Tests of efficiency of two-point paraxial traveltime formula in inhomogeneous anisotropic media

Umair bin Waheed[1], Vlastislav Červený[2], Einar Iversen[3] & Ivan Pšenčík[4]

[1]*Physical Sciences and Engineering Division, KAUST, Saudi Arabia*
[2]*Charles University, Faculty of Mathematics and Physics, Department of Geophysics, Ke Karlovu 3, 121 16 Praha 2, Czech Republic*
[3]*Norsar, P.O. Box 53, 2027 Kjeller, Norway*
[4]*Institute of Geophysics, Acad.Sci. of Czech Republic, Boční II, 141 31 Praha 4, Czech Republic*

1 Introduction

We study efficiency and accuracy of the two-point paraxial traveltime formula (Červený, Iversen and Pšenčík, 2011). The formula can be used for the approximate determination of traveltimes between two points arbitrarily chosen in a paraxial vicinity of a single reference ray, along which ray propagator matrix in ray-centered coordinates has been determined. The reference ray can be traced in an arbitrary, laterally varying, layered medium of arbitrary anisotropy.

For testing purposes, we concentrate on computation of traveltimes of P waves propagating in smooth 2D models of inhomogeneous, isotropic or weakly anisotropic media. We start with tests on the isotropic model used for testing of another method (Alkhalifah and Fomel, 2010), and proceed to a weakly anisotropic model using first-order ray tracing approach (Pšenčík and Farra, 2005).

2 Two-point paraxial travetime formula

The two-point paraxial travetime formula (Červený et al., 2011) reads:

$$T(R',S') = T(R,S) + [x_i(R') - x_i(R)]p_i(R) - [x_i(S') - x_i(S)]p_i(S)$$

$$+\tfrac{1}{2}[x_i(R') - x_i(R)][f^R_{Mi}(\mathbf{P}^{(q)}_2\mathbf{Q}^{(q)-1}_2)_{MN}f^R_{Nj} + (p_i\eta_j + p_j\eta_i - p_ip_j\mathcal{U}_k\eta_k)_R][x_j(R') - x_j(R)]$$

$$+\tfrac{1}{2}[x_i(S') - x_i(S)][f^S_{Mi}(\mathbf{Q}^{(q)-1}_2\mathbf{Q}^{(q)}_1)_{MN}f^S_{Nj} - (p_i\eta_j + p_j\eta_i - p_ip_j\mathcal{U}_k\eta_k)_S][x_j(S') - x_j(S)]$$

$$-[x_i(S') - x_i(S)]f^S_{Mi}(\mathbf{Q}^{(q)-1}_2)_{MN}f^R_{Nj}[x_j(R') - x_j(R)].$$

The upper-case indices M, N take values 1 and 2, and the lower-case indices i, j take values 1,2 and 3. The Einstein summation convention is used. In the formula, $x_i(S)$ and $x_i(R)$ are Cartesian coordinates of two points, S and R, on the reference ray Ω; $x_i(S')$ and $x_i(R')$ are coordinates of points S' and R' situated in close vicinities of S and R, respectively. The symbols $\mathbf{Q}^{(q)}_1 = \mathbf{Q}^{(q)}_1(R,S)$, $\mathbf{Q}^{(q)}_2 = \mathbf{Q}^{(q)}_2(R,S)$, and $\mathbf{P}^{(q)}_2 = \mathbf{P}^{(q)}_2(R,S)$ denote 2×2 submatrices of the 4×4 ray propagator matrix in ray-centred coordinates calculated along Ω from S to R by dynamic ray tracing. The symbols f^S_{Mi} and f^R_{Mi} denote Cartesian components of vectors perpendicular to Ω at S and R, respectively. For their determination, a vectorial, ordinary differential equation must be solved along Ω. The symbols p_i, $\mathcal{U}_i$ and η_i denote Cartesian components of slowness vector, ray-velocity vector and the vector $\mathrm{d}\mathbf{p}(\tau)/\mathrm{d}\tau$, respectively, determined during tracing the reference ray. The indices S and R indicate if the corresponding quantities are considered at point S or R. For more details see Červený et al. (2011).

Once the reference ray Ω and the above-mentioned quantities calculated along it are available, two-point paraxial traveltimes can be calculated easily. Let us note that the above formula will fail to work properly if model parameters variation is too strong or if the matrix $\mathbf{Q}_2^{(q)}$ is singular at point R. The latter problem occurs when there is a caustic at the point R.

In the isotropic medium, exact ray tracing and dynamic ray tracing are used. In the weakly anisotropic medium, the first-order ray tracing approach (Pšenčík and Farra, 2005) is used. Along first-order rays, we perform first-order dynamic ray tracing in ray-centered coordinates. For the determination of two-point paraxial traveltime, we use the formulae given by Červený et al. (2011), in which we replace the exact quantities by their first-order counterparts, computed along Ω.

3 Test examples

As an illustration of the accuracy of the paraxial two-point traveltime formula, we present several results of tests performed, for simplicity, on 2D models of isotropic and weakly anisotropic (HTI), vertically inhomogeneous media. The models are covered by rectangular grids with 0.1 km separation in x and z direction. Receivers are situated at grid points.

Figure 1a illustrates efficiency of the above two-point traveltime formula for the case $S' \equiv S$ and points R' situated at all grid points. The points S (0,0) and R (1,1) are situated on the reference ray Ω (white curve). The isotropic model (Alkhalifah and Fomel, 2010) with P-wave velocity of 2 km/s at $z = 0$ km and constant vertical gradient of 0.7 s^{-1} is used. Figure 1a shows differences of paraxial and exact traveltimes. We can see that highly accurate results are obtained around the ray Ω between S and R, and around the wavefront passing through the point R. Figure 1b shows differences of paraxial and exact traveltimes in the same model as in Figure 1a, but for the source and receivers shifted by 0.2 km in the positive z direction so that the perturbed source is at (0,0). The plot can be compared with Figure 2b of Alkhalifah and Fomel (2010) obtained for the same model and configuration by an approach based on the use of the eikonal solver. Note that positive z-axis points down in our plot.

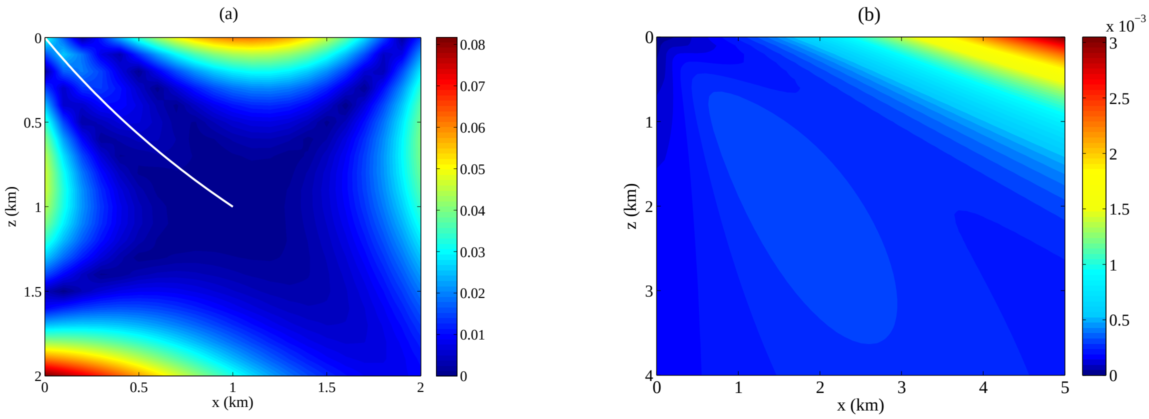

Figure 1: Differences of paraxial and exact two-point P-wave traveltimes for an isotropic model with velocity 2 km/s at $z = 0$ km and linear vertical gradient of 0.7 s^{-1}. (a) The paraxial traveltimes to all the grid points are computed from the point R (1,1) on the ray (white curve) connecting S (0,0) and R. (b) The source and receivers perturbed vertically by 0.2 km (cf. Figure 2b of Alkhalifah and Fomel, 2010). Perturbed source at (0,0).

By comparing Figure 1b with Figure 2b of Alkhalifah and Fomel (2010), we can conclude that the errors of the paraxial traveltime formula: a) are in most of the studied region by nearly one order lower (less than 0.001 s) than those of the eikonal-based approach, the highest errors (in the right upper corner) being around 0.003 s; b) vary negligibly in the whole studied region. CPU time necessary for computing paraxial two-point traveltimes in Figure 1b is about 0.004% of CPU time necessary for computing exact rays to all the grid points.

Figure 2 shows maps of traveltime errors for a larger shift of the source and receivers (a) and for a shift of receivers different from the source (b). In Figure 2a, the source and receivers were shifted in the direction of positive z by 0.8 km. In great part of studied region the traveltime error is around 0.05 s. Again, variation of the error is very small in great part of the region. In Figure 2b, the source is shifted vertically by 0.2 km and receivers by 0.4 km. We can observe larger errors in a close vicinity of the perturbed source (0,0). In substantial part of the model, the errors are well under 0.01 s.

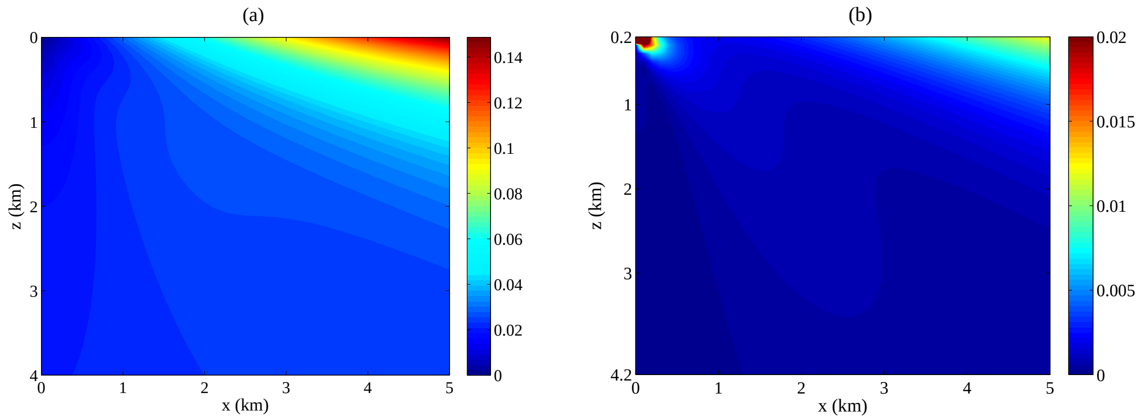

Figure 2: Differences of paraxial and exact two-point P-wave traveltimes as in Fig. 1b, but (a) with the source and receivers perturbed vertically by 0.8 km and (b) with source perturbed vertically by 0.2 km and receivers by 0.4 km in the same direction. The perturbed source at (0,0).

In Figure 3, we show results similar to those in Figures 1 and 2, but for the model of a transversely isotropic medium with horizontal axis of symmetry (HTI) varying linearly with depth ("twisted crystal" model). Density-normalized elastic moduli, measured in (km/s)^2, are obtained by linear variation of their values between depths $z = 0$ and $z = 3$ km. They are:

$$\begin{pmatrix} 14.485 & 4.525 & 4.755 & 0 & 0 & -0.58 \\ & 14.485 & 4.755 & 0 & 0 & -0.58 \\ & & 15.71 & 0 & 0 & -0.295 \\ & & & 5.155 & -0.175 & 0 \\ & & & & 5.155 & 0 \\ & & & & & 5.045 \end{pmatrix}$$

at $z = 0$ km and

$$\begin{pmatrix} 30.12750 & 10.0350 & 10.0350 & 0 & 0 & 0 \\ & 35.3475 & 11.3625 & 0 & 0 & 0 \\ & & 35.3475 & 0 & 0 & 0 \\ & & & 11.9925 & 0 & 0 \\ & & & & 11.2050 & 0 \\ & & & & & 11.2050 \end{pmatrix}$$

at $z = 3$ km. Figure 3a shows traveltime errors for the source and receivers shifted by 0.2 km in the direction of positive z. Figure 3b shows the same, but for the shift of 0.8 km. We can see that accuracy of the paraxial two-point traveltime formula is even higher in this case than in the isotropic model although the gradient is higher than in Figs 1 and 2. In contrast to the isotropic case, there are stronger variations of accuracy within the model. The CPU time necessary for computing paraxial two-point traveltimes in Figure 3 was about 0.00066% of CPU time necessary for computing exact rays to all the grid points.

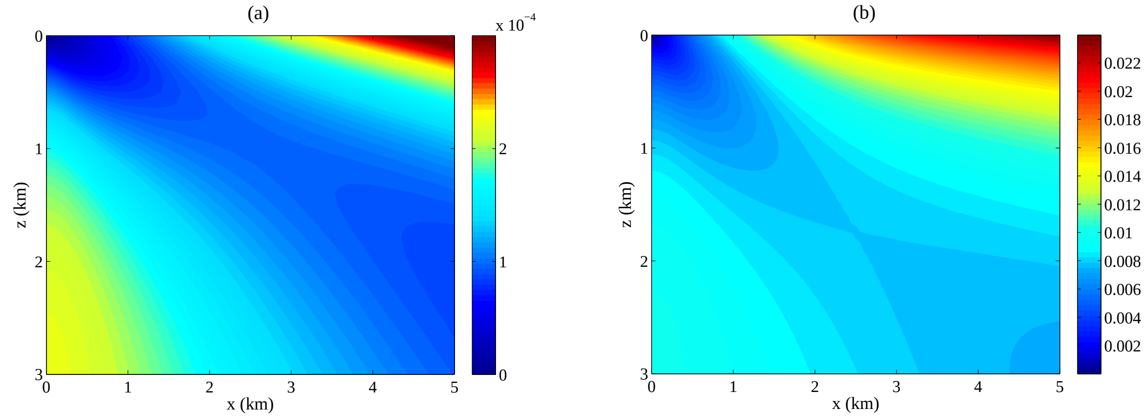

Figure 3: Differences of paraxial and exact two-point P-wave traveltimes for the HTI "twisted crystal" model with a linear vertical gradient. For details see the text. Source and receivers perturbed by (a) 0.2 km, (b) 0.8 km. The perturbed source at (0,0).

4 Concluding remarks

The above test examples are very simple. Nevertheless, they show great potential of the two-point paraxial traveltime formula. Its advantage, when compared, for example, with approaches based on most eikonal solvers, is that it provides travetimes related to energetic arrivals. As shown in Figure 2b, sources and receivers might be perturbed in a different way. The approach is expected not to work properly in strongly varying media, in which multipathing occurs. Generalization to more realistic models is straightforward. We are going to consider 2D or 3D models of laterally inhomogeneous isotropic or anisotropic media. We are also going to study cases, in which both the source and receiver positions are perturbed. Let us note that layered media can also be considered. It is only necessary to take into account effects of interfaces in ray tracing and dynamic ray tracing. Results of such tests will be presented at the 15th IWSA.

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References

Alkhalifah, T., and Fomel, S., 2010. An eikonal-based formulation for traveltime perturbation with respect to the source location. *Geophysics*, **75**, T175–T183.

Červený, V., Iversen, E., and Pšenčík, I., 2011. Two-point paraxial travel times in an inhomogeneous anisotropic medium. *Geophys.J.Int.*, submitted.

Pšenčík, I. and Farra, V., 2005. First-order ray tracing for qP waves in inhomogeneous weakly anisotropic media. *Geophysics*, **70**, D65-D75.