Coupling ray series

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ABSTRACT

We propose a coupling ray series, which yields the coupling ray theory in a similar way as the standard ray series yields the standard ray–theory solution of the elastodynamic equation. We insert the coupling ray series into the elastodynamic equation and obtain the coupling ray theory as the zero–order high–frequency asymptotic approximation.

Keywords: elastic waves, anisotropy, heterogeneous media, ray series, coupling ray theory, travel time, amplitude, polarization

1. INTRODUCTION

There are two different standard zero–order high–frequency asymptotic ray theories for S waves in elastic media: the *isotropic ray theory* based on the assumption of equal velocities of both S waves, and the *anisotropic ray theory* assuming both S waves being decoupled. Note that here the term "different" means that the anisotropic ray theory is not a generalization of the isotropic ray theory and that both theories yield different S waves in the same "weakly anisotropic" velocity model.

In the isotropic ray theory, the S–wave polarization vectors do not rotate about the ray, whereas in the anisotropic ray theory they coincide with the eigenvectors of the Christoffel matrix, which may rotate rapidly about the ray. *Thomson et al. (1992)* demonstrated analytically that the high–frequency asymptotic error of the anisotropic–ray–theory wave field is inversely proportional to the second, or higher, root of the frequency, if the ray passes through the point of equal S–wave eigenvalues of the Christoffel matrix even in otherwise strongly anisotropic media.

In "weakly anisotropic" media, at moderate frequencies, the S–wave polarization tends to remain unrotated about the ray, but is partly attracted by the rotation of the eigenvectors of the Christoffel matrix. The intensity of the attraction increases with frequency. This behaviour of the S–wave polarization is described by the *coupling ray theory* proposed by *Coates and Chapman (1990)*. The frequency–dependent coupling ray theory is the generalization of both the zero–order isotropic and anisotropic ray theories, and provides continuous transition between them. The coupling ray theory is applicable to S waves for all degrees of anisotropy, including isotropic media.

S–wave coupling was first studied in the quasi–isotropic approximation, in which the S wave in a weakly anisotropic medium was approximately composed of two S waves calculated in a reference isotropic medium. The quasi–isotropic approximation for electromagnetic waves in a weakly anisotropic 3–D heterogeneous medium was proposed by *Kravtsov (1968)*. The quasi–isotropic approximation was applied to elastic waves in a weakly anisotropic 3–D heterogeneous medium, e.g., by *Naida (1977)*, *Sharafutdinov (1994)*, *Kravtsov et al. (1996)*, *Pšenčík (1998)*, *Zillmer et al. (1998)*, *Pšenčík and Dellinger (2001)*, *Červený (2001, Sec. 5.4.6.4)* and *Chapman (2004, Sec. 10.2.2.4)*. Since the S waves in an anisotropic medium are polarized in a different plane than in an isotropic medium, the quasi–isotropic approximation considerably decreases the accuracy of the coupling ray theory, especially by mixing S waves with a P wave. Various errors of the quasi–isotropic approximation were discussed and demonstrated by *Bulant and Klimeš (2002; 2004)*, *Bulant et al. (2004)*, *Klimeš (2004)*, and *Klimeš and Bulant (2004)*.

The coupling ray theory has been demonstrated to be considerably more accurate than the standard isotropic or anisotropic ray theory in many numerical examples (e.g., *Zillmer et al., 1998; Pšenčík and Dellinger, 2001; Bulant et al., 2004; Červený et al., 2007; Pšenčík et al., 2012*). The accuracy of the coupling ray theory has been tested by comparisons with exact analytical solutions (e.g., *Bulant et al., 2004; Bulant and Klimeš, 2004*) or with other, more accurate, numerical methods (e.g., *Bulant et al., 2011; Pšenčík et al., 2012*).

Klimeš and Bulant (2012) demonstrated that, within a limited frequency band, the frequency–domain coupling–ray–theory S–wave Green tensor may efficiently and accurately be approximated by two Green tensors corresponding to two S waves, described by their coupling–ray–theory travel times and the corresponding vectorial amplitudes, calculated for the prevailing frequency. The coupling ray theory can thus efficiently increase accuracy in all applications where we currently use the standard ray theory to calculate S waves.

Both the isotropic ray theory and the anisotropic ray theory can be derived as the zero–order terms of the standard ray series applied to the elastodynamic equation, which was demonstrated by *Babich (1956)* and *Karal and Keller (1959)* in isotropic media, and by *Babich (1961)* and *Červený (1972)* in anisotropic media.

In this paper, we propose the coupling ray series, which yields the coupling ray theory in a similar way as the standard ray series yields the standard ray–theory solution of the elastodynamic equation. We insert the coupling ray series into the elastodynamic equation and obtain the coupling ray theory as the zero–order high–frequency asymptotic approximation. It is obvious from our derivation, which terms have been neglected in deriving the commonly used coupling equation, and in which ways it can be improved.

The Einstein summation over repetitive lower–case Roman subscripts corresponding to the three spatial coordinates is used throughout the paper. The Einstein summation does not apply to the upper–case Roman superscripts denoting the wave polarizations.

2. STANDARD RAY SERIES

The elastodynamic equation for displacement u_i in the frequency domain reads

$$\varrho^{-1}(\varrho\, a_{ijkl} u_{k,l})_{,j} + \omega^2 u_i = 0 \quad , \tag{1}$$

where

$$a_{ijkl} = \varrho^{-1} c_{ijkl} \tag{2}$$

are the density–normalized elastic moduli.

For each elementary wave J, where $J = 1$ or $J = 2$ for S waves and $J = 3$ for a P wave, we express the displacement in terms of travel time τ^J and vectorial amplitude

$$U_i^J = \sum_{n=0}^{\infty} (\mathrm{i}\omega)^{-n}\, U_i^{J\,[n]} \quad . \tag{3}$$

The standard anisotropic ray series and isotropic ray series for elementary wave J reads

$$u_i = \sum_{n=0}^{\infty} (\mathrm{i}\omega)^{-n}\, U_i^{J\,[n]} \exp\bigl(\mathrm{i}\omega\tau^J\bigr) \tag{4}$$

(*Červený, 2001, Eq. 5.7.1* but the opposite sign of the odd–order terms).

Inserting series (4) into elastodynamic equation (1), we obtain the elastodynamic equation in the form

$$\sum_{n=0}^{\infty} \Bigl[(\mathrm{i}\omega)^{2-n}\, N_i\bigl(U_k^{J\,[n]}, \tau^J_{,l}\bigr) + (\mathrm{i}\omega)^{1-n}\, M_i\bigl(U_k^{J\,[n]}, \tau^J_{,l}\bigr) + (\mathrm{i}\omega)^{-n}\, L_i\bigl(U_k^{J\,[n]}\bigr) \Bigr] = 0 \quad . \tag{5}$$

Operators N_i, M_i, L_i are defined by relations

$$N_i\bigl(U_m, \tau^J_{,n}\bigr) = a_{ijkl}\tau^J_{,j}\tau^J_{,l} U_k - U_i \quad , \tag{6}$$

$$M_i\bigl(U_m, \tau^J_{,n}\bigr) = \varrho^{-1}\bigl(\varrho\, a_{ijkl}\tau^J_{,l} U_k\bigr)_{,j} + a_{ijkl}\tau^J_{,j} U_{k,l} \quad , \tag{7}$$

$$L_i(U_m) = \varrho^{-1}(\varrho\, a_{ijkl} U_{k,l})_{,j} \tag{8}$$

(*Červený, 2001, Eq. 2.4.41* but density normalized).

Sorting series (5) according to the order of ω, we obtain the system of equations (*Červený, 2001, Eq. 5.7.3*)

$$N_i\bigl(U_k^{J\,[n]}, \tau^J_{,l}\bigr) + M_i\bigl(U_k^{J\,[n-1]}, \tau^J_{,l}\bigr) + L_i\bigl(U_k^{J\,[n-2]}\bigr) = 0 \quad , \quad n = 0, 1, 2, \ldots \quad , \tag{9}$$

where $U_k^{J\,[-1]} = 0$ and $U_k^{J\,[-2]} = 0$, i.e., operator M_i is missing in Eq. (9) for $n = 0$ and operator L_i is missing in Eq. (9) for $n = 0, 1$.

2.1. Anisotropic Ray Theory

Equation (9) for $n = 0$ constitutes the matrix Christoffel equation

$$\Gamma_{ik}\, U_k^{J[0]} - U_i^{J[0]} = 0 \quad , \tag{10}$$

where

$$\Gamma_{ik} = a_{ijkl}\, \tau^J_{,j} \tau^J_{,l} \tag{11}$$

is the Christoffel matrix.

We select unit eigenvector g_i^J and the corresponding eigenvalue G^J of Γ_{ik}. Travel time τ^J is the solution of eikonal equation

$$G^J\big(x_m, \tau^J_{,n}\big) = 1 \quad , \tag{12}$$

where $G^J(x_m, \tau^J_{,n})$ is the eigenvalue of Christoffel matrix $\Gamma_{ik}(x_m, \tau^J_{,n})$ corresponding to eigenvector $g_i^J(x_m, \tau^J_{,n})$. In general, we denote by g_i^{KJ} the three eigenvectors and by G^{KJ} the corresponding eigenvalues of Christoffel matrix $\Gamma_{ik}(x_m, \tau^J_{,n})$. In this notation, $g_i^J = g_i^{JJ}$ and $G^J = G^{JJ}$.

The zero–order vectorial amplitude then reads

$$U_i^{J[0]} = A^J\, g_i^J \quad , \tag{13}$$

where the zero–order ray–theory amplitude A^J is determined by the transport equation.

To obtain the transport equation, we multiply the three equations (9) for $n = 1$ from the left by eigenvector g_i^J. Since $g_i^J N_i(U_k^{J[n]}, \tau^J_{,l}) = 0$, the transport equation for the zero–order ray–theory amplitude A^J reads

$$g_i^J\, M_i\big(A^J g_k^J, \tau^J_{,l}\big) = 0 \quad . \tag{14}$$

Inserting Eq. (7) into Eq. (14), the transport equation becomes

$$\big[\varrho\, g_i^J a_{ijkl} \tau^J_{,l}\, g_k^J \big(A^J\big)^2\big]_{,j} = 0 \quad . \tag{15}$$

For $n > 0$, we define amplitude components $U^{KJ[n]}$ with respect to the eigenvectors of the Christoffel matrix (*Červený, 2001, Eq. 5.7.9*):

$$U_i^{J[n]} = \sum_K U^{KJ[n]}\, g_i^{KJ} \quad . \tag{16}$$

We decompose each higher–order vectorial amplitude $U_i^{J[n]}$, $n > 0$, into principal component $U^{JJ[n]}$ and two additional components $U^{KJ[n]}$, $K \neq J$:

$$U_i^{J[n]} = U^{JJ[n]}\, g_i^J + \sum_{K \neq J} U^{KJ[n]}\, g_i^{KJ} \quad . \tag{17}$$

One of the three equations (9) for $n = 1$ is transport equation (14) for amplitude $A^J = U^{JJ[0]}$. The two other equations (9) for $n = 1$,

$$g_i^{KJ}\big[N_i\big(U_k^{J[1]}, \tau^J_{,l}\big) + M_i\big(U_k^{J[0]}, \tau^J_{,l}\big)\big] = 0 \quad , \tag{18}$$

can be satisfied by additional components $U^{KJ[1]}$, $K \neq J$. Since

$$N_i\big(g_m^{KJ}, \tau_{,n}^J\big) = \big(G^{KJ} - 1\big)\, g_i^{KJ} \quad , \tag{19}$$

Eq. (18) yields

$$U^{KJ[1]} = -\frac{g_i^{KJ} M_i(A^J g_m^J, \tau_{,n}^J)}{G^{KJ} - 1} \tag{20}$$

(*Červený, 2001, Eq. 5.7.14*).

There are additional components in the first–order solution, but there is no coupling between the S waves in the zero–order solution. The coupling between S waves can be calculated as the series composed of approximate higher–order additional components, but this series often does not converge or converges slowly. *Vavryčuk (1999)* demonstrated the convergence on a numerical example of two S waves propagating vertically in a weakly transversely isotropic medium with the axis of symmetry rotating in the horizontal plane.

2.2. Isotropic Ray Theory

The anisotropic ray theory is applicable fully to P waves in isotropic media. There is no difference between the anisotropic and isotropic ray theories for P waves in isotropic media.

In an isotropic medium, which may also serve as an approximation to a weakly anisotropic medium, the eigenvalues of the Christoffel matrix corresponding to S waves are equal,

$$G^1 = G^2 \quad . \tag{21}$$

Thus, the travel times are equal,

$$\tau^1 = \tau^2 \quad , \tag{22}$$

and amplitude A^2 is proportional to amplitude A^1. The vectorial amplitude of S waves is expressed as

$$U_i^{1[0]} + U_i^{2[0]} = \big(r^{1[0]}\, g_i^1 + r^{2[0]}\, g_i^2\big)\, A^1 \quad , \tag{23}$$

and we obtain two equations

$$g_i^K \sum_{J=1}^{2} M_i\big(r^{J[0]} A^1 g_k^J, \tau_{,l}^1\big) = 0 \quad , \tag{24}$$

which represent the transport equation for A^1 and the decoupling equation for the 2–D unit vector $(r^{1[0]}, r^{2[0]})$. The solution of the decoupling equation guarantees that the S–wave polarization vector

$$e_i = r^{1[0]}\, g_i^1 + r^{2[0]}\, g_i^2 \tag{25}$$

does not rotate about the ray.

In the isotropic ray theory, for each $n > 0$, we have two principal S–wave amplitude components $U^{1J[n]}$ and $U^{2J[n]}$ and one additional component $U^{3J[n]}$.

An isotropic medium may also serve as an approximation for a weakly anisotropic medium. In a weakly anisotropic medium, we may apply two different standard high–frequency asymptotic ray theories for S waves: the isotropic ray theory based on the assumption of equal velocities of both S waves, and the anisotropic ray theory assuming both S waves to be decoupled. In the isotropic ray theory, the S–wave polarization vectors do not rotate about the ray, whereas in the anisotropic ray theory they coincide with the eigenvectors of the Christoffel matrix, which may rotate rapidly about the ray.

3. COUPLING RAY SERIES

We aim to propose the coupling ray series, which yields the coupling ray theory in a similar way as the standard ray series yields the standard ray–theory solution of the elastodynamic equation. We follow *Coates and Chapman (1990, Eq. 15)*, who proposed the coupling ray theory, in which the S wave in a weakly anisotropic medium was composed of two S waves calculated by the anisotropic ray theory.

In this paper, we consider the coupling ray series composed of the principal amplitude components only, without additional components. This series is sufficient for deriving the coupling ray theory for waves propagating with similar velocities. On the other hand, this series does not yield the additional components of the standard ray theory for waves propagating with considerably different velocities. Using this series, the additional components can be calculated as the series composed of the solutions of the coupling equations, which may converge slowly, see *Klimeš (2007, Sec. 4.3)*.

We define the coupling ray series

$$u_i = \sum_{n=0}^{\infty} \sum_{J} (\mathrm{i}\omega)^{-n} \, U_i^{J\,[n]} \exp\!\left(\mathrm{i}\omega\tau^J\right) \quad , \tag{26}$$

where

$$U_i^{J\,[n]} = r^{J\,[n]} \, A^J \, g_i^J \quad , \tag{27}$$

i.e.,

$$u_i = \sum_{n=0}^{\infty} \sum_{J} (\mathrm{i}\omega)^{-n} \, r^{J\,[n]} \, A^J \, g_i^J \, \exp\!\left(\mathrm{i}\omega\tau^J\right) \quad . \tag{28}$$

Here travel time τ^J is the solution of eikonal equation (12) corresponding to unit eigenvector g_i^J, and the zero–order ray–theory amplitude A^J is determined by transport equation (14). Coupling ray series (28) is applicable to the coupling of all 3 elementary waves, with $J = 1, 2, 3$. Since the coupling between the P wave and S waves is usually weak, the coupling is often restricted to S waves only, with $J = 1, 2$.

Unfortunately, coupling ray series (28) is not applicable if eigenvectors g_i^J corresponding to different elementary waves are close to parallel, which makes coupling impossible. In this special case, a rough common–ray approximation of the coupling ray theory, which approximates eigenvectors g_i^J by the unperturbed eigenvectors corresponding to the common S–wave reference ray, may yield much better results. These unperturbed eigenvectors are considerably inaccurate but mutually perpendicular.

The Christoffel equation is satisfied by all terms of the coupling ray series,

$$N_i\big(r^{J\,[n]}A^J g_k^J,\tau_{,l}^J\big)=0\quad . \tag{29}$$

Inserting coupling ray series (28) into elastodynamic equation (1), we obtain the elastodynamic equation in the form

$$\sum_{n=0}^{\infty}\sum_J\Big[(\mathrm{i}\omega)^{1-n}\,M_i\big(r^{J\,[n]}A^J g_k^J,\tau_{,l}^J\big)+(\mathrm{i}\omega)^{-n}\,L_i\big(r^{J\,[n]}A^J g_k^J\big)\Big]\exp\big(\mathrm{i}\omega\tau^J\big)=0\quad . \tag{30}$$

4. COUPLING EQUATIONS

Considering definition (7), we express

$$M_i\big(r^{J\,[n]}A^J g_m^J,\tau_{,n}^J\big)=r^{J\,[n]}\,M_i\big(A^J g_m^J,\tau_{,n}^J\big)+(a_{ijkl}+a_{ilkj})\,g_k^J\,r_{,j}^{J\,[n]}\,\tau_{,l}^J\,A^J\,. \tag{31}$$

We wish to calculate $r^{J\,[n]}$ along a *reference ray* corresponding to *reference travel time* $\widetilde{\tau}^J$. For example, the reference travel time and ray may coincide with anisotropic–ray–theory travel time τ^J and its corresponding ray, but most frequently, we use common S–wave reference travel time $\widetilde{\tau}$ and the common S–wave reference ray (*Bakker, 2002; Klimeš, 2006; Klimeš and Bulant, 2006*). Note that the isotropic–ray–theory reference ray, traced in a reference isotropic medium approximating the anisotropic medium, considerably reduces the accuracy of the coupling ray theory (*Klimeš and Bulant, 2004; Bulant and Klimeš, 2008*).

Of spatial derivatives $r_{,j}^{J\,[n]}$, we can adjust by the coupling equation only derivative

$$r^{J\,[n]\prime}=\frac{\mathrm{d}r^{J\,[n]}}{\mathrm{d}\widetilde{\tau}^J}=r_{,j}^{J\,[n]}\,\frac{\mathrm{d}x^j}{\mathrm{d}\widetilde{\tau}^J} \tag{32}$$

in the direction of the reference ray. We thus decompose spatial derivatives $r_{,j}^{J\,[n]}$ into component

$$\widetilde{\tau}_{,j}^J r^{J\,[n]\prime} \tag{33}$$

perpendicular to the reference wavefront, and component

$$r_{,j}^{J\,[n]\perp}=r_{,j}^{J\,[n]}-\widetilde{\tau}_{,j}^J r^{J\,[n]\prime} \tag{34}$$

perpendicular to the reference ray.

The part of Eq. (31) related to component $r_{,j}^{J\,[n]\perp}$ reads

$$R_i^J\big(r^{J\,[n]}\big)=(a_{ijkl}+a_{ilkj})\,g_k^J\left(r_{,j}^{J\,[n]}-\widetilde{\tau}_{,j}^J\frac{\mathrm{d}x^s}{\mathrm{d}\widetilde{\tau}^J}r_{,s}^{J\,[n]}\right)\tau_{,l}^J\,A^J\quad . \tag{35}$$

Equation (31) with $r^{J[n]}_{,j}$ decomposed into components (33) and (34) reads

$$M_i\big(r^{J[n]}A^J g^J_m,\tau^J_{,n}\big) = r^{J[n]}\,M_i\big(A^J g^J_m,\tau^J_{,n}\big) + (a_{ijkl}+a_{ilkj})\,g^J_k\,\widetilde{\tau}^J_{,j}\,\tau^J_{,l}\,A^J\,r^{J[n]\prime} + R^J_i\big(r^{J[n]}\big) \quad . \tag{36}$$

We shall multiply elastodynamic equation (30) from the left by eigenvector g^K_i, and express it in terms of matrices

$$B^{KJ} = \tfrac{1}{2}\,g^K_i\,(a_{ijkl}+a_{ilkj})\,\widetilde{\tau}^J_{,j}g^J_k\,\tau^J_{,l} \quad , \tag{37}$$

$$C^{KJ} = -\tfrac{1}{2}\,g^K_i\,M_i\big(A^J g^J_m,\tau^J_{,n}\big)\,(A^J)^{-1} \quad , \tag{38}$$

and

$$D^K\big(r^{J[n-1]}\big) = -\tfrac{1}{2}\,g^K_i\left[\mathrm{i}\omega\,R^J_i\big(r^{J[n-1]}\big) + L_i\big(r^{J[n-1]}A^J g^J_m,\tau^J_{,n}\big)\right]\big(A^J\big)^{-1} \quad . \tag{39}$$

Note that if $\widetilde{\tau}^J_{,j} = \tau^J_{,j}$, then $B^{KJ} = g^K_i g^J_i$. The diagonal elements of matrix (38) satisfy identity

$$C^{JJ} = 0 \tag{40}$$

following from transport equation (14).

Elastodynamic equation (30) multiplied from the left by eigenvector g^K_i then reads

$$\sum_{n=0}^{\infty}\sum_J\Big[(\mathrm{i}\omega)^{1-n}\,\big(B^{KJ}\,r^{J[n]\prime} - C^{KJ}\,r^{J[n]}\big) - (\mathrm{i}\omega)^{-n}\,D^K\big(r^{J[n]}\big)\Big]A^J\exp\big(\mathrm{i}\omega\tau^J\big) = 0 \quad . \tag{41}$$

Equation (41) is satisfied if we choose

$$\sum_J\Big[B^{KJ}\,r^{J[n]\prime} - C^{KJ}\,r^{J[n]} - D^K\big(r^{J[n-1]}\big)\Big]A^J\exp\big(\mathrm{i}\omega\tau^J\big) = 0 \quad , \qquad n=0,1,2,\dots \quad , \tag{42}$$

where $r^{J[n-1]} = 0$, i.e., operator D^K is missing in Eq. (42) for $n=0$. From Eq. (42), we calculate

$$\boxed{r^{K[n]\prime} = \sum_L\sum_J \exp\big[\mathrm{i}\omega(\tau^J-\tau^K)\big]\,(A^K)^{-1}A^J\,(B^{-1})^{KL}\big[C^{LJ}\,r^{J[n]} + D^L\big(r^{J[n-1]}\big)\big] \quad ,} \tag{43}$$

where $(B^{-1})^{KJ}$ are the elements of the matrix inverse to B^{KJ}.

Coupling equation (43) represents one of the various forms of the coupling equation corresponding to coupling ray series (28), without any approximation. If we constrain matrices B^{LJ} and C^{LJ} in coupling equation (43) to be frequency–independent, this coupling equation with matrices (37)–(39) would probably yield the most accurate zero–order coupling–ray–theory solution with respect to the constraint.

5. INCREASING THE ACCURACY OF THE ZERO–ORDER COUPLING RAY THEORY

It is possible to slightly enhance the accuracy of coupling equation (43), proposed in the preceding section, for the zero–order solution. However, we do not know yet how much the improvements proposed in this section will increase the accuracy of the zero–order solution. That is why we left the basic version of coupling equation (43) without the following improvements.

Considering definition (8), we express

$$\begin{aligned} L_i\big(r^{J[n]}U^J_m\big) &= L_i\big(U^J_m\big)\, r^{J[n]} \\ &+ \big[(a_{ijkl}+a_{ilkj})\, U^J_{k,j} + \varrho^{-1}(\varrho\, a_{ijkl})_{,j}\, U^J_k\big]\, r^{J[n]}_{,l} + a_{ijkl}\, U^J_k\, r^{J[n]}_{,jl} \quad , \end{aligned} \tag{44}$$

where

$$U^J_m = A^J\, g^J_m \quad . \tag{45}$$

In Eq. (44), we decompose derivative $r^{J[n]}_{,l}$ into components (33) and (34):

$$r^{J[n]}_{,j} = \widetilde{\tau}^J_{,j} r^{J[n]\prime} + r^{J[n]\perp}_{,j} \quad . \tag{46}$$

In elastodynamic equation (41), we move the terms with $r^{J[n]\prime}$ from expression $D^K\big(r^{J[n]}\big)$ to expression $B^{KJ}r^{J[n]\prime}$, move the terms with $r^{J[n]}$ from expression $D^K\big(r^{J[n]}\big)$ to expression $C^{KJ}r^{J[n]}$, and leave only the terms with $r^{J[n]\perp}_{,j}$ and $r^{J[n]}_{,jl}$ in expression $D^K\big(r^{J[n]}\big)$.

The new matrices (37)–(39), used in coupling equation (43), then read

$$\begin{aligned} B^{KJ} &= \tfrac{1}{2}\, g^K_i\, (a_{ijkl}+a_{ilkj})\, \widetilde{\tau}^J_{,j} \big[g^J_k\, \tau^J_{,l} + (\mathrm{i}\omega A^J)^{-1}(g^J_k A^J)_{,l}\big] \\ &+ \tfrac{1}{2}\, g^K_i\, (\mathrm{i}\omega\varrho)^{-1}(\varrho\, a_{ijkl})_{,j}\, \widetilde{\tau}^J_{,l}\, g^J_k \quad , \end{aligned} \tag{47}$$

$$C^{KJ} = -\tfrac{1}{2}\, g^K_i \big[M_i\big(A^J g^J_m, \tau^J_{,n}\big) + (\mathrm{i}\omega)^{-1} L_i\big(A^J g^J_m, \tau^J_{,n}\big)\big]\, (A^J)^{-1} \quad , \tag{48}$$

and

$$\begin{aligned} D^K\big(r^{J[n-1]}\big) &= -\tfrac{1}{2}\, g^K_i \Big\{(a_{ijkl}+a_{ilkj})\, \big[\mathrm{i}\omega\, \tau^J_{,j}\, g^J_k + (g^J_k A^J)_{,j}\, (A^J)^{-1}\big]\, r^{J[n-1]\perp}_{,l} \\ &+ \varrho^{-1}(\varrho\, a_{ijkl})_{,j}\, g^J_k\, r^{J[n-1]\perp}_{,l} + a_{ijkl}\, g^J_k\, r^{J[n-1]}_{,jl} \Big\} \quad . \end{aligned} \tag{49}$$

In contrast with matrix (38), the diagonal elements of matrix (48) do not satisfy identity (40),

$$C^{JJ} \neq 0 \quad . \tag{50}$$

Note that, in coupling equation (43), we may consider all modifications (47)–(49) of matrices (37)–(39), or only some of them, which provides us a considerable amount of freedom in choosing various forms of the coupling equation of different accuracies.

6. DECREASING THE ACCURACY OF THE ZERO–ORDER COUPLING RAY THEORY

Although the accuracy of coupling equation (43) can be enhanced with respect to the basic version proposed in the Section 4, it is common to decrease the accuracy of the coupling equation for the zero–order solution by various simplifications of matrix (38).

Equation (7) with $U_m = A^J g^J_m$ yields

$$M_i\big(A^J g^J_m, \tau^J_{,n}\big) = \varrho^{-1}\big(\varrho\, a_{ijkl}\tau^J_{,l}\big)_{,j} g^J_k A^J + (a_{ijkl}+a_{ilkj})\,\tau^J_{,j}\big[g^J_k A^J_{,l} + g^J_{k,l}A^J\big] \quad . \tag{51}$$

We may remove some part $\Delta M_i(A^J g^J_m, \tau^J_{,n})$ from $M_i(A^J g^J_m, \tau^J_{,n})$ in definition (38) for $K \neq J$. Then

$$C^{KJ} \;\longrightarrow\; C^{KJ} + \tfrac{1}{2}\, g^K_i\, \Delta M_i\big(A^J g^J_m, \tau^J_{,n}\big)\,\big(A^J\big)^{-1} \tag{52}$$

and

$$D^K\big(r^{J[n-1]}\big) \;\longrightarrow\; D^K\big(r^{J[n-1]}\big) - \tfrac{1}{2}\,\mathrm{i}\omega\, g^K_i\, \Delta M_i\big(A^J g^J_m, \tau^J_{,n}\big)\,\big(A^J\big)^{-1} r^{J[n-1]} \quad . \tag{53}$$

The most important phenomenon for the coupling ray theory is the rotation of the eigenvector of the Christoffel matrix about the reference ray. The derivative of the eigenvector of the Christoffel matrix in the direction of the reference ray is

$$g^{J\prime}_k = \frac{\mathrm{d}g^J_k}{\mathrm{d}\widetilde{\tau}^J} = g^J_{k,j}\frac{\mathrm{d}x^j}{\mathrm{d}\widetilde{\tau}^J} \quad . \tag{54}$$

We thus decompose spatial derivatives $g^J_{k,j}$ into the component perpendicular to the reference ray and component

$$\widetilde{\tau}^J_{,l}\, g^{J\prime}_k \tag{55}$$

perpendicular to the reference wavefront. If we remove nearly the whole of expression $M_i(A^J g^J_m, \tau^J_{,n})$, and leave only the part with the component (55) of derivative $g^J_{k,l}$ perpendicular to the reference wavefront,

$$\Delta M_i\big(A^J g^J_m, \tau^J_{,n}\big) = M_i\big(A^J g^J_m, \tau^J_{,n}\big) - (a_{ijkl}+a_{ilkj})\,\tau^J_{,j}\,\widetilde{\tau}^J_{,l}\, g^{J\prime}_k\, A^J \quad , \tag{56}$$

we obtain, for $L \neq J$,

$$C^{LJ} = -\tfrac{1}{2}\, g^L_i\,(a_{ijkl}+a_{ilkj})\,\tau^J_{,j}\,\widetilde{\tau}^J_{,l}\, g^{J\prime}_k \quad , \tag{57}$$

and coupling equation (43) for the zero–order solution becomes

$$r^{K[0]\prime} = \sum_L \sum_J \exp\big[\mathrm{i}\omega\big(\tau^J - \tau^K\big)\big]\,(A^K)^{-1} A^J\,(B^{-1})^{KL}\, C^{LJ}\, r^{J[0]} \tag{58}$$

with matrix B^{LJ} given by Eq. (37) and matrix C^{LJ} given by approximation (57). We leave the diagonal elements of matrix C^{LJ} zero, see identity (40).

7. ZERO–ORDER PERTURBATION APPROXIMATION ALONG THE COMMON REFERENCE RAY

Although the coupling equation with matrix (57) is less accurate than the coupling equation with matrix (38), it is common to decrease its accuracy further by means of the first–order or second–order perturbation approximation of travel times and the zero–order perturbation approximation of other quantities.

We shall consider S waves only, with the common S–wave reference ray and common S–wave reference travel time $\widetilde{\tau}$. Then

$$\widetilde{\tau}^J_{,i} = \widetilde{\tau}_{,i} \quad , \qquad J = 1, 2 \quad . \tag{59}$$

The zero–order perturbation approximation corresponds to the reference quantities. We consider the reference Christoffel matrix calculated for the reference slowness vector and determine its reference eigenvalues $\widetilde{G}^J$ and reference eigenvectors $\widetilde{g}^J_i$. We leave the reference slowness vector unperturbed,

$$\tau^J_{,i} \approx \widetilde{\tau}_{,i} \quad , \tag{60}$$

and the reference eigenvectors likewise unperturbed,

$$g^J_i \approx \widetilde{g}^J_i \quad . \tag{61}$$

We also leave the reference amplitude unperturbed,

$$A^J \approx \widetilde{A} \quad . \tag{62}$$

Matrix (37) then reads

$$B^{KJ} \approx \delta^{KJ} \, \widetilde{G}^J \quad , \tag{63}$$

and matrix (57) becomes

$$C^{LJ} \approx -\widetilde{G}^L \, \widetilde{g}^L_k \widetilde{g}^{J\prime}_k \quad . \tag{64}$$

Then

$$\sum_L (B^{-1})^{KL} \, C^{LJ} \approx -\widetilde{g}^K_k \, \widetilde{g}^{J\prime}_k \quad . \tag{65}$$

We insert approximations (62) and (65) into coupling equation (58) and obtain the simplified coupling equation

$$r^{K[0]\prime} \approx -\sum_J \exp\big[\mathrm{i}\omega\big(\tau^J - \tau^K\big)\big] \, \widetilde{g}^K_k \, \widetilde{g}^{J\prime}_k \, r^{J[0]} \quad , \tag{66}$$

which is identical to equations (30) and (31) of *Coates and Chapman (1990)*. Here $J, K = 1, 2$.

Note that all numerical applications of the coupling ray theory up to the present are based on approximate coupling equation (66) or on its further approximations.

8. DISCUSSION AND CONCLUSIONS

The standard anisotropic ray series fails in the description of S–wave coupling. We have proposed coupling ray series (28) which yields the coupling ray theory. The coupling ray series is applicable to coupling of all 3 elementary waves. Since the coupling between the P wave and S waves is usually weak, the coupling ray series is often restricted to S waves only. If the coupling ray series converges and we consider coupling of all 3 elementary waves, the coupling ray series satisfying coupling equation (43) with consistently selected coefficient matrices B^{LJ} and C^{LJ} and operator D^L yields the exact solution. If we consider coupling of S waves only, the additional components in the direction of P–wave polarization are missing.

There is a variety of options how to select coefficient matrices B^{LJ} and C^{LJ} of coupling equation (43). Different options yield zero–order coupling–ray–theory solutions of different accuracies and may influence the convergence of the coupling ray series.

We have demonstrated that the commonly used coupling equation (66) is the low–order perturbation approximation of coupling equation (43). It is obvious, which terms have been neglected in deriving the commonly used coupling equation (66).

The proposed coupling ray series (28) is not applicable if eigenvectors g_i^J corresponding to different elementary waves are close to parallel, which makes coupling impossible. In this special case, the rough common–ray approximation of the coupling ray theory described in Section 7, which approximates eigenvectors g_i^J by unperturbed eigenvectors $\widetilde{g}_i^J$ corresponding to the common S–wave reference ray, may yield much better results. Unperturbed eigenvectors $\widetilde{g}_i^J$ are considerably inaccurate but mutually perpendicular.

The proposed coupling ray series (28) is composed of the principal amplitude components only, without additional components. This series does not yield the additional components of the standard ray theory for waves propagating with considerably different velocities. The additional components can be calculated as the series composed of the solutions of the coupling equations as demonstrated by *Klimeš (2007, Sec. 4.3)*, but this series may converge slowly. To generalize both the standard ray theory and the coupling ray theory at the same time, we may wish to propose a more general coupling ray series with additional components, see *Klimeš (2007, Sec. 5)*.

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