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Perturbation expansions of complex-valued traveltime along real-valued reference rays

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SUMMARY

The eikonal equation in an attenuating medium has the form of a complex-valued Hamilton–Jacobi equation and must be solved in terms of the complex-valued traveltime (complex-valued action function). A very suitable approximate method for calculating the complex-valued traveltime right in real space is represented by the perturbation from the reference traveltime calculated along real-valued reference rays to the complex-valued traveltime defined by the complex-valued Hamilton–Jacobi equation.

The real-valued reference rays are calculated using the reference Hamiltonian function. The perturbation Hamiltonian function is parametrized by one or more perturbation parameters, and smoothly connects the reference Hamiltonian function with the Hamiltonian function corresponding to a given complex-valued Hamilton–Jacobi equation. Both the reference Hamiltonian function and the perturbation Hamiltonian function may be constructed in different ways, yielding differently accurate perturbation expansions of traveltime. All present perturbation methods use reference rays calculated in a reference anisotropic non-attenuating medium.

In this paper, the reference Hamiltonian function is constructed directly using the Hamiltonian function corresponding to a given complex-valued Hamilton–Jacobi equation, and the perturbation Hamiltonian function is linear with respect to the perturbation parameter. The direct construction of the reference Hamiltonian function from the given complex-valued Hamilton–Jacobi equation is very general and accurate, especially for homogeneous Hamiltonian functions of degree $N = -1$ with respect to the slowness vector.

Key words: Elasticity and anelasticity; Body waves; Seismic anisotropy; Seismic attenuation; Theoretical seismology; Wave propagation.

1 INTRODUCTION

Attenuation is a very important phenomenon in wave propagation, especially in the propagation of seismic or electromagnetic waves, and is essential whenever the intensity of waves matters.

The eikonal equation in an attenuating medium has the form of a complex-valued Hamilton–Jacobi equation and must be solved in terms of the complex-valued traveltime (complex-valued action function).

The solution of the complex-valued Hamilton–Jacobi equation for complex-valued traveltime by Hamilton's (1837) equations of rays would require complex-valued rays (complex-valued geodesics). Since the material properties are known in real space only, we cannot calculate complex-valued rays. We thus need to calculate the complex-valued traveltime right in real space. A very suitable approximate method for this purpose is represented by the perturbation from the reference traveltime calculated along real-valued reference rays to the complex-valued traveltime defined by the complex-valued Hamilton–Jacobi equation.

For this perturbation from the reference traveltime to the complex-valued traveltime, we need a complex-valued perturbation Hamiltonian function, i.e. a family of complex-valued Hamiltonian functions smoothly parametrized by one or more perturbation parameters. The perturbation Hamiltonian function must smoothly connect the reference Hamiltonian function with the Hamiltonian function corresponding to a given complex-valued Hamilton–Jacobi equation, and Hamilton's equations corresponding to the reference Hamiltonian function must yield real-valued reference rays. In order to be able to perform the perturbation from the reference traveltime to the complex-valued traveltime, we need the perturbation Hamiltonian function to be a holomorphic function of the complex slowness vector. This paper is devoted to the construction of the reference Hamiltonian function for a given complex-valued Hamilton–Jacobi equation, and to the construction of the corresponding complex-valued perturbation Hamiltonian function.

The perturbation Hamiltonian function may be constructed in different ways. Different forms of the perturbation Hamiltonian function yield different expressions for the perturbation expansion of traveltime of a given order, with considerably different accuracies. In this paper, we construct the reference Hamiltonian function directly using the Hamiltonian function corresponding to a given complex-valued Hamilton–Jacobi equation, whereas Červený *et al.* (2008) and Červený & Pšenčík (2009) considered the complex-valued Hamilton–Jacobi equation for traveltime in an anisotropic attenuating medium, selected a reference anisotropic non-attenuating medium, and then put the reference Hamiltonian function equal to the Hamiltonian function corresponding to the reference anisotropic non-attenuating medium. The direct construction of the reference Hamiltonian function from a given complex-valued Hamilton–Jacobi equation is more general and should usually be more accurate, which was numerically demonstrated by Vavryčuk (2009) in a special case of an isotropic attenuating medium.

In this paper, we propose to prefer homogeneous Hamiltonian functions of degree $N = -1$ with respect to the slowness vector. Homogeneous Hamiltonian functions of degree $N = -1$ usually yield the most accurate linear perturbations of traveltime, which was theoretically explained by Klimeš (2002, section 4.4), numerically demonstrated by Bulant & Klimeš (2008) in examples of perturbations from isotropic reference rays and common anisotropic reference rays in an anisotropic elastic medium, and also numerically demonstrated by Vavryčuk (2009) in examples of perturbations from real-valued reference rays to the complex-valued traveltime in two isotropic attenuating media.

When a perturbation Hamiltonian function is constructed, we can calculate the perturbation derivatives (derivatives with respect to perturbation parameters) of traveltime according to equations by Klimeš (2002), and construct the perturbation expansion (Taylor expansion with respect to perturbation parameters) of traveltime. For the calculation of the *n*th order perturbation derivatives of traveltime, we need the phase-space, perturbation and mixed derivatives of the perturbation Hamiltonian function up to the *n*th order at the real-valued reference rays. The perturbation derivatives of traveltime of all orders are calculated by simple numerical quadratures along unperturbed reference rays.

In Section 2, which represents the key part of the paper, we propose a general method for the construction of the reference Hamiltonian function for an arbitrary given complex-valued Hamilton–Jacobi equation, and for the construction of the corresponding complex-valued perturbation Hamiltonian function. The subsequent sections are devoted to the application of this general method. In Section 3, we review general Hamilton's equations for reference rays and the corresponding general Hamiltonian equations of geodesic deviation (dynamic ray tracing system), and apply them to the reference Hamiltonian function derived in Section 2. In Section 4, we review the general equations for the first-order and second-order perturbation derivatives of traveltime, and apply them to the perturbation Hamiltonian function derived in Section 2. In Section 5, we apply the final general equations of Sections 3 and 4 to the complex-valued eikonal equation (Hamilton–Jacobi equation) for viscoelastic waves propagating in an anisotropic attenuating medium. The final equations for the approximate calculation of the complex-valued traveltime of viscoelastic waves, propagating in an anisotropic attenuating medium, along real-valued reference rays can be found in Section 5.2.

Under phase space, we understand a spatial manifold parametrized by coordinates x^i with cotangent spaces parametrized by slowness-vector components p_i. We use the componential notation for vectors and matrices. For example, p_i stands for the covariant vector with components p_i. The Einstein summation over repetitive lower-case Roman indices is used throughout the paper. The summation does not apply to subscripts α corresponding to the derivatives with respect to the perturbation parameter.

2 CONSTRUCTION OF THE PERTURBATION HAMILTONIAN FUNCTION CORRESPONDING TO A GIVEN COMPLEX-VALUED HAMILTONIAN FUNCTION

The equations of Klimeš (2002) for perturbation expansions of traveltime are applicable to the complex-valued traveltime and complex slowness vector, if the complex-valued perturbation Hamiltonian function is a holomorphic function of the slowness vector. For the perturbation expansions of complex-valued traveltime along real-valued rays (geodesics), we thus need a holomorphic perturbation Hamiltonian function which yields real-valued reference rays.

2.1 Complex-valued Hamiltonian function

We consider the complex-valued Hamiltonian function

$$H(x^m, p_n) \tag{1}$$

of real spatial coordinates x^m and of complex slowness vector p_n. We assume that $H(x^m, p_n)$ is a holomorphic function of p_n in the domain of our interest. The corresponding Hamilton–Jacobi equation for complex-valued traveltime $\tau = \tau(x^k)$ reads

$$H(x^m, \tau_{,n}(x^k)) = C, \tag{2}$$

where

$$\tau_{,i}(x^k) = \frac{\partial}{\partial x^i}\tau(x^k). \tag{3}$$

The value of constant C is determined by the form and meaning of the Hamiltonian function.

2.2 Reference Hamiltonian function

In order to be able to perform the perturbation from the reference traveltime to the complex-valued traveltime, we need the reference Hamiltonian function $\widetilde{H}(x^m, p_n)$ to be a holomorphic function of the complex slowness vector.

We assume that the reference traveltime is real-valued. In order to obtain real-valued reference rays, the reference Hamiltonian function $\widetilde{H}(x^m, p_n)$ should take real values for real slowness vectors p_n. Since the reference Hamiltonian function $\widetilde{H}(x^m, p_n)$ should be as close to the given Hamiltonian function $H(x^m, p_n)$ as possible, we want $\widetilde{H}(x^m, p_n)$ to be equal to the real part $\mathrm{Re}\,[H(x^m, p_n)]$ for real p_n. Requiring that the reference Hamiltonian function $\widetilde{H}(x^m, p_n)$ should be a holomorphic function of the slowness vector then determines $\widetilde{H}(x^m, p_n)$ uniquely.

The reference Hamiltonian function may be constructed in the following way. We choose a real slowness vector p_n^0, and take the infinite Taylor expansion of the complex-valued Hamiltonian function $H(x^m, p_n)$ with respect to p_i at phase-space point (x^m, p_n^0),

$$H(x^m, p_n) = \sum_{\Omega=0}^{+\infty} \frac{1}{\Omega!} H^{,k_1k_2\ldots k_\Omega}\left(x^m, p_n^0\right) \left(p_{k_1} - p_{k_1}^0\right) \left(p_{k_2} - p_{k_2}^0\right) \ldots \left(p_{k_\Omega} - p_{k_\Omega}^0\right), \tag{4}$$

where

$$H^{,k_1k_2\ldots k_\Omega}(x^m, p_n) = \frac{\partial}{\partial p_{k_1}} \frac{\partial}{\partial p_{k_2}} \cdots \frac{\partial}{\partial p_{k_\Omega}} H(x^m, p_n). \tag{5}$$

The reference Hamiltonian function $\widetilde{H}(x^m, p_n)$ is obtained just by replacing the coefficients of the Taylor series by their real parts

$$\widetilde{H}(x^m, p_n) = \sum_{\Omega=0}^{+\infty} \frac{1}{\Omega!} \mathrm{Re}\left[H^{,k_1k_2\ldots k_\Omega}\left(x^m, p_n^0\right)\right] \left(p_{k_1} - p_{k_1}^0\right) \left(p_{k_2} - p_{k_2}^0\right) \ldots \left(p_{k_\Omega} - p_{k_\Omega}^0\right). \tag{6}$$

For real p_i, eq. (6) represents the real part of the Taylor expansion of $H(x^m, p_n)$ with respect to p_i at phase-space point (x^m, p_n^0), and its value is independent of the choice of real p_n^0 within the domain of convergence.

For each p_n^0, eq. (6) defines a holomorphic function of complex p_n in the domain of convergence. Since the values of functions (6) for two different p_n^0 are equal at real p_n from the intersection of the corresponding domains of convergence, they are also equal at complex p_n from the intersection of the corresponding domains of convergence. Taking the expansion at two different real points p_n^0 close to one another will thus yield the same holomorphic function $\widetilde{H}(x^m, p_n)$. Eq. (6) with proper choices of real vectors p_n^0 thus defines a holomorphic function of complex p_n in the vicinity of all real p_n. This holomorphic function coincides with $\mathrm{Re}\,[H(x^m, p_n)]$ at real p_n.

In order to secure the convergence of (6) in the vicinity of real p_n, we therefore equivalently express $\widetilde{H}(x^m, p_n)$ as a series at $p_n^0 = \mathrm{Re}(p_n)$ for every complex vector p_n sufficiently close to real one. This leads to the following more convenient form

$$\widetilde{H}(x^m, p_n) = \sum_{\Omega=0}^{+\infty} \frac{\mathrm{i}^\Omega}{\Omega!} \mathrm{Re}[H^{,k_1k_2\ldots k_\Omega}(x^m, \mathrm{Re}\, p_n)]\, \mathrm{Im}(p_{k_1})\, \mathrm{Im}(p_{k_2}) \ldots \mathrm{Im}(p_{k_\Omega}). \tag{7}$$

We shall assume that the modified Taylor series (7) is convergent in the domain of our interest. On this condition, the reference Hamiltonian function $\widetilde{H}(x^m, p_n)$ will be defined and will be a holomorphic function of complex p_n in the domain of our interest.

2.3 Perturbation Hamiltonian function

For a convenient perturbation from Hamiltonian function $\widetilde{H}(x^m, p_n)$ to Hamiltonian function $H(x^m, p_n)$, we define the one-parametric perturbation Hamiltonian function

$$H(x^m, p_n, \alpha) = \widetilde{H}(x^m, p_n) + [H(x^m, p_n) - \widetilde{H}(x^m, p_n)]\, \alpha, \tag{8}$$

linear with respect to perturbation parameter α.

For each value of α, Hamilton–Jacobi eq. (2) with the corresponding Hamiltonian function $H(x^m, p_n, \alpha)$ defines the complex-valued traveltime $\tau = \tau(x^k, \alpha)$.

We now calculate derivatives

$$H^{,k_1k_2\ldots k_\Omega}_{,j_1j_2\ldots j_\Phi\alpha\ldots\alpha}(x^m, p_n, \alpha) = \frac{\partial}{\partial x^{j_1}} \frac{\partial}{\partial x^{j_2}} \cdots \frac{\partial}{\partial x^{j_\Phi}} \frac{\partial}{\partial p_{k_1}} \frac{\partial}{\partial p_{k_2}} \cdots \frac{\partial}{\partial p_{k_\Omega}} \frac{\partial}{\partial \alpha} \cdots \frac{\partial}{\partial \alpha} H(x^m, p_n, \alpha) \tag{9}$$

of Hamiltonian function (8) at real p_m and $\alpha = 0$.

For real p_m, the perturbation Hamiltonian function and its derivatives with respect to complex p_m read

$$H(x^m, p_n, 0) = \mathrm{Re}[H(x^m, p_n)] \tag{10}$$

and

$$H^{,k_1k_2\ldots k_\Omega}(x^m, p_n, 0) = \mathrm{Re}[H^{,k_1k_2\ldots k_\Omega}(x^m, p_n)]. \tag{11}$$

For real p_m, the first-order perturbation derivative of the perturbation Hamiltonian function and of its derivatives with respect to complex p_m read

$$H_{,\alpha}(x^m, p_n, 0) = \mathrm{i}\, \mathrm{Im}[H(x^m, p_n)] \tag{12}$$

and

$$H_{,\alpha}^{,k_1k_2\ldots k_\Omega}(x^m, p_n, 0) = \mathrm{i}\,\mathrm{Im}[H^{,k_1k_2\ldots k_\Omega}(x^m, p_n)]. \tag{13}$$

The second-order and higher-order perturbation derivatives of the perturbation Hamiltonian function vanish,

$$H_{,\alpha\alpha}(x^m, p_n, 0) = H_{,\alpha\alpha\alpha}(x^m, p_n, 0) = \cdots = 0 \tag{14}$$

and

$$H_{,\alpha\alpha}^{,k_1k_2\ldots k_\Omega}(x^m, p_n, 0) = H_{,\alpha\alpha\alpha}^{,k_1k_2\ldots k_\Omega}(x^m, p_n, 0) = \cdots = 0. \tag{15}$$

The spatial derivatives and other mixed derivatives of perturbation Hamiltonian function (8) can be obtained by differentiating the above eqs (10)–(15) with respect to x^i.

These derivatives of the perturbation Hamiltonian function can be used for calculating the perturbation derivatives of traveltime $\tau(x^k, \alpha)$ (Klimeš 2002, eqs 19–21, 23).

We see that all phase-space derivatives of the complex-valued perturbation Hamiltonian function at $\alpha = 0$ are real-valued, which secures the reference rays and reference traveltime $\tau(x^k, 0)$ to be real-valued.

3 REFERENCE RAYS AND REFERENCE TRAVELTIME

3.1 Reference rays

Reference rays are determined by Hamilton's equations

$$\frac{\mathrm{d}x^i}{\mathrm{d}\gamma} = H^{,i}(x^m, p_n, 0), \tag{16}$$

$$\frac{\mathrm{d}p_i}{\mathrm{d}\gamma} = -H_{,i}(x^m, p_n, 0), \tag{17}$$

where the meaning of parameter γ along rays is determined by the form of the Hamiltonian function.

The initial conditions for p_n in eqs (16) and (17) are determined by the initial traveltime specified along the initial surface and by condition

$$H(x^m, p_n, 0) = C, \tag{18}$$

where constant C is defined by Hamilton–Jacobi eq. (2).

We insert relations (10) and (11) into the above equations. Hamilton's eqs (16) and (17) for reference rays then read

$$\frac{\mathrm{d}x^i}{\mathrm{d}\gamma} = \mathrm{Re}[H^{,i}(x^m, p_n)], \tag{19}$$

$$\frac{\mathrm{d}p_i}{\mathrm{d}\gamma} = -\mathrm{Re}[H_{,i}(x^m, p_n)]. \tag{20}$$

The initial conditions for p_n in eqs (19) and (20) are determined by the initial traveltime specified along the initial surface and by condition

$$\mathrm{Re}[H(x^m, p_n, 0)] = C \tag{21}$$

following from (18). Constant C is defined by Hamilton–Jacobi eq. (2).

3.2 Reference traveltime

Reference traveltime τ^0 is determined by equation

$$\frac{\mathrm{d}\tau^0}{\mathrm{d}\gamma} = p_i\, H^{,i}(x^m, p_n, 0). \tag{22}$$

We insert relation (11), and eq. (22) for reference traveltime τ^0 reads

$$\frac{\mathrm{d}\tau^0}{\mathrm{d}\gamma} = p_i\, \mathrm{Re}[H^{,i}(x^m, p_n)]. \tag{23}$$

3.3 Ray parameters and Hamiltonian equations of geodesic deviation

For calculating the second-order perturbation derivatives of traveltime, we need the second-order spatial derivatives of traveltime. These second-order spatial derivatives are most conveniently calculated using the Hamiltonian equations of geodesic deviation (dynamic ray tracing system) by Červený (1972).

In D-dimensional space, the initial conditions for reference rays, which start from the initial surface with the initial traveltime for Hamilton–Jacobi eq. (2), may be parametrized by $D-1$ ray parameters $\gamma_1, \gamma_2, \ldots, \gamma_{D-1}$. These $D-1$ ray parameters together with parameter $\gamma_D = \gamma$ along the rays form D ray coordinates.

We define the matrices of the partial derivatives of coordinates x^i and of slowness-vector components $p_i = \tau_{,i}$ with respect to the ray coordinates,

$$Q^i{}_a = \frac{\partial x^i}{\partial \gamma_a}, \tag{24}$$

$$P_{ia} = \frac{\partial \tau_{,i}}{\partial \gamma_a}, \tag{25}$$

where $\frac{\partial}{\partial \gamma_D}$ is identical to $\frac{\mathrm{d}}{\mathrm{d}\gamma}$ used in other equations. Paraxial matrices $Q^i{}_a$ and P_{ia} describe, by definition, the properties of the orthonomic system of rays corresponding to the traveltime under consideration. Let us emphasize that the definition of paraxial matrices $Q^i{}_a$ and P_{ia} depends on the kind of parameter $\gamma = \gamma_D$ along rays, which is in turn determined by the form of the Hamiltonian function.

The system

$$\frac{\mathrm{d}}{\mathrm{d}\gamma} Q^i{}_a = H^{\cdot i}_{,j}(x^q, p_r, 0)\, Q^j{}_a + H^{\cdot ij}(x^q, p_r, 0)\, P_{ja}, \tag{26}$$

$$\frac{\mathrm{d}}{\mathrm{d}\gamma} P_{ia} = -H_{,ij}(x^q, p_r, 0)\, Q^j{}_a - H^{\cdot j}_{,i}(x^q, p_r, 0)\, P_{ja} \tag{27}$$

of Hamiltonian equations of geodesic deviation (dynamic ray tracing system) of the reference rays can be obtained by differentiating Hamilton's eqs (16)–(17) with respect to γ_a (Červený 1972).

Equation

$$P_{ia} = \tau_{,ij}\, Q^j{}_a \tag{28}$$

is a direct consequence of definitions (24)–(25).

We insert relations (10) and (11) into the above equations. Hamiltonian eqs (26) and (27) of the geodesic deviation of the reference rays then read

$$\frac{\mathrm{d}}{\mathrm{d}\gamma} Q^i{}_a = \mathrm{Re}\left[H^{\cdot i}_{,j}(x^q, p_r)\right] Q^j{}_a + \mathrm{Re}[H^{\cdot ij}(x^q, p_r)]\, P_{ja}, \tag{29}$$

$$\frac{\mathrm{d}}{\mathrm{d}\gamma} P_{ia} = -\mathrm{Re}[H_{,ij}(x^q, p_r)]\, Q^j{}_a - \mathrm{Re}\left[H^{\cdot j}_{,i}(x^q, p_r)\right] P_{ja}. \tag{30}$$

4 PERTURBATION EXPANSION OF COMPLEX-VALUED TRAVELTIME

The perturbation expansion of complex-valued traveltime $\tau = \tau(x^k, \alpha)$ is its Taylor expansion

$$\tau(x^m, \alpha) \approx \tau(x^m, 0) + \tau_{,\alpha}(x^m, 0)\,\alpha + \tfrac{1}{2}\tau_{,\alpha\alpha}(x^m, 0)\,\alpha^2 + \tfrac{1}{6}\tau_{,\alpha\alpha\alpha}(x^m, 0)\,\alpha^3 + \cdots \tag{31}$$

with respect to perturbation parameter α. The Greek subscripts following a comma denote partial derivatives with respect to perturbation parameter α, here referred to as perturbation derivatives.

Since perturbation Hamiltonian function (8) yields the given Hamiltonian function (1) for $\alpha = 1$, perturbation expansion (31) yields the solution of Hamilton–Jacobi eq. (2) for $\alpha = 1$.

The zero-order term

$$\tau(x^m, 0) = \tau^0 \tag{32}$$

in perturbation expansion (31) is the real-valued reference traveltime determined by eq. (23).

Eqs (10)–(15) and their derivatives with respect to real x^i can be inserted into the equations of Klimeš (2002, eqs 19–21, 23) for calculating the perturbation derivatives of traveltime. As a consequence of this insertion, the even-order terms in the perturbation expansion (31) of traveltime are real-valued, and the odd-order terms in the perturbation expansion (31) of traveltime are purely imaginary.

We shall now explicitly express the equations for calculating the linear and quadratic terms in perturbation expansion (31) of traveltime.

4.1 First-order perturbation derivative of traveltime

The first-order perturbation derivative $\tau_{,\alpha}$ in the perturbation expansion (31) of traveltime is determined by equation

$$\frac{\mathrm{d}\tau_{,\alpha}}{\mathrm{d}\gamma} = -H_{,\alpha}(x^m, p_n, 0) \tag{33}$$

(Klimeš 2002, eq. 25).

We insert relation (12), and eq. (33) for the first-order perturbation derivative $\tau_{,\alpha}$ in the perturbation expansion (31) of traveltime reads

$$\frac{\mathrm{d}\tau_{,\alpha}}{\mathrm{d}\gamma} = -\mathrm{i}\,\mathrm{Im}[H(x^m, p_n)]. \tag{34}$$

The first-order term in the perturbation expansion (31) of traveltime is purely imaginary.

4.2 Second-order perturbation derivative of traveltime

To calculate the second-order perturbation derivative $\tau_{,\alpha\alpha}$ of traveltime, we need to calculate the second-order mixed derivatives $\tau_{,i\alpha}$ first.

The first-order perturbation derivative $\tau_{,i\alpha}$ of the spatial traveltime gradient is determined by equation

$$\tau_{,i\alpha}(x^m, 0) = T_{a\alpha}\, Q^{-1}_{ai} \tag{35}$$

(Klimeš 2002, eq. 20), where Q^{-1}_{ai} are the elements of the matrix inverse to matrix $Q^i{}_a$. The covariant derivatives $T_{a\alpha}$ of $\tau_{,\alpha}$ with respect to ray coordinates γ_a can be calculated using equation

$$\frac{\mathrm{d}T_{,a\alpha}}{\mathrm{d}\gamma} = -H_{,j\alpha}(x^m, p_n, 0)\; Q^j{}_a - H^{j}_{,\alpha}(x^m, p_n, 0)\; P_{ja} \tag{36}$$

(Klimeš 2002, eqs 19, 27). Since

$$T_{,D\alpha} = -H_{,\alpha}(x^m, p_n, 0) \tag{37}$$

(Klimeš 2002, eq. 28), the quadrature of eq. (36) is unnecessary for $a = D$.

The second-order perturbation derivative $\tau_{,\alpha\alpha}$ in the perturbation expansion (31) of traveltime is determined by equation

$$\frac{\mathrm{d}\tau_{,\alpha\alpha}}{\mathrm{d}\gamma} = -H_{,\alpha\alpha}(x^m, p_n, 0) - 2H^{\cdot j}_{,\alpha}(x^m, p_n, 0)\,\tau_{,j\alpha}(x^m, 0) - H^{\cdot jk}(x^m, p_n, 0)\,\tau_{,j\alpha}(x^m, 0)\,\tau_{,k\alpha}(x^m, 0) \tag{38}$$

(Klimeš 2002, eqs 19, 20, 30).

We insert relations (11)–(14) into eqs (36)–(38). Eqs (36) and (37) then read

$$\frac{\mathrm{d}T_{,a\alpha}}{\mathrm{d}\gamma} = -\mathrm{i}\,\mathrm{Im}[H_{,j}(x^m, p_n)]\; Q^j{}_a - \mathrm{i}\,\mathrm{Im}[H^{\cdot j}(x^m, p_n)]\; P_{ja} \tag{39}$$

and

$$T_{,D\alpha} = -\mathrm{i}\,\mathrm{Im}[H(x^m, p_n)]. \tag{40}$$

The first-order perturbation derivative $\tau_{,i\alpha}$ of the spatial traveltime gradient is then determined by eq. (35). We see that $T_{,i\alpha}$ and $\tau_{,i\alpha}$ are purely imaginary. Eq. (38) then reads

$$\frac{\mathrm{d}\tau_{,\alpha\alpha}}{\mathrm{d}\gamma} = -\,2\,\mathrm{i}\,\mathrm{Im}[H^{\cdot j}(x^m, p_n, 0)]\,\tau_{,j\alpha}(x^m, 0) - \,\mathrm{Re}[H^{\cdot jk}(x^m, p_n)]\,\tau_{,j\alpha}(x^m, 0)\,\tau_{,k\alpha}(x^m, 0). \tag{41}$$

The second-order term in the perturbation expansion (31) of traveltime is real-valued.

4.3 Transformation of the perturbation derivatives of traveltime at structural interfaces

The previous equations are applicable to Hamiltonian functions $H(x^m, p_n)$ which are sufficiently smooth (i.e. differentiable) with respect to real spatial coordinates x^m. If the Hamiltonian function is discontinuous at a smooth interface, all spatial and perturbation derivatives of traveltime may be transformed at the interface using the equations by Klimeš (2010).

5 APPLICATION TO VISCOELASTIC WAVES IN ANISOTROPIC ATTENUATING MEDIA

We now consider a time-harmonic wave, which propagates in an anisotropic attenuating medium described by the complex-valued density-reduced viscoelastic moduli $a_{ijkl}(x^m)$ specified in Cartesian coordinates x^m.

The density-reduced viscoelastic moduli $a_{ijkl}(x^m)$ obey symmetry relations

$$a_{ijkl}(x^m) = a_{jikl}(x^m) = a_{ijlk}(x^m). \tag{42}$$

We assume in this paper that the density-reduced viscoelastic moduli also obey symmetry relation

$$a_{ijkl}(x^m) = a_{klij}(x^m). \tag{43}$$

The eikonal equation (Hamilton–Jacobi equation) for the corresponding complex-valued traveltime reads

$$G(x^m, \tau_{,n}(x^k)) = 1, \tag{44}$$

where the selected eigenvalue $G(x^m, p_n)$ of the symmetric complex-valued Christoffel matrix

$$\Gamma_{ik}(x^m, p_n) = a_{ijkl}(x^m)\; p_j\; p_l \tag{45}$$

is a homogeneous function of degree 2 with respect to slowness vector p_n. The eigenvalue can also be expressed as

$$G(x^m, p_n) = a_{ijkl}(x^m)\, g_i\, p_j\, g_k\, p_l, \tag{46}$$

where $g_j = g_j(x^m, p_n)$ is the unit complex-valued eigenvector,

$$g_j\, g_j = 1, \tag{47}$$

corresponding to the eigenvalue.

The first-order phase-space derivatives of the eigenvalue read

$$G^{,i}(x^m, p_n) = 2\, a_{aikl}(x^m)\, g_a\, g_k\, p_l, \tag{48}$$

$$G_{,i}(x^m, p_n) = a_{ajkl,i}(x^m)\, g_a\, p_j\, g_k\, p_l. \tag{49}$$

For the calculation of the second-order phase-space derivatives $G^{,ij}$, $G^{,i}_{,j}$ and $G_{,ij}$ of the eigenvalue refer, for example, to Červený (2001, eq. 4.14.7) or to Klimeš (2006, eq. 23).

5.1 Homogeneous Hamiltonian function for anisotropic attenuating media

We choose the complex-valued Hamiltonian function $H(x^m, p_n)$ homogeneous of degree N with respect to slowness vector p_n,

$$H(x^m, p_n) = \frac{1}{N}\, [G(x^m, p_n)]^{\frac{N}{2}}, \tag{50}$$

where the selected eigenvalue $G(x^m, p_n)$ of the Christoffel matrix is given by (46), and is a homogeneous function of degree 2 with respect to slowness vector p_n. For this choice (50), the constant in Hamilton–Jacobi eq. (2) has the value

$$C = \frac{1}{N}. \tag{51}$$

For a homogeneous Hamiltonian function, parameter γ along the rays coincides with the reference traveltime determined by eq. (23),

$$\gamma = \tau^0, \tag{52}$$

and Hamilton's eqs (19)–(20) for reference rays read

$$\frac{\mathrm{d}x^i}{\mathrm{d}\tau^0} = \frac{1}{2}\, \mathrm{Re}\big\{[G(x^m, p_n)]^{\frac{N-2}{2}}\, G^{,i}(x^m, p_n)\big\}, \tag{53}$$

$$\frac{\mathrm{d}p_i}{\mathrm{d}\tau^0} = -\frac{1}{2}\, \mathrm{Re}\big\{[G(x^m, p_n)]^{\frac{N-2}{2}}\, G_{,i}(x^m, p_n)\big\}. \tag{54}$$

The initial conditions for eqs (53) and (54) should be chosen in such a way that

$$\mathrm{Re}[H(x^m, p_n)] = \frac{1}{N}. \tag{55}$$

Paraxial matrices $Q^i{}_a$ and P_{ia} of the reference rays can be calculated using the linear Hamiltonian eqs (29) and (30) of geodesic deviation, with the second-order phase-space derivatives $H^{,ij}$, $H^{,i}_{,j}$ and $H_{,ij}$ of the complex-valued Hamiltonian function (50) expressed according to Klimeš (2006, eq. 27).

Eq. (34) for the first-order perturbation derivative $\tau_{,\alpha}$ in the perturbation expansion (31) of traveltime reads

$$\frac{\mathrm{d}\tau_{,\alpha}}{\mathrm{d}\tau^0} = -\mathrm{i}\, \frac{1}{N}\, \mathrm{Im}\big\{[G(x^m, p_n)]^{\frac{N}{2}}\big\}. \tag{56}$$

The first-order perturbation derivative $\tau_{,i\alpha}$ of the spatial traveltime gradient is determined by eq. (35), with

$$\frac{\mathrm{d}T_{,a\alpha}}{\mathrm{d}\gamma} = -\frac{\mathrm{i}}{2}\, \mathrm{Im}\big\{[G(x^m, p_n)]^{\frac{N-2}{2}}\, G_{,j}(x^m, p_n)\big\}\, Q^j{}_a - \frac{\mathrm{i}}{2}\, \mathrm{Im}\big\{[G(x^m, p_n)]^{\frac{N-2}{2}}\, G^{,j}(x^m, p_n)\big\}\, P_{ja} \tag{57}$$

and

$$T_{,D\alpha} = -\mathrm{i}\, \frac{1}{N}\, \mathrm{Im}\big\{[G(x^m, p_n)]^{\frac{N}{2}}\big\}, \tag{58}$$

see (39) and (40). Note that $T_{,i\alpha}$ and $\tau_{,i\alpha}$ are purely imaginary. Eq. (41) then reads

$$\frac{\mathrm{d}\tau_{,\alpha\alpha}}{\mathrm{d}\gamma} = -\,\mathrm{i}\, \mathrm{Im}\big\{[G(x^m, p_n)]^{\frac{N-2}{2}}\, G^{,j}(x^m, p_n)\big\}\, \tau_{,j\alpha}(x^m, 0) - \mathrm{Re}[H^{,jk}(x^m, p_n)]\, \tau_{,j\alpha}(x^m, 0)\, \tau_{,k\alpha}(x^m, 0). \tag{59}$$

For the calculation of the second-order phase-space derivatives $H^{,ij}$ of complex-valued Hamiltonian function (50) refer, for example, to Klimeš (2006, eq. 27).

5.2 Homogeneous Hamiltonian function of degree −1 for anisotropic attenuating media

The most accurate linear perturbations of traveltime are usually obtained for the homogeneous Hamiltonian function of degree $N = -1$ with respect to the slowness vector (Klimeš 2002, section 4.4; Bulant & Klimeš 2008; Vavryčuk 2009).

For complex-valued Hamiltonian function $H(x^m, p_n)$ homogeneous of degree $N = -1$ with respect to slowness vector p_n, eq. (50) reads

$$H(x^m, p_n) = -[G(x^m, p_n)]^{-\frac{1}{2}}, \tag{60}$$

where the selected complex-valued eigenvalue $G(x^m, p_n)$ of the Christoffel matrix is given by (46). For this choice, the complex-valued traveltime satisfies Hamilton–Jacobi eq. (2) with constant

$$C = -1. \tag{61}$$

Hamilton's eqs (53)–(54) for the real-valued reference rays and reference traveltime τ^0 read

$$\frac{\mathrm{d}x^i}{\mathrm{d}\tau^0} = \frac{1}{2}\,\mathrm{Re}\big\{[G(x^m, p_n)]^{-\frac{3}{2}}\, G^{,i}(x^m, p_n)\big\}, \tag{62}$$

$$\frac{\mathrm{d}p_i}{\mathrm{d}\tau^0} = -\frac{1}{2}\,\mathrm{Re}\big\{[G(x^m, p_n)]^{-\frac{3}{2}}\, G_{,i}(x^m, p_n)\big\}, \tag{63}$$

where $G^{,i}(x^m, p_n)$ and $G_{,i}(x^m, p_n)$ are given by (48) and (49). The initial conditions for eqs (62) and (63) should satisfy condition

$$\mathrm{Re}\big\{[G(x^m, p_n)]^{-\frac{1}{2}}\big\} = 1. \tag{64}$$

Paraxial matrices $Q^i{}_a$ and P_{ia} of the reference rays can be calculated using linear Hamiltonian eqs (29) and (30) of geodesic deviation, with the second-order phase-space derivatives $H^{,ij}$, $H^{,i}_{,j}$ and $H_{,ij}$ of the complex-valued Hamiltonian function (60) expressed according to Klimeš (2006, eq. 27).

Eq. (56) for the first-order perturbation derivative $\tau_{,\alpha}$ in the perturbation expansion (31) of traveltime reads

$$\frac{\mathrm{d}\tau_{,\alpha}}{\mathrm{d}\tau^0} = \mathrm{i}\,\mathrm{Im}\big\{[G(x^m, p_n)]^{-\frac{1}{2}}\big\}. \tag{65}$$

The first-order perturbation derivative $\tau_{,i\alpha}$ of the spatial traveltime gradient is determined by eq. (35), with

$$\frac{\mathrm{d}T_{,a\alpha}}{\mathrm{d}\gamma} = -\frac{\mathrm{i}}{2}\,\mathrm{Im}\big\{[G(x^m, p_n)]^{-\frac{3}{2}}\, G_{,j}(x^m, p_n)\big\}\, Q^j_{\ a} - \frac{\mathrm{i}}{2}\,\mathrm{Im}\big\{[G(x^m, p_n)]^{-\frac{3}{2}}\, G^{,j}(x^m, p_n)\big\}\, P_{ja} \tag{66}$$

and

$$T_{,D\alpha} = \mathrm{i}\,\mathrm{Im}\big\{[G(x^m, p_n)]^{-\frac{1}{2}}\big\}, \tag{67}$$

see (57) and (58). Here D is the number of spatial dimensions, usually $D = 3$. Note that $T_{,i\alpha}$ and $\tau_{,i\alpha}$ are purely imaginary. Eq. (59) for the second-order perturbation derivative of traveltime, with $N = -1$, reads

$$\frac{\mathrm{d}\tau_{,\alpha\alpha}}{\mathrm{d}\gamma} = -\,\mathrm{i}\,\mathrm{Im}\big\{[G(x^m, p_n)]^{-\frac{3}{2}}\, G^{,j}(x^m, p_n)\big\}\, \tau_{,j\alpha}(x^m, 0) - \mathrm{Re}[H^{,jk}(x^m, p_n)]\, \tau_{,j\alpha}(x^m, 0)\, \tau_{,k\alpha}(x^m, 0). \tag{68}$$

For the calculation of the second-order phase-space derivatives $H^{,ij}$ of the complex-valued Hamiltonian function (60) refer, e.g. to Klimeš (2006, eq. 27).

5.3 Homogeneous Hamiltonian function of degree −1 for isotropic attenuating media

In an isotropic medium with complex-valued propagation velocity $v(x^m)$, the corresponding complex-valued eigenvalue of the Christoffel matrix reads

$$G(x^m, p_n) = [v(x^m)]^2\, p_k p_k. \tag{69}$$

Hamilton's eqs (62)–(63) for real-valued reference rays and reference traveltime τ^0 then read

$$\frac{\mathrm{d}x^i}{\mathrm{d}\tau^0} = \mathrm{Re}[[v(x^m)]^{-1}]\,(p_k p_k)^{-\frac{3}{2}}\, p_i, \tag{70}$$

$$\frac{\mathrm{d}p_i}{\mathrm{d}\tau^0} = -\mathrm{Re}[[v(x^m)]^{-2} v_{,i}(x^n)]\,(p_k p_k)^{-\frac{1}{2}}. \tag{71}$$

The initial conditions for eqs (70) and (71) should satisfy condition (64) which now reads

$$\mathrm{Re}[[v(x^m)]^{-1}]\,(p_k p_k)^{-\frac{1}{2}} = 1. \tag{72}$$

This condition is then satisfied along the whole reference rays, and Hamilton's eqs (70)–(71) read

$$\frac{\mathrm{d}x^i}{\mathrm{d}\tau^0} = \{\mathrm{Re}[[v(x^m)]^{-1}]\}^{-2}\, p_i, \tag{73}$$

$$\frac{\mathrm{d}p_i}{\mathrm{d}\tau^0} = -\{\mathrm{Re}[[v(x^m)]^{-1}]\}^{-1}\, \mathrm{Re}[[v(x^m)]^{-2} v_{,i}(x^n)]. \tag{74}$$

Eq. (65) for the first-order perturbation derivative $\tau_{,\alpha}$ in the perturbation expansion (31) of traveltime reads

$$\frac{\mathrm{d}\tau_{,\alpha}}{\mathrm{d}\tau^0} = \mathrm{i}\,\mathrm{Im}[[v(x^m)]^{-1}]\,(p_k p_k)^{-\frac{1}{2}}. \tag{75}$$

Since condition (72) is satisfied along the reference rays,

$$\frac{\mathrm{d}\tau_{,\alpha}}{\mathrm{d}\tau^0} = \mathrm{i}\,\mathrm{Im}[[v(x^m)]^{-1}]\,\{\mathrm{Re}[[v(x^m)]^{-1}]\}^{-1} = -\mathrm{i}\,\mathrm{Im}[v(x^m)]\,\{\mathrm{Re}[v(x^m)]\}^{-1}. \tag{76}$$

The first-order perturbation equations (73), (74) and (76) have already been proposed by Vavryčuk (2009, eq. 15). Vavryčuk (2009) numerically demonstrated that these equations yield considerably more accurate first-order perturbation expansions of complex-valued traveltime than the commonly used equations corresponding to the homogeneous Hamiltonian function of the second degree with respect to the slowness vector.

6 CONCLUSIONS

Experience with perturbations from real-valued reference rays to the complex-valued traveltime in isotropic media and experience with other applications of traveltime perturbations in anisotropic media suggest that the approach proposed in this paper is likely to be more accurate than the present approaches based on the reference rays calculated in a reference anisotropic non-attenuating medium.

The order of applied perturbation expansion and its accuracy will depend on details of the particular application and should be the responsibility of those who actually apply the method to a real-world case.

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