{S_{KK}} = \sigma\ L\ \sqrt{\frac{2}{(1+2N)(2+2N)}}\ \left(\frac{|\mathbf{x}_1 - \mathbf{x}_2|}{L}\right)^{1+N} \tag{39}$$

(Klimeš 1996) of travel times from the mean travel–time curve and other statistical properties derived from the correlation function should be in accordance with measured travel times.

q	N_0	$N_{\rm min}$	$\sigma(N_{\rm min})$	$y(N_{\rm min})$
0.50	-0.24	-0.24	0.0092	0.805
0.60	-0.22	-0.23	0.0092	0.850
0.70	-0.18	-0.21	0.0093	0.905
0.75 ←	-0.19	-0.20	0.0094	0.921
0.80	-0.14	-0.17	0.0097	1.006
0.85	-0.12	-0.15	0.0100	1.053
0.90 ←	-0.10	-0.12	0.0106	1.099
0.95	-0.10	-0.09	0.0117	1.121
0.98	-0.10	-0.08	0.0122	1.196

Table 2. The dependence of the minimum of objective function $y(N)$ on the selection of q, for $\sigma_{\rm err} = 0.001558\,{\rm s\,km^{-1}}$ and σ_0 corresponding to our choice of N_0. $N_{\rm min}$ and $\sigma(N_{\rm min})$ are the respective parameters of the medium correlation function. The arrows denote options (37) and (38).

The estimated statistical properties of the medium are applicable at distances between the *inner and outer cut-off scales* (Mandelbrot 1977), i.e. for epicentral distances roughly from 0.2 km to 50 km in this example. Because the medium covariance function has been determined using the straight–ray approximation, it is applicable to horizontal directions, but not vertical.

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see "http:/seis.karlov.mff.cuni.cz/consort/main.htm".

* Available online at "http://seis.karlov.mff.cuni.cz/consort/main.htm".