Geophys. J. Int. (2008) **175,** 357–374 doi: 10.1111/j.1365-246X.2008.03968.x

Numerical comparison of the isotropic-common-ray and anisotropic-common-ray approximations of the coupling ray theory

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Accepted 2006 November 16. Received 2006 November 4; in original form 2006 April 21

SUMMARY

In the common-ray approximation of the coupling ray theory, the *S*-wave traveltimes in the coupling equations are approximated by the first-order perturbation expansion from the common reference ray. The common-ray approximation eliminates problems with ray tracing through *S*-wave singularities and also considerably simplifies the numerical algorithm of the coupling ray theory for *S* waves, but may introduce errors in traveltimes due to the perturbation from the common reference ray. The errors of *S*-wave traveltimes due to the common-ray approximation may be approximated by the second-order terms in the perturbation expansion. The first-order and second-order terms in the perturbation expansion can be calculated by simple numerical quadratures along the common reference ray.

The common reference ray may be represented by the isotropic common ray or by the anisotropic common ray. The anisotropic-common-ray approximation of the coupling ray theory is more accurate than the isotropic-common-ray approximation. The equations for the first-order and second-order perturbation expansions of traveltime from both the isotropic and anisotropic common rays to the anisotropic-ray-theory rays have already been derived. We numerically test the equations in three smooth 1-D velocity models of differing degrees of anisotropy.

We first compare the second-order perturbation expansions of traveltime from the isotropic common rays to the anisotropic common rays and vice versa. This comparison not only demonstrates the accuracy of the perturbation expansions, but also allows us to estimate the order of magnitude of the third-order, fourth-order and fifth-order terms in the perturbation expansions.

The first-order and second-order perturbation expansions of traveltime from the isotropic common rays and from the anisotropic common rays to the anisotropic-ray-theory rays are then compared with the correct anisotropic-ray-theory traveltimes in order to illustrate the accuracy of the perturbation expansions and reliability of the error estimates.

Finally, the synthetic seismograms obtained by the isotropic-common-ray and anisotropic-common-ray approximations of the coupling ray theory are compared with the more accurate coupling-ray-theory synthetic seismograms simulated by the second-order perturbation expansion of traveltime from the anisotropic common rays. In the numerical examples, the errors of the anisotropic-common-ray approximation of the coupling ray theory are considerably smaller than the errors of the isotropic-common-ray approximation.

Key words: common-ray approximation, coupling ray theory, heterogeneous media, perturbation theory, seismic anisotropy, traveltime.

1 INTRODUCTION

Traditionally, there are two different high-frequency asymptotic ray theories: *isotropic ray theory* based on the assumption of equal velocities of both *S* waves, and *anisotropic ray theory* assuming both *S* waves strictly decoupled. Note that here the term 'different' means that the anisotropic ray theory is not a generalization of the isotropic ray theory and that both theories yield really different *S* waves in equal velocity

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models. In this sense, for example, the ray theory for viscoelastic media is just a generalization of the ray theory for elastic media and we do not need to use the term 'different' for them.

In the isotropic ray theory, the *S*-wave polarization vectors do not rotate about the ray, whereas in the anisotropic ray theory they coincide with the eigenvectors of the Christoffel matrix which may rotate rapidly about the ray. Thomson *et al.* (1992) demonstrated analytically that the high-frequency asymptotic error of the anisotropic-ray-theory wavefield is inversely proportional to the second or higher root of the frequency if a ray passes through the point of equal *S*-wave eigenvalues of the Christoffel matrix even in otherwise strongly anisotropic media.

In 'weakly anisotropic' models, at moderate frequencies, the *S*-wave polarization tends to remain unrotated round the ray, but is partly attracted by the rotation of the eigenvectors of the Christoffel matrix. The intensity of the attraction increases with frequency. This behaviour of the *S*-wave polarization is described by the *coupling ray theory* proposed by Coates & Chapman (1990). The frequency-dependent coupling ray theory is the generalization of both the zero-order isotropic and anisotropic ray theories and provides continuous transition between them. The coupling ray theory is applicable to *S* waves at all degrees of anisotropy, from isotropic to considerably anisotropic velocity models. The numerical algorithm for calculating the frequency-dependent complex-valued *S*-wave polarization vectors of the coupling ray theory has been designed by Bulant & Klimeš (2002).

S-wave coupling is thus important if the eigenvectors of the Christoffel matrix rotate about the ray. This rotation may be caused by the spatial variation of the orientation of anisotropy axes, or for fixed anisotropy axes by the ray curvature. For example, the depth variation of anisotropy axes due to the preferred orientation of the olivine lattice in the oceanic upper mantle was described by Rümpker *et al.* (1999). The rotation of the eigenvectors of the Christoffel matrix due to the ray curvature does not occur in 1-D velocity models which are simultaneously horizontally isotropic and horizontally homogeneous, but occurs in most models which are not isotropic and homogeneous in the same plane. *S*-wave coupling is thus usually important in the presence of the depth variation of anisotropy axes, in the presence of considerably inclined velocity gradients, or if the medium is not horizontally isotropic. The numerical examples presented in this paper represent the latter case.

There are many commonly used *quasi-isotropic approximations* of the coupling ray theory (e.g. Pšenčík 1998a), which diminish the accuracy of the coupling ray theory both with increasing frequency and increasing degree of anisotropy. Refer to Bulant & Klimeš (2002, 2004) and to Klimeš & Bulant (2004) for the description of the individual quasi-isotropic approximations and for the examples of their impact on synthetic seismograms. Most of these quasi-isotropic approximations can be avoided with minimal effort except for the common-ray approximation for *S* waves.

In the *common-ray approximation*, only one reference ray is traced for both anisotropic-ray-theory *S* waves, and both *S*-wave anisotropic-ray-theory traveltimes in the coupling equations are approximated by the first-order perturbation expansion from the common reference ray. Whereas tracing the continuous system of anisotropic-ray-theory rays may be very difficult in the vicinity of an *S*-wave singularity at which the *S*-wave slowness surfaces coincide (Vavryčuk 2003), this problem does not occur in the common-ray approximation. The common-ray approximation thus eliminates problems with ray tracing through *S*-wave singularities and also considerably simplifies coding of the coupling ray theory and numerical calculations, but may introduce errors in traveltimes due to the perturbation. These traveltime errors can deteriorate the coupling-ray-theory solution at high frequencies. It is thus of principal importance in numerical applications to estimate the traveltime errors due to the common-ray approximation, and then the related error of the wavefield.

The common reference rays have routinely been represented by the *isotropic common rays* (ICRs) calculated in the isotropic reference model, but they would be better represented by the *anisotropic common rays* (ACRs). The ACRs are traced in the anisotropic model, using the averaged Hamiltonian of both anisotropic-ray-theory *S* waves (Bakker 2002). Interpolating all non-zero density-reduced elastic moduli instead of interpolating the *S*-wave velocity only thus makes tracing the ACRs more costly than tracing the ICRs. The algorithm of the dynamic ray tracing corresponding to the ACRs has been proposed by Klimeš (2006b). The numerical results of Klimeš & Bulant (2004) suggest that the *ACR approximation* may be considerably more accurate than the *ICR approximation*, which we demonstrate on the numerical examples in this paper. For a more detailed description of the common-ray approximations of the coupling ray theory refer to Section 2.1.

In the common-ray approximation, the *S*-wave traveltimes in the coupling equations are approximated by the first-order perturbation expansion from the common reference ray, see Section 2.2. The common reference ray, represented by the ICR or by the ACR, is traced using the reference Hamiltonian, see Section 2.3. The errors of *S*-wave traveltimes may then be approximated by the second-order terms in the perturbation expansion as explained in Section 2.4. Note that the perturbation expansion is the Taylor expansion with respect to the perturbation parameters, see eq. (9), which parametrize the Hamiltonian, see eq. (6). We refer here to the partial derivatives with respect to the perturbation parameters as perturbation derivatives, in order to distinguish them from the partial derivatives with respect to the spatial coordinates.

Klimeš & Bulant (2004) derived the equations for estimating the traveltime errors due to the ICR approximation (and roughly also due to the ACR approximation) by numerical quadrature along the *ICRs* in smooth velocity models without interfaces.

Klimeš & Bulant (2006) derived the equations for estimating the traveltime errors due to the ACR approximation (and roughly also due to the ICR approximation) by numerical quadrature along the *ACRs* in smooth velocity models without interfaces.

The derivations by Klimeš & Bulant (2004, 2006) are based on the general equations for the second-order perturbations of traveltime by Klimeš (2002). The equations for estimating the traveltime errors due to the common-ray approximations of the coupling ray theory for *S* waves are derived in a form suitable for dynamic ray tracing in ray-centred coordinates attached to the common reference rays. The accuracy of both the ICR approximation and the ACR approximation can thus be checked routinely.

In this paper, we numerically test the derived equations in three smooth 1-D velocity models of differing degrees of anisotropy. The models are described in Sections 3.1 and 3.2.

In Section 3.4, we compare the second-order perturbation expansions of traveltime from the ICRs to the ACRs and vice versa. This comparison not only demonstrates the accuracy of the perturbation expansions, but also allows us to estimate the order of magnitude of the third-order, fourth-order and fifth-order terms in the perturbation expansions.

In Section 3.5, the first-order and second-order perturbation expansions of traveltime from the ICRs and from the ACRs to the anisotropic-ray-theory rays are compared with the correct anisotropic-ray-theory traveltimes in order to illustrate the accuracy of the perturbation expansions and reliability of the error estimates. We observe that the traveltime errors due to the ACR approximation of the coupling ray theory are considerably smaller than the errors due to the ICR approximation.

Since the second-order perturbation expansion of traveltime from the ACRs to the anisotropic-ray-theory rays is sufficiently accurate in our three velocity models, we may use this expansion for simulating the accurate coupling-ray-theory synthetic seismograms. These seismograms are used in Section 3.5 for comparison with the synthetic seismograms obtained by the ICR and ACR approximations of the coupling ray theory to demonstrate the errors of the two common-ray approximations.

This paper represents a continuation of the numerical examples by Klimeš & Bulant (2004) who could not trace the ACRs and thus performed all first-order and second-order perturbation expansions of traveltime from the ICRs. We now present numerical examples of the ACR approximation of the coupling ray theory and perform the first-order and second-order perturbation expansions of traveltime from the ACRs using the equations by Klimeš & Bulant (2006). We have applied the algorithms of ACR tracing and of the corresponding dynamic ray tracing described by Klimeš (2006a,b).

2 COMMON-RAY APPROXIMATIONS

For the derivation of the coupling ray theory refer to Coates & Chapman (1990) and to Červený (2001). For the description of the numerical algorithm of the coupling ray theory refer to Bulant & Klimeš (2002). The numerical algorithm of the coupling ray theory by Bulant & Klimeš (2002) is independent of the reference ray. The algorithm is thus applicable to the ICRs, to the ACRs, and also to the anisotropic-ray-theory reference rays.

The algorithms of ACR tracing in smooth anisotropic models without interfaces and of the corresponding dynamic ray tracing in ray-centred coordinates are described by Klimeš (2006a,b). We thus have all equations necessary for the ACR approximation of the coupling ray theory in smooth anisotropic models without interfaces.

Klimeš & Bulant (2004) derived the equations for estimating the traveltime errors due to the ICR approximation (and roughly also due to the ACR approximation) by numerical quadrature along the ICRs. Klimeš & Bulant (2006) derived the equations for estimating the traveltime errors due to the ACR approximation (and roughly also due to the ICR approximation) by numerical quadrature along the ACRs.

2.1 Selection of the reference ray

The isotropic ray theory is always the limiting case of the coupling ray theory for decreasing anisotropy at a fixed frequency. On the other hand, the high-frequency limit of the coupling ray theory at a fixed anisotropy depends on the choice of the reference ray, and even on the choice of the *system* of reference rays, because the amplitudes are determined by the paraxial reference rays. The reference ray should be 'close to the ray of the coupled *S* wave under study'. Unfortunately, this is but a rough statement, because the ray of the coupled *S* wave has not been defined yet.

From the point of view of the high-frequency asymptotic validity, the frequency-independent reference ray is best represented by the *anisotropic-ray-theory reference ray*, provided we choose the initial condition for the polarization vector in the coupling equation given by the eigenvector of the Christoffel matrix corresponding to the reference ray. The anisotropic-ray-theory traveltime corresponding to the selected polarization is then exact, and only the difference between the two anisotropic-ray-theory *S*-wave traveltimes is approximate. The coupling ray theory may then also be used at high frequencies because the approximate traveltime difference influences only the coupling due to low-frequency scattering. The coupling ray theory then correctly converges to the anisotropic ray theory for high frequencies. For other choices of reference rays, the high-frequency limit of the coupling ray theory at a fixed anisotropy is incorrect, although the differences may be small at the finite frequencies under consideration. Note that the anisotropic-ray-theory reference ray can be traced only if the eigenvectors of the Christoffel matrix vary continuously along the whole ray (Vavryčuk 2001).

In the *ACR approximation*, the common reference ray is traced using the averaged Hamiltonian of both anisotropic-ray-theory *S* waves (Bakker 2002; Klimeš 2006b). This is probably the best common-ray approximation. In the numerical examples presented in this paper, the errors due to the ACR approximation of the coupling ray theory are considerably smaller than the errors due to the routinely used ICR approximation.

In the less accurate *ICR approximation*, the reference ray is traced in the reference isotropic model. Moreover, the reference isotropic model may be selected in different ways, yielding quasi-isotropic approximations of differing accuracies.

2.2 Traveltimes in the coupling ray theory, anisotropic-ray-theory Hamiltonians

Let $a_{ijkl} = a_{ijkl}(x_m)$ be the density-normalized elastic moduli describing a *smooth anisotropic model*, in which the *ACRs* are traced using the averaged Hamiltonian of both anisotropic-ray-theory *S* waves. We also introduce *S*-wave velocity $v_0 = v_0(x_m)$ in the smooth reference isotropic model in which the ICRs are traced.

Assume a phase-space *reference ray*, parametrized by reference traveltime τ, with reference slowness vectors $p_i(\tau)$ known at all its points $x_j(\tau)$. In this Section 2.2, the reference ray is general, including the ACR and ICR as special cases. Using the reference slowness vectors, we can calculate the reference Christoffel matrix

$$\Gamma_{jk}(\tau) = p_i(\tau)\, a_{ijkl}(x_m(\tau))\, p_l(\tau) \tag{1}$$

and its eigenvectors $g_{i\alpha}(\tau)$, $\alpha = 1, 2, 3$ along the reference ray. Whereas the Einstein summation over the pairs of identical Roman indices (both subscripts and superscripts) $i, j, k, \ldots = 1, 2, 3$ or $I, J, K, \ldots = 1, 2$ is used throughout this paper, no implicit summation applies to Greek subscripts $\alpha, \beta, \ldots$ indexing the eigenvectors of the Christoffel matrix or the perturbation parameters. Assume that eigenvectors $g_{i1}(\tau)$ and $g_{i2}(\tau)$ correspond to the *S* waves. For application of the coupling ray theory, the eigenvectors should vary continuously along the reference ray (Bulant & Klimeš 2002). This condition is not required in regions where the two *S*-wave eigenvalues of the Christoffel matrix are approximately equal.

Let us denote by $\tau_\alpha(\tau)$ the anisotropic-ray-theory traveltime corresponding to the selected eigenvector $g_{i\alpha}(\tau)$ of the Christoffel matrix. It may be approximated by a quadrature along the unperturbed reference ray,

$$\frac{\mathrm{d}\tau_\alpha}{\mathrm{d}\tau} = (\Gamma_{jk} g_{j\alpha} g_{k\alpha})^{-\frac{1}{2}}. \tag{2}$$

Traveltime approximation (2), suggested for the coupling ray theory by Bulant & Klimeš (2002), would become exact for a reference ray following the path of the respective anisotropic-ray-theory ray. Traveltime approximation (2) can be derived as the first-order part of perturbation expansion (9), corresponding to Hamiltonian

$$H_\alpha(x_m, p_n) = -[G_\alpha(x_m, p_n)]^{-\frac{1}{2}}, \qquad \alpha = 1, 2, \tag{3}$$

where $G_\alpha(x_m, p_n)$ is the eigenvalue of Christoffel matrix (1), corresponding to eigenvector $g_{i\alpha}$. Hamiltonian (3) is a homogeneous function of the minus first degree with respect to the slowness vector.

2.3 Reference Hamiltonian, isotropic-common-ray Hamiltonian, anisotropic-common-ray Hamiltonian

We denote by $H_0(x_m, p_n)$ the *reference Hamiltonian* used to trace the reference ray. Since the equations for the perturbation derivatives of traveltime were derived by Klimeš (2002) on the condition that the Hamiltonian has an equal value for all orthonomic systems of rays under consideration, reference Hamiltonian $H_0(x_m, p_n)$ must be a homogeneous function of the same degree with respect to the slowness vector as anisotropic-ray-theory Hamiltonians (3). We thus select the reference Hamiltonian so that it is also a homogeneous function of the minus first degree with respect to the slowness vector.

For the ICR, the reference Hamiltonian equals Hamiltonian

$$H^{\mathrm{ICR}}(x_m, p_n) = -[v_0(x_m)]^{-1}(p_i p_i)^{-\frac{1}{2}} \tag{4}$$

for the reference isotropic medium.

For the ACR, the reference Hamiltonian is equal to the averaged Hamiltonian

$$H^{\mathrm{ACR}}(x_m, p_n) = -\left\{\tfrac{1}{2}\left\{[G_1(x_m, p_n)]^{\frac{N}{2}} + [G_2(x_m, p_n)]^{\frac{N}{2}}\right\}\right\}^{-\frac{1}{N}} \tag{5}$$

of both *S* waves (Klimeš 2006b, eqs 28 and 36), where constant *N* specifies the way of averaging. Note that Bakker (2002) considered $N = 2$, Klimeš & Bulant (2004) considered $N = -1$, whereas Klimeš (2006b) and Klimeš & Bulant (2006) considered general *N*. Hamiltonian (5) is a homogeneous function of the minus first degree with respect to the slowness vector. The value of reference Hamiltonian (4) or (5) along the respective reference ray is $H_0(x_m, p_n) = -1$.

Note that had we used some sort of weighted average in (5), Hamiltonian $H^{\mathrm{ACR}}(x_m, p_n)$ would be closer to one of the Hamiltonians (3) than to the other. The error of approximation (2) would then decrease for one traveltime and increase for the other, but the relative error of the time-harmonic Green function (Klimeš & Bulant 2004, eq. 67) would increase. On the other hand, some sort of weighted average in (5) could be useful if the polarization of the *S* wave were close to one of the *S*-wave eigenvectors of the Christoffel matrix (1), that is, if we traced different rays for different *S*-wave polarizations instead of the common reference rays.

2.4 Parametric system of the Hamiltonians and errors due to the common-ray approximations

We consider the parametric set $H = H(x_m, p_n, f_\alpha)$ of Hamiltonians parametrized by f_α, $\alpha = 1, 2, 3$

$$H(x_m, p_n, f_\alpha) = H_0(x_m, p_n) + \sum_{\beta=1}^{3}[H_\beta(x_m, p_n) - H_0(x_m, p_n)]\, f_\beta, \tag{6}$$

where $H_1(x_m, p_n)$ and $H_2(x_m, p_n)$ are the anisotropic-ray-theory Hamiltonians (3). Parameters f_α are called *perturbation parameters* (Baumgärtel 1985; Murdock 1999; Klimeš & Bulant 2004, 2006) or model parameters (Tarantola 1987; Klimeš 2002). The parametric system (6) of the Hamiltonians then generates the parametric system $\tau(x_m, f_\alpha)$ of the traveltime fields corresponding to the individual Hamiltonians $H(x_m, p_n, f_\alpha)$.

We obtain the reference Hamiltonian for $(f_1, f_2, f_3) = (0, 0, 0)$, the Hamiltonian corresponding to the first anisotropic-ray-theory *S* wave for $(f_1, f_2, f_3) = (1, 0, 0)$, the Hamiltonian corresponding to the second anisotropic-ray-theory *S* wave for $(f_1, f_2, f_3) = (0, 1, 0)$.

For the ICR, we choose

$$H_0(x_m, p_n) = H^{\mathrm{ICR}}(x_m, p_n), \qquad H_3(x_m, p_n) = H^{\mathrm{ACR}}(x_m, p_n). \tag{7}$$

In this case, we obtain the Hamiltonian corresponding to the ACR for $(f_1, f_2, f_3) = (0, 0, 1)$.

For the ACR, we choose

$$H_0(x_m, p_n) = H^{\mathrm{ACR}}(x_m, p_n), \qquad H_3(x_m, p_n) = H^{\mathrm{ICR}}(x_m, p_n). \tag{8}$$

In this case, we obtain the Hamiltonian corresponding to the isotropic-ray-theory ray traced in the reference isotropic model for $(f_1, f_2, f_3) = (0, 0, 1)$.

The Taylor expansion with respect to perturbation parameters f_α is referred to here as the *perturbation expansion*. The second-order perturbation expansion of traveltime is

$$\tau(x_m, f_\gamma) \approx \tau(x_m) + \sum_{\alpha=1}^{3} \tau_{,\alpha}(x_m) f_\alpha + \frac{1}{2} \sum_{\alpha=1}^{3} \sum_{\beta=1}^{3} \tau_{,\alpha\beta}(x_m) f_\alpha f_\beta, \tag{9}$$

where the Greek subscripts following a comma denote partial derivatives with respect to perturbation parameters f_α, hereinafter called *perturbation derivatives*. Note that Klimeš (2002) refers to the perturbation derivatives briefly as 'perturbations'. Here, traveltime $\tau(x_m)$ and its perturbation derivatives $\tau_{,\alpha}(x_m)$, $\tau_{,\alpha\beta}(x_m)$, with arguments f_α omitted, correspond to the reference system of rays $(f_1, f_2, f_3) = (0,0,0)$.

Traveltime approximation (2) corresponds to the first-order part of perturbation expansion (9). The error of traveltime approximation (2) may thus be approximated by the quadratic term in perturbation expansion (9) with Hamiltonians (6).

The errors of traveltime approximation (2) from reference ray $(f_1, f_2, f_3) = (0, 0, 0)$ to anisotropic-ray-theory ray $(f_1, f_2, f_3) = (1, 0, 0)$ and to anisotropic-ray-theory ray $(f_1, f_2, f_3) = (0, 1, 0)$ are approximately

$$\delta\tau_\alpha^{\mathrm{CR}} = \frac{\tau_{,\alpha\alpha}}{2}, \qquad \alpha = 1, 2. \tag{10}$$

If the reference ray is the ICR, errors (10) represent the errors $\delta\tau_\alpha^{\mathrm{ICR}}$ of the ICR approximation. If the reference ray is the ACR, errors (10) represent the errors $\delta\tau_\alpha^{\mathrm{ACR}}$ of the ACR approximation.

We may also use the second-order perturbation derivatives of traveltime, calculated along the ICR, to estimate the accuracy of the ACR approximation, and vice versa. The estimates of the errors of traveltime approximation (2) from the common ray $(f_1, f_2, f_3) = (0, 0, 1)$ other than the reference ray to anisotropic-ray-theory ray $(f_1, f_2, f_3) = (1, 0, 0)$ and to anisotropic-ray-theory ray $(f_1, f_2, f_3) = (0, 1, 0)$ are

$$\delta\tau_\alpha^{\mathrm{OCR}} = \tfrac{1}{2}\tau_{,\alpha\alpha} - \tau_{,\alpha 3} + \tfrac{1}{2}\tau_{,33}, \qquad \alpha = 1, 2. \tag{11}$$

If the reference ray is the ICR, errors (11) represent rough estimates of the errors $\delta\tau_\alpha^{\mathrm{ACR}}$ of the ACR approximation. The accuracy of the ACR approximation can thus be studied approximately along the ICRs, without tracing the ACRs. If the reference ray is the ACR, errors (11) represent rough estimates of the errors $\delta\tau_\alpha^{\mathrm{ICR}}$ of the ICR approximation.

To calculate the first-order and second-order perturbation derivatives $\tau_{,\alpha}$ and $\tau_{,\alpha\beta}$ of traveltime, we need the first-order and second-order phase-space and perturbation derivatives of Hamiltonian (6) at the reference ray. These derivatives are simply composed of the first-order phase-space derivatives of Hamiltonians (3), (4) and (5), and of the second-order phase-space derivatives of the reference Hamiltonian (Klimeš & Bulant 2004, 2006).

3 NUMERICAL EXAMPLES

ACR tracing in smooth anisotropic models without interfaces and corresponding dynamic ray tracing in ray-centred coordinates according to Klimeš (2006a,b) have been coded, added to the Fortran 77 package CRT, debugged and partly numerically tested (Bucha & Bulant 2004). The numerical algorithm of the coupling ray theory by Bulant & Klimeš (2002), coded in the Fortran 77 package CRT, can be applied not only to the ICRs, but also to the ACRs (Bucha & Bulant 2004). The calculation of the second-order traveltime perturbation expansion from the ACR to the anisotropic-ray-theory rays in smooth velocity models without interfaces have also been coded and added to the Fortran 77 package CRT (Bucha & Bulant 2005).

The two-point ACRs are traced in four 1-D models QI, QI2, QI4 and QI8 of differing degrees of anisotropy. The first-order and second-order ACR approximations of the anisotropic-ray-theory traveltimes are compared with the correct anisotropic-ray-theory traveltimes in order to demonstrate the errors of the ACR approximation and the applicability of the perturbation expansion. The coupling-ray-theory synthetic seismograms calculated using the ICR approximation and ACR approximation in models QI, QI2 and QI4 are then compared with the synthetic seismograms calculated by the second-order approximation from the ACRs.

The data for models QI, QI2, QI4 and QI8 have been released on the compact disk of Bucha & Bulant (2002). Numerical examples in this paper have been calculated using the software and data by Bucha & Bulant (2005). Note that both the ACRs and anisotropic-ray-theory rays in the anisotropic 1-D models are three-dimensional because of anisotropy.

3.1 Model QI

A vertically heterogeneous 1-D anisotropic model QI was provided by Pšenčík & Dellinger (2001, model WA rotated by 45°) who performed the coupling-ray-theory calculations using the programs of package ANRAY (Pšenčík 1998b) and compared the results with the reflectivity method. The same model has been used by Bulant & Klimeš (2002) and Klimeš & Bulant (2004) to demonstrate the coupling ray theory and its quasi-isotropic approximations. For comparison with the isotropic and anisotropic ray theory seismograms in model QI and for a more detailed discussion and description of this model refer to Pšenčík & Dellinger (2001). Model QI is transversely isotropic in a vertical plane which forms a 45° angle with the source-receiver vertical plane.

The density-normalized elastic moduli a_{ijkl} in km^2 s^{-2} at the surface (zero depth) are

$$\begin{array}{c} \\ 11 \\ 22 \\ 33 \\ 23 \\ 13 \\ 12 \end{array} \begin{array}{c} \begin{array}{cccccc} 11 & 22 & 33 & 23 & 13 & 12 \end{array} \\ \begin{pmatrix} 14.48500 & 4.52500 & 4.75000 & 0.00000 & 0.00000 & -0.58000 \\ & 14.48500 & 4.75000 & 0.00000 & 0.00000 & -0.58000 \\ & & 15.71000 & 0.00000 & 0.00000 & -0.29000 \\ & & & 5.15500 & -0.17500 & 0.00000 \\ & & & & 5.15500 & 0.00000 \\ & & & & & 5.04500 \end{pmatrix} \end{array}, \qquad (12)$$

and at the depth of 1 km they are

$$\begin{array}{c} \\ 11 \\ 22 \\ 33 \\ 23 \\ 13 \\ 12 \end{array} \begin{array}{c} \begin{array}{cccccc} 11 & 22 & 33 & 23 & 13 & 12 \end{array} \\ \begin{pmatrix} 22.08963 & 6.90063 & 7.24375 & 0.00000 & 0.00000 & -0.88450 \\ & 22.08963 & 7.24375 & 0.00000 & 0.00000 & -0.88450 \\ & & 23.95775 & 0.00000 & 0.00000 & -0.44225 \\ & & & 7.86138 & -0.26688 & 0.00000 \\ & & & & 7.86138 & 0.00000 \\ & & & & & 7.69363 \end{pmatrix} \end{array}. \qquad (13)$$

Here the rows correspond to the first couple of indices of a_{ijkl}, the columns correspond to the second couple of indices. For the vertical sections through the *S*-wave phase-velocity surfaces at the depths of 0 and 0.6 km refer to Pšenčík & Dellinger (2001, fig. 7). The reference isotropic model is given by

$$v_{\mathrm{P}}^2 = 15.00\,\mathrm{km}^2\,\mathrm{s}^{-2}, \qquad v_{\mathrm{S}}^2 = 5.10\,\mathrm{km}^2\,\mathrm{s}^{-2} \qquad (14)$$

at the surface, and

$$v_{\mathrm{P}}^2 = 23.00\,\mathrm{km}^2\,\mathrm{s}^{-2}, \qquad v_{\mathrm{S}}^2 = 7.79\,\mathrm{km}^2\,\mathrm{s}^{-2} \qquad (15)$$

at the depth of 1 km. All the above values are interpolated linearly with depth. The density is constant.

The synthetic seismograms, corresponding to vertical force $\mathbf{F} = (0, 0, 100)^{\mathrm{T}}$ at position $(50, 50, 0)^{\mathrm{T}}$, are calculated at 29 receivers $(51, 50, 0.010)^{\mathrm{T}}$, $(51, 50, 0.030)^{\mathrm{T}}$, $(51, 50, 0.050)^{\mathrm{T}}$, ..., $(51, 50, 0.570)^{\mathrm{T}}$ located in a vertical well (distances in km), see Fig. 1. The source time function is the Gabor signal $\cos(2\pi f t)\exp[-(2\pi f t/4)^2]$ with reference frequency $f = 50$ Hz, bandpass filtered by a cosine filter given by frequencies 0, 5, 60 and 100 Hz.

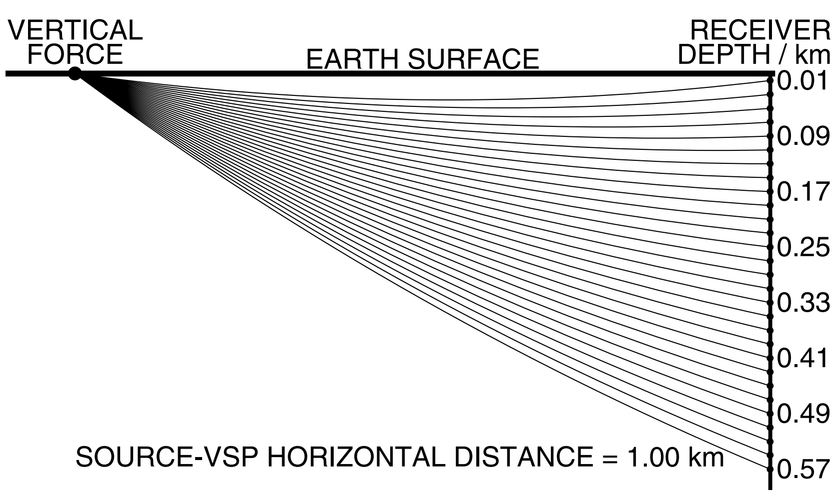

Figure 1. The source and receiver configuration in models QI, QI2, QI4 and QI8.

For comparison with the isotropic-ray-theory and anisotropic-ray-theory seismograms in model QI and for a more detailed discussion and description of this model refer to Pšenčík & Dellinger (2001).

3.2 Models QI2, QI4 and QI8

To emphasize the effects of perturbations of traveltime, new models with an increased degree of anisotropy have been derived from the QI model.

The differences of the elastic moduli of model QI2 from the elastic moduli of the reference isotropic model (14), (15) are exactly twice larger than the differences of model QI. The density-normalized elastic moduli a_{ijkl} of model QI2 in km^{2}s^{-2} at the surface (zero depth) are

$$
\begin{array}{c} \\ 11 \\ 22 \\ 33 \\ 23 \\ 13 \\ 12 \end{array}
\begin{array}{c}
\begin{array}{cccccc} 11 & 22 & 33 & 23 & 13 & 12 \end{array} \\
\begin{pmatrix}
13.97000 & 4.25000 & 4.70000 & 0.00000 & 0.00000 & -1.16000 \\
 & 13.97000 & 4.70000 & 0.00000 & 0.00000 & -1.16000 \\
 & & 16.42000 & 0.00000 & 0.00000 & -0.58000 \\
 & & & 5.21000 & -0.35000 & 0.00000 \\
 & & & & 5.21000 & 0.00000 \\
 & & & & & 4.99000
\end{pmatrix}
\end{array}, \qquad (16)
$$

and at the depth of 1 km they are

$$
\begin{array}{c} \\ 11 \\ 22 \\ 33 \\ 23 \\ 13 \\ 12 \end{array}
\begin{array}{c}
\begin{array}{cccccc} 11 & 22 & 33 & 23 & 13 & 12 \end{array} \\
\begin{pmatrix}
21.17926 & 6.38126 & 7.06750 & 0.00000 & 0.00000 & -1.76900 \\
 & 21.17926 & 7.06750 & 0.00000 & 0.00000 & -1.76900 \\
 & & 24.91550 & 0.00000 & 0.00000 & -0.88450 \\
 & & & 7.93276 & -0.53376 & 0.00000 \\
 & & & & 7.93276 & 0.00000 \\
 & & & & & 7.59726
\end{pmatrix}
\end{array}. \qquad (17)
$$

Analogously, the differences of the elastic moduli of model QI4 from the elastic moduli of the reference isotropic model (14), (15) are exactly four times larger than the differences of model QI. The density-normalized elastic moduli a_{ijkl} of model QI4 in km^2 s^{-2} at the surface (zero depth) are

$$
\begin{array}{c} \\ 11 \\ 22 \\ 33 \\ 23 \\ 13 \\ 12 \end{array}
\begin{array}{c}
\begin{array}{cccccc} 11 & 22 & 33 & 23 & 13 & 12 \end{array} \\
\begin{pmatrix}
12.94000 & 3.70000 & 4.60000 & 0.00000 & 0.00000 & -2.32000 \\
 & 12.94000 & 4.60000 & 0.00000 & 0.00000 & -2.32000 \\
 & & 17.84000 & 0.00000 & 0.00000 & -1.16000 \\
 & & & 5.32000 & -0.70000 & 0.00000 \\
 & & & & 5.32000 & 0.00000 \\
 & & & & & 4.88000
\end{pmatrix}
\end{array}, \qquad (18)
$$

and at the depth of 1 km they are

$$
\begin{array}{c} \\ 11 \\ 22 \\ 33 \\ 23 \\ 13 \\ 12 \end{array}
\begin{array}{c}
\begin{array}{cccccc} 11 & 22 & 33 & 23 & 13 & 12 \end{array} \\
\begin{pmatrix}
19.35852 & 5.34252 & 6.71500 & 0.00000 & 0.00000 & -3.53800 \\
 & 19.35852 & 6.71500 & 0.00000 & 0.00000 & -3.53800 \\
 & & 26.83100 & 0.00000 & 0.00000 & -1.76900 \\
 & & & 8.07552 & -1.06752 & 0.00000 \\
 & & & & 8.07552 & 0.00000 \\
 & & & & & 7.40452
\end{pmatrix}
\end{array}. \qquad (19)
$$

Note that we have also traced the ACRs in model QI8, in which the differences of the elastic moduli from the elastic moduli of the reference isotropic model (14), (15) are exactly 8 times larger than the differences of model QI.

3.3 Effects of common-ray approximations

The two-point ACRs have been traced from the source to the receivers using the CRT program (Červený *et al.* 1988), and the first-order and second-order perturbation derivatives of traveltime have been calculated along these rays. Eq. (10) has then been used to estimate the traveltime errors $\delta\tau_1^{\mathrm{ACR}}$ and $\delta\tau_2^{\mathrm{ACR}}$ of the ACR approximation. Eq. (10) has also been used to estimate the second-order term $\delta\tau_3^{\mathrm{ACR}}$ in the perturbation expansion of traveltime from the ACR to the ICR. If required, the traveltime errors $\delta\tau_1^{\mathrm{ICR}}$ and $\delta\tau_2^{\mathrm{ICR}}$ of the ICR approximation can also be roughly estimated using eq. (11). The projection of the traveltime errors on the relative error of the coupled *S* wavefield depends on the polarization. For the projection of the traveltime errors on the relative errors of the time-harmonic Green tensor refer to Klimeš & Bulant (2004, section 3.5).

3.4 Perturbation expansion from isotropic common rays to anisotropic common rays and vice versa

To check the convergency and accuracy of the second–order perturbation expansion (9), we first compare the perturbation expansion from the ICR to the ACR with the perturbation expansion from the ACR to the ICR. The results of the comparison for the ACRs traced using the reference Hamiltonian (5) averaged with degree $N = -1$ are displayed in Tables 1–3. Only the results at the 1st, 8th, 15th, 22nd and 29th receivers (see Fig. 1) are shown, because the variation of the quantities along the vertical profile in models QI, QI2 and QI4 is very moderate. Note that the neglected third-order and higher odd-order perturbation corrections map onto the differences between the ACR and ICR remaining terms in Tables 1–3, whereas the neglected fourth-order and higher even-order perturbation corrections map onto the average of the ACR and ICR remaining terms. We may approximate the third-order terms in the perturbation expansions using the differences between the ACR and ICR quadratic terms. If we subtract these approximate third-order terms from the ACR and ICR remaining terms, the differences between the ACR and ICR remaining terms decrease to the level of numerical errors. This means that the fifth-order terms in the perturbation expansion are not detectable in Tables 1–3. The average of the ACR and ICR remaining terms in Table 3 is mostly caused by the fourth-order terms in the perturbation expansions.

Table 1. Perturbation expansion of traveltime from the isotropic common ray to the anisotropic common ray and from the anisotropic common ray to the isotropic common ray in model QI.

Rec. dep.	ICR time	ICR linear term	ICR quadratic term	ICR remaining term	ACR time	ACR linear term	ACR quadratic term	ACR remaining term
0.01	0.440993	0.000796	0.000060	0.000001	0.441850	−0.000917	0.000062	−0.000001
0.15	0.438077	0.000743	0.000058	0.000002	0.438880	−0.000862	0.000061	−0.000002
0.29	0.443550	0.000575	0.000056	0.000002	0.444183	−0.000692	0.000059	−0.000001
0.43	0.456339	0.000328	0.000054	0.000000	0.456721	−0.000440	0.000057	0.000000
0.57	0.475205	0.000037	0.000050	0.000003	0.475296	−0.000143	0.000054	−0.000003

Rec. dep.: the receiver depth along the vertical profile (see Fig. 1). *ICR time*: the traveltime along the isotropic common ray traced in the reference isotropic model. *ICR linear term*: the linear term in perturbation expansion (9) of traveltime from the isotropic common ray to the anisotropic common ray. *ICR quadratic term*: the quadratic term in the perturbation expansion from the isotropic common ray to the anisotropic common ray. *ICR remaining term*: the difference between the traveltime along the anisotropic common ray and the second-order perturbation expansion from the isotropic common ray (*ICR time* +*ICR linear term* +*ICR quadratic term* +*ICR remaining term* = *ACR time*). *ACR time*: the traveltime along the anisotropic common ray. *ACR linear term*: the linear term in perturbation expansion (9) of traveltime from the anisotropic common ray to the isotropic common ray. *ACR quadratic term*: the quadratic term in the perturbation expansion from the anisotropic common ray to the isotropic common ray. *ACR remaining term*: the difference between the traveltime along the isotropic common ray and the second-order perturbation expansion from the anisotropic common ray (*ACR time* +*ACR linear term* +*ACR quadratic term* +*ACR remaining term* = *ICR time*). The ICR remaining term and ACR remaining term represent both the inaccuracy of numerical ray tracing and the third-order and higher-order terms in the perturbation expansion. The third-order and fourth-order terms are significant in model QI4 only.

Table 2. Perturbation expansion of traveltime from the isotropic common ray to the anisotropic common ray and from the anisotropic common ray to the isotropic common ray in model QI2.

Rec. dep.	ICR time	ICR linear term	ICR quadratic term	ICR remaining term	ACR time	ACR linear term	ACR quadratic term	ACR remaining term
0.01	0.440993	0.003429	0.000250	−0.000005	0.444667	−0.003921	0.000240	0.000006
0.15	0.438077	0.003310	0.000248	−0.000002	0.441634	−0.003802	0.000241	0.000004
0.29	0.443550	0.002982	0.000252	0.000005	0.446788	−0.003485	0.000248	−0.000001
0.43	0.456339	0.002500	0.000255	−0.000002	0.459093	−0.003018	0.000258	0.000005
0.57	0.475205	0.001926	0.000257	0.000004	0.477392	−0.002453	0.000266	0.000000

For notes relating to Table 2, see footnote of Table 1.

Table 3. Perturbation expansion of traveltime from the isotropic common ray to the anisotropic common ray and from the anisotropic common ray to the isotropic common ray in model QI4.

Rec. dep.	ICR time	ICR linear term	ICR quadratic term	ICR remaining term	ACR time	ACR linear term	ACR quadratic term	ACR remaining term
0.01	0.440993	0.016119	0.001701	−0.000241	0.458572	−0.018809	0.001016	0.000214
0.15	0.438077	0.015763	0.001735	−0.000231	0.455345	−0.018545	0.001059	0.000218
0.29	0.443550	0.014973	0.001861	−0.000212	0.460171	−0.018043	0.001196	0.000226
0.43	0.456339	0.013833	0.002024	−0.000175	0.472022	−0.017316	0.001407	0.000226
0.57	0.475205	0.012454	0.002173	−0.000117	0.489715	−0.016385	0.001661	0.000214

For notes relating to Table 3, see footnote of Table 1.

3.5 Perturbation expansions from isotropic and anisotropic common rays to anisotropic-ray-theory rays

For convenient comparison of the ICR and ACR approximations, the perturbation expansion of traveltime from the ICR to the anisotropic-ray-theory rays is compared with the correct anisotropic-ray-theory traveltimes in Tables 4–6. The exact anisotropic-ray-theory traveltimes have been calculated by the ANRAY program (Gajewski & Pšenčík 1990; Pšenčík 1998b). Note that the quadratic terms in analogous tables 1–3 by Klimeš & Bulant (2004) differ slightly from the corresponding terms in our Tables 4–6 because of a bug which has been fixed by Bucha & Bulant (2005).

The estimated ACR quadratic terms in Tables 4–6 represent the estimate of the error due to the ACR approximation of the anisotropic-ray-theory traveltimes calculated using eq. (11) along the ICRs. The accuracy of the ACR approximation can thus be studied approximately along the ICRs, without tracing the ACRs. This was used by Klimeš & Bulant (2004) who could not trace the ACRs.

The perturbation expansion of traveltime from the ACR to the anisotropic-ray-theory rays is then compared with the correct anisotropic-ray-theory traveltimes. The results of the comparison for the ACRs traced using the reference Hamiltonian (5) averaged with degree $N = -1$ are displayed in Tables 7–9. The results of the comparison for the ACRs traced using the reference Hamiltonian (5) averaged with degree $N = 2$ (Bakker 2002) are displayed in Tables 10–12.

The quadratic terms in the perturbation expansions displayed in Tables 4–12 represent the estimates of the errors due to the ICR and ACR approximations of the anisotropic-ray-theory traveltimes.

Table 4. Perturbation expansion of traveltime from the isotropic common ray to the anisotropic-ray-theory rays in model QI.

Rec. dep.	ICR time	ICR linear terms		ICR quadratic terms		ICR remaining terms		Est. ACR q. terms
0.01	0.440993	−0.002392	0.003983	0.000251	0.000000	−0.000001	−0.000001	0.000065
0.15	0.438077	−0.002518	0.004003	0.000248	0.000000	0.000009	−0.000001	0.000066
0.29	0.443550	−0.002967	0.004117	0.000249	0.000001	0.000009	0.000001	0.000069
0.43	0.456339	−0.003661	0.004317	0.000251	0.000001	0.000010	−0.000002	0.000072
0.57	0.475205	−0.004520	0.004595	0.000250	0.000003	0.000022	0.000006	0.000076

Rec. dep.: the receiver depth along the vertical profile (see Fig. 1). *ICR time*: the reference traveltime along the isotropic common ray traced in the reference isotropic model. *ICR linear terms*: the linear terms in perturbation expansion (9) of traveltime from the isotropic common ray to the anisotropic-ray-theory rays. They represent the traveltime corrections considered in the isotropic-common-ray approximation of the coupling ray theory. *ICR quadratic terms*: the quadratic terms in the perturbation expansion. They represent the estimates of the errors due to the isotropic-common-ray approximation of the anisotropic-ray-theory traveltimes. *ICR remaining terms*: the differences between the correct anisotropic-ray-theory traveltimes calculated by the ANRAY program version 4.40 and the second-order perturbation expansion from the isotropic common ray, in order to illustrate the reliability of the error estimates. The ICR remaining terms represent both the inaccuracy of numerical ray tracing and the third-order and higher-order terms in the perturbation expansion. *Est. ACR q. terms*: the estimate of the equal quadratic terms in the perturbation expansion from the anisotropic common ray, traced using the reference Hamiltonian (5) averaged with degree $N = -1$, to the anisotropic-ray-theory rays. The quadratic terms represent the estimate of the error due to the anisotropic-common-ray approximation of the anisotropic-ray-theory traveltimes. Note that these estimated ACR quadratic terms are approximately calculated along the isotropic common rays, that is, without tracing the anisotropic common rays. The estimated ACR quadratic terms may be compared with the actual errors of the anisotropic-common-ray approximation (Tables 7–9, columns ACR quadratic terms and ACR remaining terms). The differences between the estimated ACR quadratic terms calculated along the isotropic common rays and the ACR quadratic terms calculated along the anisotropic common rays are mostly due to the neglected third-order terms in the perturbation expansion.

Table 5. Perturbation expansion of traveltime from the isotropic common ray to the anisotropic-ray-theory rays in model QI2.

Rec. dep.	ICR time	ICR linear terms		ICR quadratic terms		ICR remaining terms		Est. ACR q. terms
0.01	0.440993	−0.004746	0.011604	0.000971	0.000000	0.000008	−0.000007	0.000235
0.15	0.438077	−0.004992	0.011612	0.000959	0.000000	0.000023	0.000001	0.000231
0.29	0.443550	−0.005874	0.011837	0.000960	0.000001	0.000036	0.000005	0.000229
0.43	0.456339	−0.007234	0.012235	0.000958	0.000001	0.000054	0.000001	0.000224
0.57	0.475205	−0.008912	0.012764	0.000945	0.000002	0.000069	0.000000	0.000217

For notes relating to Table 5, see footnote of Table 4.

Table 6. Perturbation expansion of traveltime from the isotropic common ray to the anisotropic-ray-theory rays in model QI4.

Rec. dep.	ICR time	ICR linear terms		ICR quadratic terms		ICR remaining terms		Est. ACR q. terms
0.01	0.440993	−0.009341	0.041580	0.003643	0.000488	0.000119	−0.000137	0.000365
0.15	0.438077	−0.009816	0.041343	0.003589	0.000548	0.000164	−0.000142	0.000333
0.29	0.443550	−0.011517	0.041462	0.003554	0.000711	0.000249	−0.000167	0.000272
0.43	0.456339	−0.014129	0.041796	0.003496	0.000953	0.000352	−0.000203	0.000200
0.57	0.475205	−0.017335	0.042243	0.003394	0.001224	0.000455	−0.000224	0.000136

For notes relating to Table 6, see footnote of Table 4.

Table 7. Perturbation expansion of traveltime from the anisotropic common ray to the anisotropic-ray-theory rays in model QI. The homogeneous Hamiltonians of the minus first degree are averaged.

Rec. dep.	ACR time	ACR linear terms		ACR quadratic terms		ACR remaining terms	
0.01	0.441850	−0.003061	0.003061	0.000066	0.000066	−0.000003	−0.000002
0.15	0.438880	−0.003135	0.003135	0.000066	0.000066	0.000004	−0.000001
0.29	0.444183	−0.003415	0.003415	0.000070	0.000070	0.000003	0.000000
0.43	0.456721	−0.003861	0.003861	0.000075	0.000075	0.000004	−0.000001
0.57	0.475296	−0.004430	0.004430	0.000079	0.000079	0.000010	0.000004

Rec. dep.: the receiver depth along the vertical profile (see Fig. 1). *ACR time*: the reference traveltime τ along the anisotropic common ray. *ACR linear terms*: the linear terms $\tau_{,1}$ and $\tau_{,2}$ in perturbation expansion (9) of traveltime from the anisotropic common ray to the anisotropic-ray-theory rays. They represent the traveltime corrections considered in the anisotropic-common-ray approximation of the coupling ray theory. *ACR quadratic terms*: quadratic terms (10) in the perturbation expansion. They represent the estimates of the errors due to the anisotropic-common-ray approximation of the anisotropic-ray-theory traveltimes. *ACR remaining terms*: the differences between the correct anisotropic-ray-theory traveltimes calculated by the ANRAY program version 4.40 and the second-order perturbation expansion from the anisotropic common ray, in order to illustrate the reliability of the error estimates. The ACR remaining terms represent both the inaccuracy of numerical ray tracing and the third-order and higher-order terms in the perturbation expansion.

Table 8. Perturbation expansion of traveltime from the anisotropic common ray to the anisotropic-ray-theory rays in model QI2. The homogeneous Hamiltonians of the minus first degree are averaged.

Rec. dep.	ACR time	ACR linear terms		ACR quadratic terms		ACR remaining terms	
0.01	0.444667	−0.007689	0.007689	0.000245	0.000245	0.000002	−0.000011
0.15	0.441634	−0.007818	0.007818	0.000244	0.000244	0.000008	−0.000004
0.29	0.446788	−0.008366	0.008366	0.000245	0.000245	0.000006	−0.000007
0.43	0.459093	−0.009240	0.009240	0.000246	0.000246	0.000017	−0.000003
0.57	0.477392	−0.010344	0.010344	0.000244	0.000244	0.000014	−0.000009

For notes relating to Table 8, see footnote of Table 7.

Table 9. Perturbation expansion of traveltime from the anisotropic common ray to the anisotropic-ray-theory rays in model QI4. The homogeneous Hamiltonians of the minus first degree are averaged.

Rec. dep.	ACR time	ACR linear terms		ACR quadratic terms		ACR remaining terms	
0.01	0.458572	−0.023856	0.023856	0.000588	0.000588	0.000110	−0.000090
0.15	0.455345	−0.024001	0.024001	0.000560	0.000560	0.000111	−0.000079
0.29	0.460171	−0.024947	0.024947	0.000505	0.000505	0.000108	−0.000067
0.43	0.472022	−0.026489	0.026489	0.000433	0.000433	0.000092	−0.000059
0.57	0.489715	−0.028424	0.028424	0.000355	0.000355	0.000072	−0.000047

For notes relating to Table 9, see footnote of Table 7.

In the strongly anisotropic model QI8, the errors of the ACR approximation of the anisotropic-ray-theory traveltimes are so large that the ACR approximation of the coupling ray theory is considerably inaccurate and should not be applied. In model QI8, both the ACRs and the errors of the ACR approximation depend considerably on degree N of the homogeneous Hamiltonians averaged in eq. (5).

3.6 Comparison of the isotropic-common-ray and anisotropic-common-ray approximations of the coupling ray theory

The ACR approximation of the coupling-ray-theory synthetic seismograms in models QI, QI2 and QI4 is compared with the more accurate coupling-ray-theory synthetic seismograms simulated by the second-order perturbation expansion of traveltime from the ACRs in Figs 3, 5 and 7 respectively. Refer to Tables 7–9 (columns ACR remaining terms) for the traveltime errors of this simulation. The ICR approximation of coupling-ray-theory synthetic seismograms is compared with these more accurate coupling-ray-theory synthetic seismograms in Figs 2, 4 and 6. Figs 2, 4 and 6 thus differ from analogous figs 1, 3 and 5 of Klimeš & Bulant (2004) in the more accurate simulation of the coupling-ray-theory synthetic seismograms without the common ray approximation. Klimeš & Bulant (2004) could not apply the second-order perturbation expansion of traveltime from the ACRs; they were only able to apply the second-order perturbation expansion of traveltime from the ICRs.

The S waves in models QI and QI2 are coupled. The change of polarization due to coupling is indicated by the increment of the transverse amplitudes at shallow receivers between models QI and QI2, whereas S-wave splitting starts to develop at deep receivers in model QI2. Clear S-wave splitting, if anisotropy is increased further, can be observed in model QI4.

Although the second-order perturbation expansion of traveltime has been applied successfully to estimating anisotropic-ray-theory traveltimes in these simple 1-D models QI, QI2 and QI4 with constant gradients of the density-normalized elastic moduli, we cannot recommend the approximation of traveltime using the second-order perturbation expansion in more complex models, because the second-order perturbations may be infinitely large in the vicinity of caustics. The second-order perturbation expansion of traveltime should be used especially for estimating and controlling the accuracy of the common-ray approximations outside caustics.

Table 10. Perturbation expansion of traveltime from the anisotropic common ray to the anisotropic-ray-theory rays in model QI. The homogeneous Hamiltonians of the second degree are averaged.

Rec. dep.	ACR time	ACR linear terms		ACR quadratic terms		ACR remaining terms	
0.01	0.441814	−0.003027	0.003090	0.000063	0.000068	0.000001	0.000002
0.15	0.438846	−0.003098	0.003165	0.000064	0.000069	0.000005	−0.000001
0.29	0.444146	−0.003372	0.003451	0.000067	0.000073	0.000000	−0.000003
0.43	0.456674	−0.003808	0.003906	0.000071	0.000078	0.000002	−0.000002
0.57	0.475234	−0.004363	0.004487	0.000075	0.000083	0.000010	0.000004

Rec. dep.: the receiver depth along the vertical profile (see Fig. 1). *ACR time*: the reference traveltime τ along the anisotropic common ray. *ACR linear terms*: the linear terms $\tau_{,1}$ and $\tau_{,2}$ in perturbation expansion (9) of traveltime from the anisotropic common ray to the anisotropic-ray-theory rays. They represent the traveltime corrections considered in the anisotropic-common-ray approximation of the coupling ray theory. *ACR quadratic terms*: quadratic terms (10) in the perturbation expansion. They represent the estimates of the errors due to the anisotropic-common-ray approximation of the anisotropic-ray-theory traveltimes. *ACR remaining terms*: the differences between the correct anisotropic-ray-theory traveltimes calculated by the ANRAY program version 4.40 and the second-order perturbation expansion from the anisotropic common ray, in order to illustrate the reliability of the error estimates. The ACR remaining terms represent both the inaccuracy of numerical ray tracing and the third-order and higher-order terms in the perturbation expansion.

Table 11. Perturbation expansion of traveltime from the anisotropic common ray to the anisotropic-ray-theory rays in model QI2. The homogeneous Hamiltonians of the second degree are averaged.

Rec. dep.	ACR time	ACR linear terms		ACR quadratic terms		ACR remaining terms	
0.01	0.444469	−0.007465	0.007861	0.000220	0.000271	0.000002	−0.000011
0.15	0.441425	−0.007586	0.007998	0.000218	0.000270	0.000010	−0.000002
0.29	0.446554	−0.008105	0.008572	0.000219	0.000273	0.000005	−0.000007
0.43	0.458815	−0.008932	0.009487	0.000218	0.000277	0.000016	−0.000004
0.57	0.477058	−0.009977	0.010647	0.000214	0.000277	0.000012	−0.000011

For notes relating to Table 11, see footnote of Table 10.

Table 12. Perturbation expansion of traveltime from the anisotropic common ray to the anisotropic-ray-theory rays in model QI4. The homogeneous Hamiltonians of the second degree are averaged.

Rec. dep.	ACR time	ACR linear terms		ACR quadratic terms		ACR remaining terms	
0.01	0.456727	−0.021838	0.025497	0.000450	0.000836	0.000075	−0.000135
0.15	0.453460	−0.021953	0.025686	0.000427	0.000800	0.000080	−0.000120
0.29	0.458162	−0.022780	0.026778	0.000383	0.000730	0.000071	−0.000114
0.43	0.469812	−0.024138	0.028541	0.000325	0.000634	0.000059	−0.000103
0.57	0.487256	−0.025848	0.030746	0.000264	0.000527	0.000047	−0.000081

For notes relating to Table 12, see footnote of Table 10.

4 CONCLUSIONS

In using any kind of the common-ray approximation of the coupling ray theory, the traveltime errors due to the applied common-ray approximation should be evaluated. The traveltime errors due to the common-ray approximations can be estimated by numerical quadrature along the isotropic common rays using the equations proposed by Klimeš & Bulant (2004), or by numerical quadrature along the anisotropic common rays using the equations proposed by Klimeš & Bulant (2006).

The numerical results demonstrate that the anisotropic-common-ray approximation by Bakker (2002) and Klimeš (2006b) is considerably more accurate than the isotropic-common-ray approximation. Whereas the first-order isotropic-common-ray approximation is considerably inaccurate in model QI4, see Fig. 6 and Table 6, the first-order anisotropic-common-ray approximation performs well in all three models QI, QI2 and QI4, see Figs 3, 5 and 7 and Tables 7–9.

For additional information, including electronic reprints, computer codes and data, refer to the consortium research project 'Seismic Waves in Complex 3-D Structures' ('http://sw3d.mff.cuni.cz').

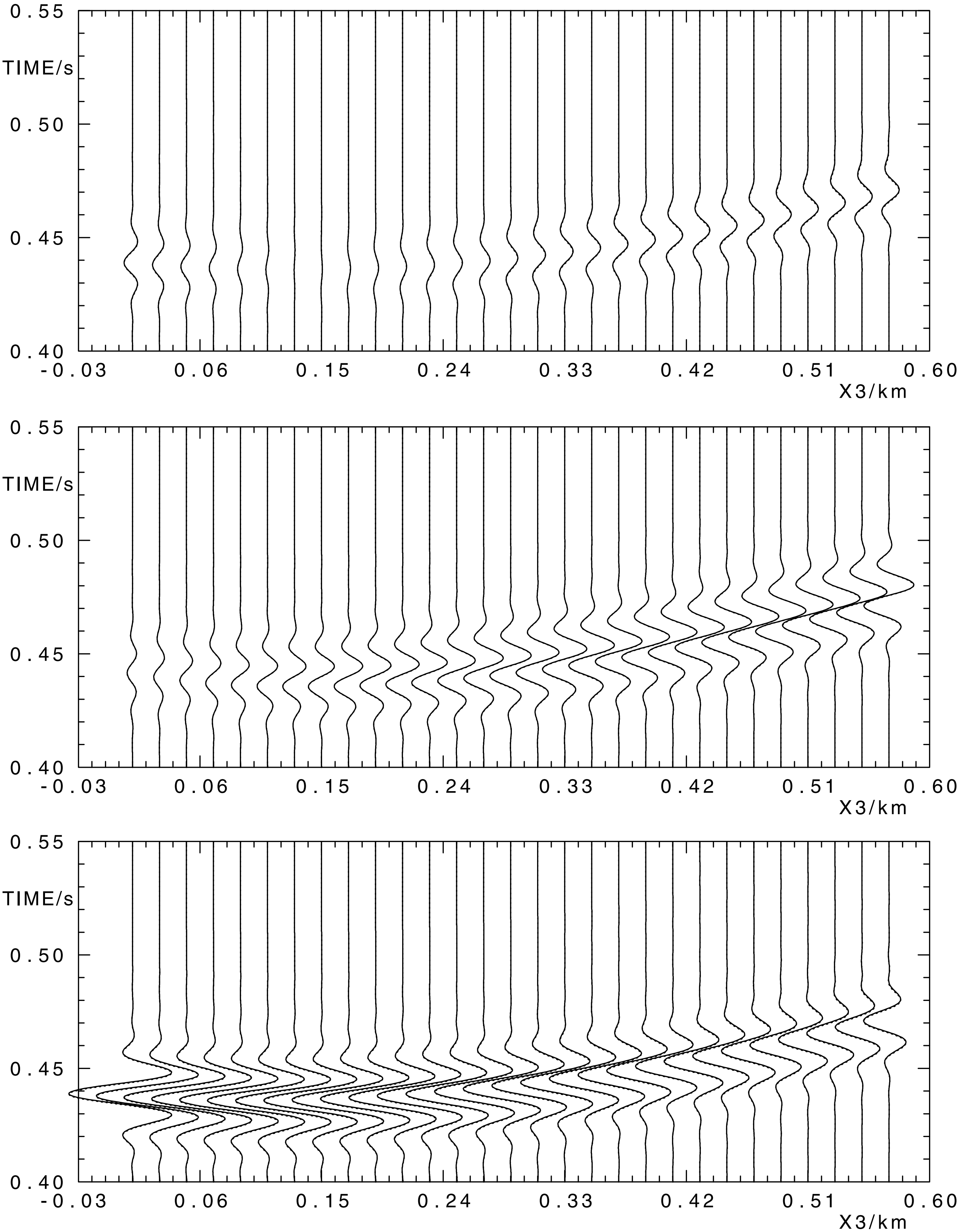

Figure 2. Inaccuracy of the isotropic-common-ray approximation of the coupling ray theory in model QI. From top to bottom: the first (radial) component, the second (transverse) component, the third (vertical) component. *Solid line*: Coupling-ray-theory synthetic seismograms without the common ray approximation, simulated by the second-order perturbation expansion of traveltime from the anisotropic common rays. For the errors of this simulation refer to the *ACR remaining terms* in Table 7. *Dotted line*: Isotropic-common-ray approximation. The dotted seismograms are mostly obscured by the solid seismograms.

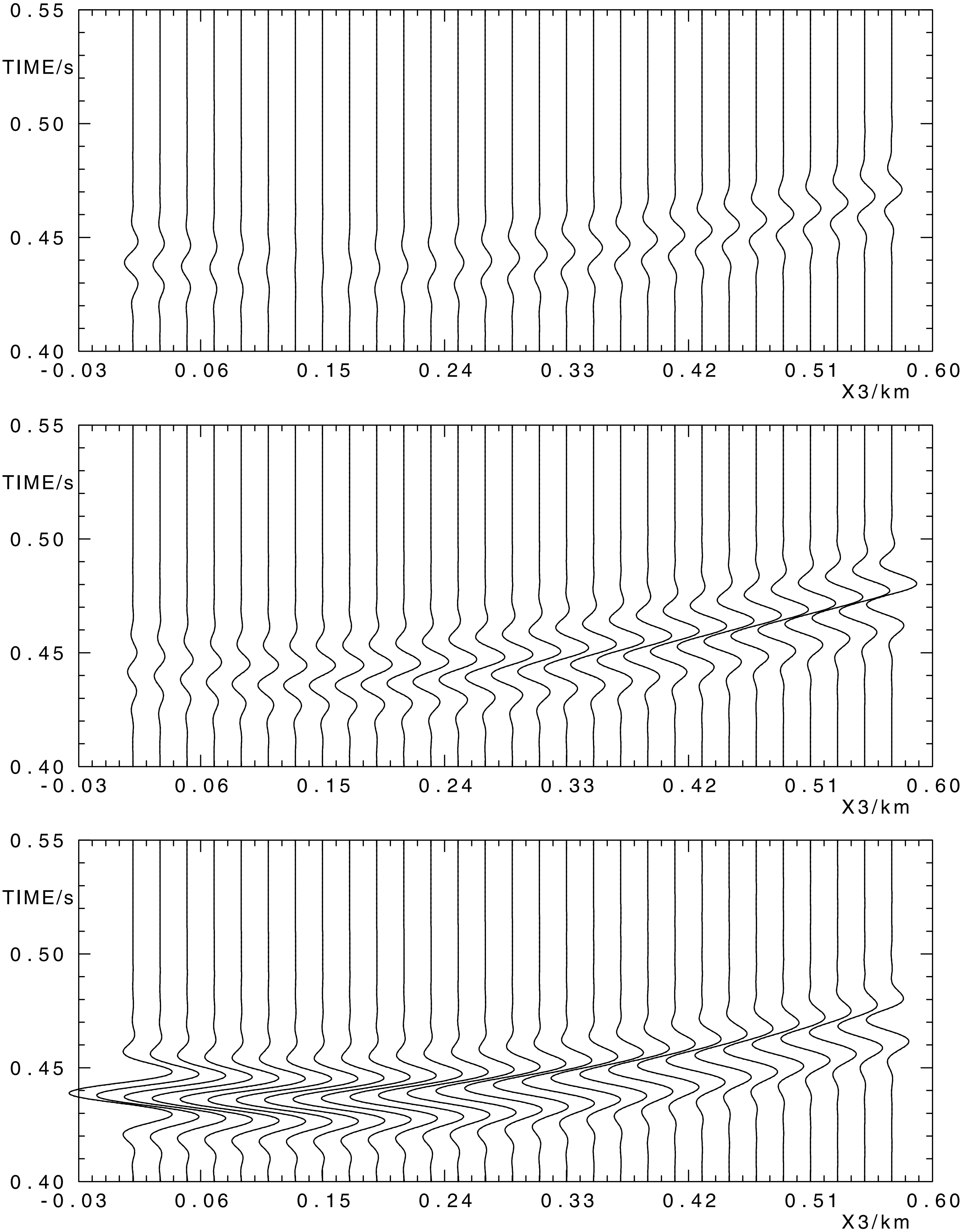

Figure 3. Inaccuracy of the anisotropic-common-ray approximation of the coupling ray theory in model QI. From top to bottom: the first (radial) component, the second (transverse) component, the third (vertical) component. *Solid line*: Coupling-ray-theory synthetic seismograms without the common ray approximation, simulated by the second-order perturbation expansion of traveltime from the anisotropic common rays. For the errors of this simulation refer to the *ACR remaining terms* in Table 7. *Dotted line*: Anisotropic-common-ray approximation. The dotted seismograms are obscured by the solid seismograms.

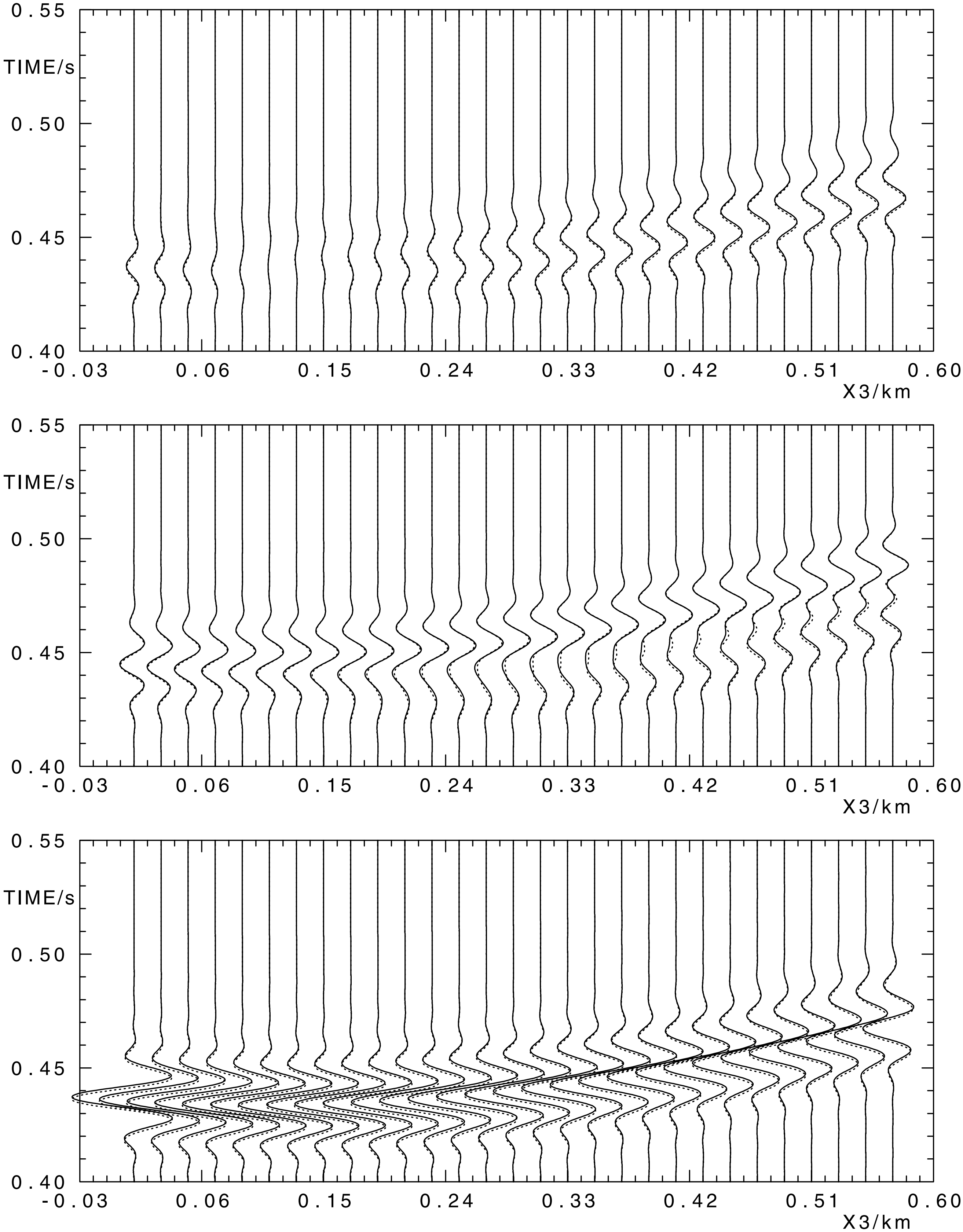

Figure 4. Inaccuracy of the isotropic-common-ray approximation of the coupling ray theory in model QI2. From top to bottom: the first (radial) component, the second (transverse) component, the third (vertical) component. *Solid line*: Coupling-ray-theory synthetic seismograms without the common ray approximation, simulated by the second-order perturbation expansion of traveltime from the anisotropic common rays. For the errors of this simulation refer to the *ACR remaining terms* in Table 8. *Dotted line*: Isotropic-common-ray approximation. The differences between the seismograms are small but already clearly visible.

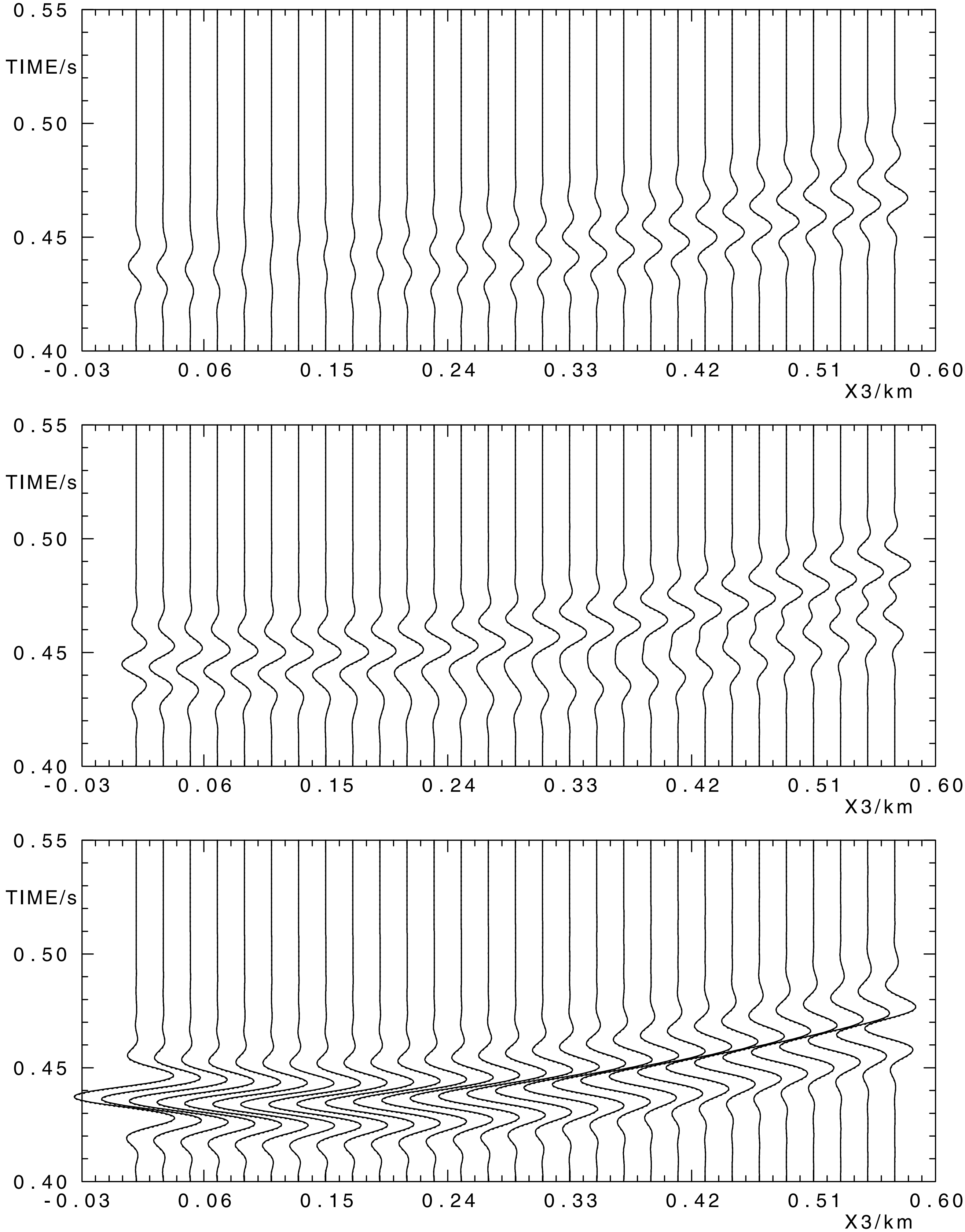

Figure 5. Inaccuracy of the anisotropic-common-ray approximation of the coupling ray theory in model QI2. From top to bottom: the first (radial) component, the second (transverse) component, the third (vertical) component. *Solid line*: Coupling-ray-theory synthetic seismograms without the common ray approximation, simulated by the second-order perturbation expansion of traveltime from the anisotropic common rays. For the errors of this simulation refer to the *ACR remaining terms* in Table 8. *Dotted line*: Anisotropic-common-ray approximation. The dotted seismograms are mostly obscured by the solid seismograms.

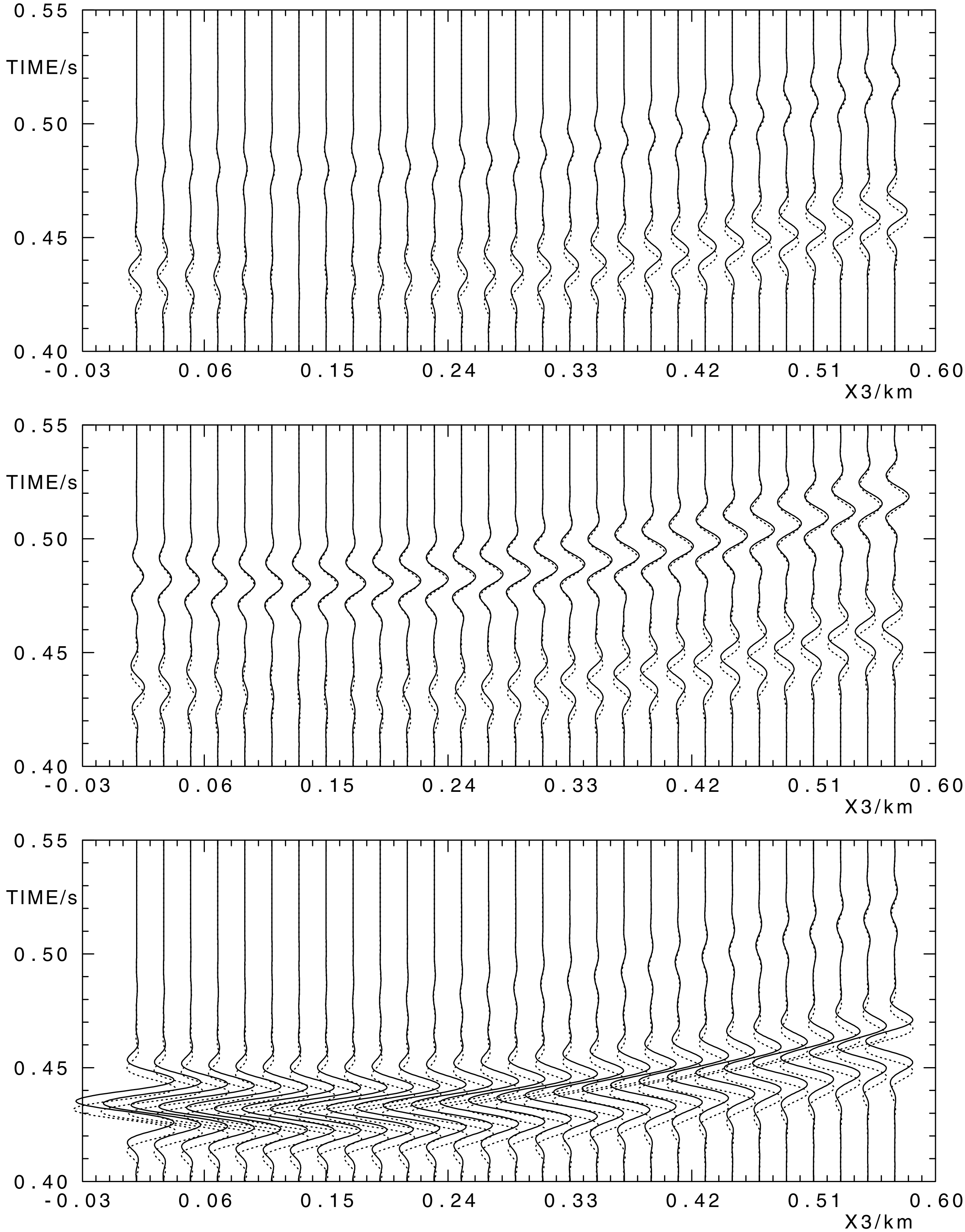

Figure 6. Inaccuracy of the isotropic-common-ray approximation of the coupling ray theory in model QI4. From top to bottom: the first (radial) component, the second (transverse) component, the third (vertical) component. *Solid line*: Coupling-ray-theory synthetic seismograms without the common ray approximation, simulated by the second-order perturbation expansion of traveltime from the anisotropic common rays. For the errors of this simulation refer to the *ACR remaining terms* in Table 9. *Dotted line*: Isotropic-common-ray approximation. The differences between the seismograms are considerable in this quite strongly anisotropic model QI4.

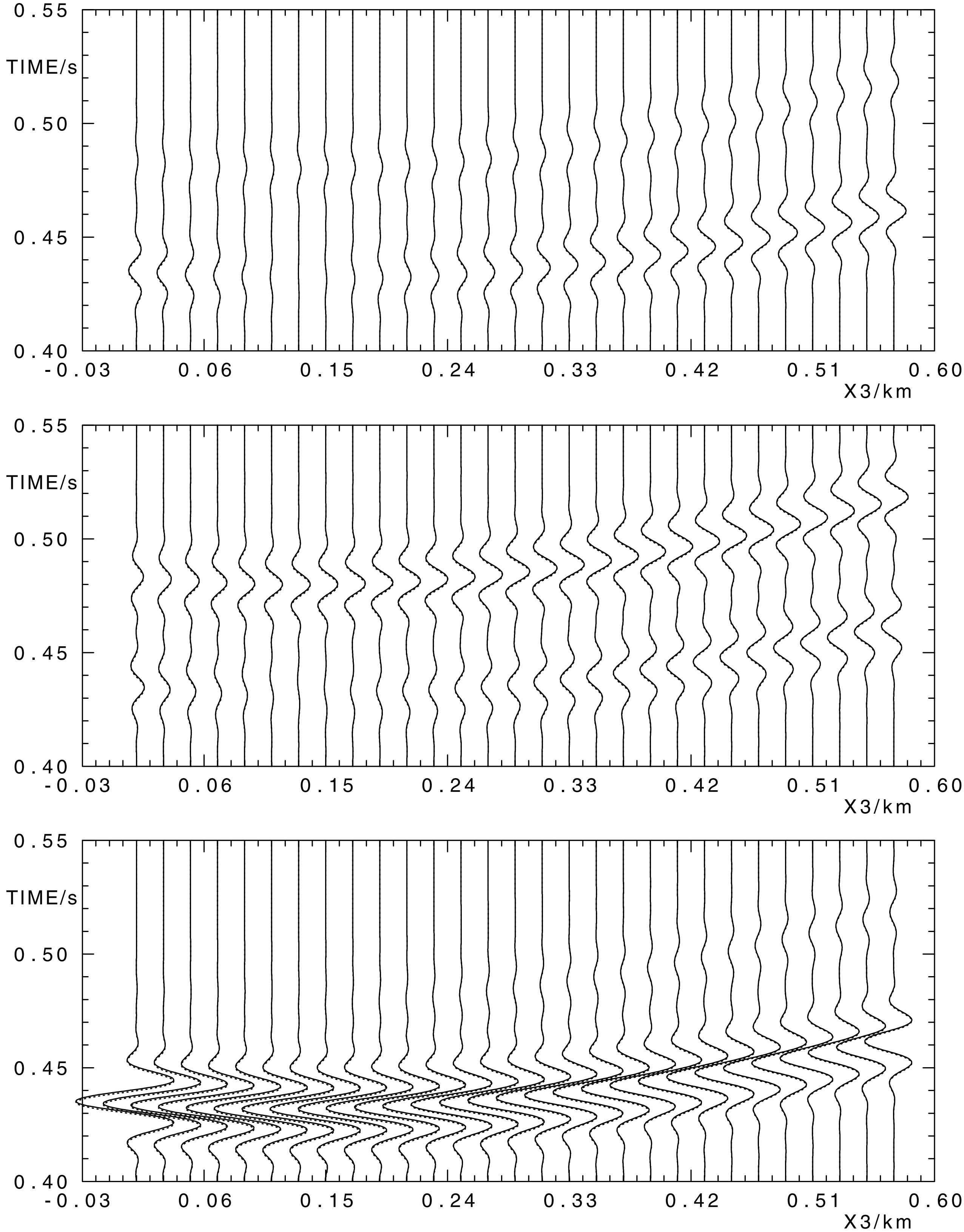

Figure 7. Inaccuracy of the anisotropic-common-ray approximation of the coupling ray theory in model QI4. From top to bottom: the first (radial) component, the second (transverse) component, the third (vertical) component. *Solid line*: Coupling-ray-theory synthetic seismograms without the common ray approximation, simulated by the second-order perturbation expansion of traveltime from the anisotropic common rays. For the errors of this simulation refer to the *ACR remaining terms* in Table 9. *Dotted line*: Anisotropic-common-ray approximation. The anisotropic-common-ray approximation in model QI4 is even more accurate than the isotropic-common-ray approximation in model QI2.

ACKNOWLEDGMENTS

The authors are grateful to Ivan Pšenčík, Mike Kendall and the anonymous reviewers, whose comments enabled the improvement of this paper.

The research has been supported by the Grant Agency of the Czech Republic under Contracts 205/04/1104 and 205/07/0032, by the Ministry of Education of the Czech Republic within Research Project MSM0021620860, by the European Commission under Contract MTKI-CT-2004-517242 (project IMAGES), and by the members of the consortium 'Seismic Waves in Complex 3-D Structures' (see 'http://sw3d.mff.cuni.cz').

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