TWO-POINT PARAXIAL TRAVEL TIME APPROXIMATION

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We study efficiency and accuracy of the two-point paraxial travel time $T(R', S')$, see formula [1]. The formula was designed for the approximate determination of travel time between two points, S' and R', arbitrarily chosen in a paraxial vicinity of two points, S and R, on a reference ray Ω, between which the travel time $T(R, S)$ is known. The reference ray can be traced in an arbitrary, laterally varying, layered medium of arbitrary anisotropy. The formula offers an efficient, although approximate, replacement of more time consuming procedures (like, e.g., two-point ray tracing) to determine travel time between S' and R' from knowledge of quantities obtained by ray tracing and dynamic ray tracing between points S and R on the reference ray Ω.

The interest in the two-point travel time computations has a long history. It has been studied already by Hamilton in the early nineteenth century [2]. Hamilton derived equations for geodesics, which correspond to ray equations. He then also studied the travel time between two points, and called it *the characteristic function*. For more details see the contribution of Klimeš [3]. Klimeš extended Hamilton's treatment of the characteristic function by providing equations of geodesic deviations (in seismic literature known as dynamic ray tracing equations). He derived relations between the 6×6 ray propagator matrix of dynamic ray tracing in Cartesian coordinates and the second-order spatial derivatives of the Hamilton's characteristic function. These derivatives play a basic role in the computation of the two-point paraxial travel time $T(R', S')$. In [1], instead of the 6×6 ray propagator matrix of dynamic ray tracing in Cartesian coordinates, the 4×4 ray propagator matrix in ray-centered coordinates was used for the evaluation of the two-point paraxial travel time $T(R', S')$.

The two-point paraxial travel time formula, see [1], has the following form:

$$\begin{aligned} T(R', S') &= T(R, S) + [x_i(R') - x_i(R)]p_i(R) - [x_i(S') - x_i(S)]p_i(S) \\ &+ \tfrac{1}{2}[x_i(R') - x_i(R)][f^R_{Mi}(\mathbf{P}^{(q)}_2 \mathbf{Q}^{(q)-1}_2)_{MN} f^R_{Nj} + (p_i\eta_j + p_j\eta_i - p_i p_j \mathcal{U}_k \eta_k)_R][x_j(R') - x_j(R)] \\ &+ \tfrac{1}{2}[x_i(S') - x_i(S)][f^S_{Mi}(\mathbf{Q}^{(q)-1}_2 \mathbf{Q}^{(q)}_1)_{MN} f^S_{Nj} - (p_i\eta_j + p_j\eta_i - p_i p_j \mathcal{U}_k \eta_k)_S][x_j(S') - x_j(S)] \\ &- [x_i(S') - x_i(S)] f^S_{Mi}(\mathbf{Q}^{(q)-1}_2)_{MN} f^R_{Nj}[x_j(R') - x_j(R)] \; . \end{aligned} \tag{1}$$

The uppercase indices M, N take values 1 and 2, and the lowercase indices i, j take values 1,2 and 3. The Einstein summation convention is used. In (1), $x_i(S)$ and $x_i(R)$ are Cartesian coordinates of two points, S and R, on the reference ray Ω; $x_i(S')$ and $x_i(R')$ are Cartesian coordinates of points S' and R' situated in close vicinities of S and R, respectively. The symbols $\mathbf{Q}^{(q)}_1 = \mathbf{Q}^{(q)}_1(R, S)$, $\mathbf{Q}^{(q)}_2 = \mathbf{Q}^{(q)}_2(R, S)$, and $\mathbf{P}^{(q)}_2 = \mathbf{P}^{(q)}_2(R, S)$ denote 2×2 submatrices of the 4×4 ray propagator matrix in ray-centred coordinates calculated along Ω from S to R by dynamic ray tracing. The symbols f^S_{Mi} and f^R_{Mi} denote Cartesian components of vectors perpendicular to Ω at S and R, respectively. For their determination, a vectorial, ordinary differential equation must be solved along Ω. The symbols p_i, $\mathcal{U}_i$ and η_i denote Cartesian components of slowness vector, ray-velocity vector and the vector $\mathrm{d}\mathbf{p}(\tau)/\mathrm{d}\tau$, respectively, determined during tracing the reference ray Ω. The symbol τ denotes travel time along Ω. The indices S and R indicate if the corresponding quantities are considered at point S or R. For more details see [1].

Once the reference ray and the above-mentioned quantities calculated along it are available, two-point paraxial traveltimes can be evaluated easily. Let us note that the formula cannot work properly if model parameters variation is too strong or if the matrix $\mathbf{Q}_2^{(q)}$ is singular at point R. The latter problem occurs when there is a caustic at the point R.

As an illustration of the accuracy of the two-point paraxial traveltime formula, we present the result of a test performed in a 2D model of a vertically inhomogeneous isotropic medium, with P-wave velocity of 2 km/s at $z = 0$ km and constant vertical gradient of 0.7 s^{-1}. The model is covered by a rectangular grid with 0.1 km spacing in both x and z directions. Point S' coincides in this test with S, and both have coordinates (0,0), see Figure 1. Point R on the reference ray Ω (white curve) has coordinate (1,1). Points R' are distributed at grid points covering the whole model. Figure 1 shows differences of paraxial and exact travel times. We can see that highly accurate results are obtained around the ray Ω between S and R and behind R, and around the wavefront passing through the point R.

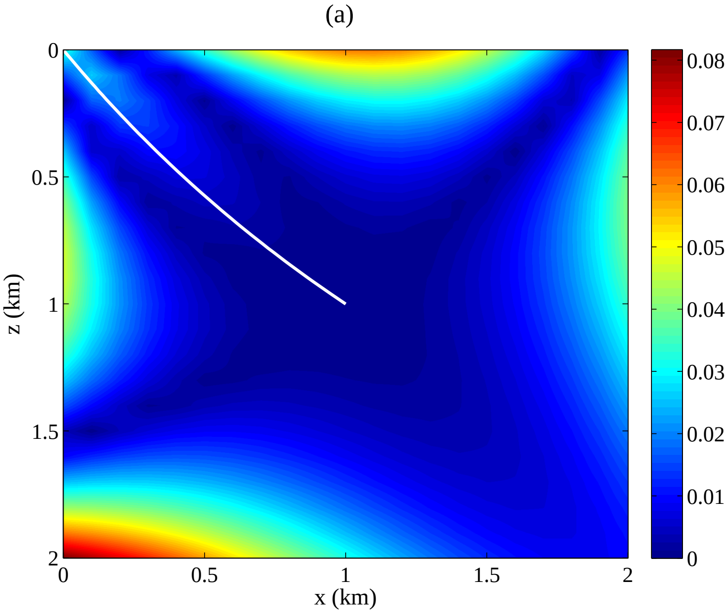

Figure 1: Differences of two-point P-wave paraxial and exact travel times computed from S (0,0) to grid points of the grid covering an isotropic model with velocity 2 km/s at $z = 0$ km and linear vertical gradient of 0.7 s^{-1}. The two-point paraxial travel times are computed from $T(R, S)$ and dynamic ray tracing quantities obtained along the reference ray (white curve) between S and R (1,1).

More details about the two-point paraxial travel time formula and tests of its efficiency on more general models will be given at the presentation.

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