Ray Series for Electromagnetic Waves in Static Heterogeneous Bianisotropic Dielectric Media

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Abstract — **We consider generally bianisotropic dielectric media. We consider the linear constitutive relations for bianisotropic media in the Boys–Post representation without spatial dispersion. We propose the high–frequency asymptotic ray series in terms of the magnetic vector potential. For the sake of simplicity, we assume that the media are static (do not change with time). In this case we can work in frequency domain, apply 3–D spatial rays, and avoid 4–D space–time rays. We assume that the media are so smoothly heterogeneous that we can apply the high–frequency ray–theory approximation. We assume the Weyl gauge (zero electric potential), which is best suited for electromagnetic wave fields.**

We derive the Hamiltonian function which specifies the rays and travel time. We then derive the transport equations for the zero–order and higher–order vectorial amplitudes.

I. Introduction

We consider linear dielectric media which are generally bianisotropic. We consider the linear constitutive relations for bianisotropic media in the Boys–Post representation without spatial dispersion (Lakhtakia, 2000; Post, 2003; Weiglhofer, 2003; Strunc, 2007). The Boys–Post representation $\mathbf{D} = \mathbf{D}(\mathbf{E}, \mathbf{B})$, $\mathbf{H} = \mathbf{H}(\mathbf{E}, \mathbf{B})$ of the constitutive relations is more natural than the Tellegen representation $\mathbf{D} = \mathbf{D}(\mathbf{E}, \mathbf{H})$, $\mathbf{B} = \mathbf{B}(\mathbf{E}, \mathbf{H})$, and is best suited for the formulation in terms of the magnetic vector potential $\mathbf{A}$. For the sake of simplicity, we assume that the media are static (do not change with time). In this case we can work in frequency domain, apply 3–D spatial rays, and avoid 4–D space–time rays. We assume that the media are so smoothly heterogeneous that we can apply the high–frequency ray–theory approximation.

There are two electromagnetic waves propagating in bianisotropic media. They differ by their polarizations. The ray theory described in this paper is applicable if these two waves are strictly decoupled and propagate with sufficiently different velocities. Whenever it is reasonable to emphasize this property, we may refer to the ray theory described in this paper as the *anisotropic ray theory*. If the velocities of the waves are not sufficiently different, the two wave polarizations are coupled (Kravtsov, 1969), and we have to replace the anisotropic ray theory by the *coupling ray theory* which will be described elsewhere. The zero–order coupling ray theory represents a generalization of the zero–order anisotropic ray theory and relies on most results of the standard anisotropic ray theory described in this paper.

Whereas the ray methods for electromagnetic waves have traditionally been expressed in terms of the electric field strength vector $\mathbf{E}$ and magnetic induction vector $\mathbf{B}$ or magnetic field strength vector $\mathbf{H}$ (Luneburg, 1944), we shall develop the ray methods in terms of the magnetic vector potential $\mathbf{A}$, which is simpler and more advantageous while inducing no drawbacks. The few authors who already considered ray methods in terms of the magnetic vector potential usually assumed the Lorenz gauge, which is best suited for vacuum and well applicable to electromagnetic fields with electrostatic components. Instead of the Lorenz gauge or the Coulomb gauge, we shall assume the Weyl gauge (zero electric potential), which is best suited for electromagnetic wave fields without significant electrostatic components. The Weyl gauge reduces the electromagnetic field variables to just 3 components of the magnetic vector potential $\mathbf{A}$ which is then parallel with the electric field strength vector $\mathbf{E}$ in the frequency domain. This reduction of ray methods from 6 components of the electric field strength vector $\mathbf{E}$ and magnetic induction vector $\mathbf{B}$ to just 3 components of the magnetic vector potential $\mathbf{A}$ represents a great advantage for both theory and numerical methods while inducing no drawbacks for the study of electromagnetic wave propagation.

The lower–case Roman indices take values 1, 2 and 3. The lower–case Greek indices take values 1, 2, 3 and 4. The Einstein summation over repetitive indices is used throughout the paper.

II. Maxwell Equations with Constitutive Relations

A. Time–domain Maxwell Equations with Constitutive Relations

Maxwell equations

$$\varepsilon^{ijk} E_{k,j} + B^i_{,4} = 0 \quad , \qquad B^k_{,k} = 0 \tag{1}$$

for *electric field strength* $E_j = E_j(x^m, x^4)$ and *magnetic induction* $B^j = B^j(x^m, x^4)$ are satisfied if we put

$$E_k = A_{4,k} - A_{k,4} \quad , \qquad B^k = \varepsilon^{klm} A_{m,l} \quad , \tag{2}$$

where $A_i = A_i(x^m, x^4)$ is the *magnetic vector potential* and $A_4 = A_4(x^m, x^4) = -\varphi(x^m, x^4)$ represents the *minus*

 978-1-5090-2501-5/16/$31.00 ©2016 IEEE

electric potential. The Maxwell equations for *electric displacement* $D^j = D^j(x^m, x^4)$ and *magnetic field strength* $H_j = H_j(x^m, x^4)$ read

$$\varepsilon^{ijk} H_{k,j} - D^i_{,4} = J^i \quad , \qquad D^k_{,k} = J^4 \quad , \tag{3}$$

where $J^4 = J^4(x^m, x^4) = \rho(x^m, x^4)$ represents the *electric charge density*, and $J^i = J^i(x^m, x^4)$ is the *electric current density*.

We thus need the *constitutive relations* which express the mutual dependence between the above mentioned quantities E^k, B^k, D^j, H_j, J^j and J^4.

In this paper, we consider *dielectric media* in which the electric current density and electric charge density vanish outside the source region,

$$J^\gamma = 0 \quad , \tag{4}$$

and 4–vector J^γ represents just the source term.

We assume the constitutive relations in the *Boys–Post representation* which express the dependence of the electric displacement D^j and magnetic field strength H_j on electric field strength E_j and magnetic induction B^j. In this paper, we consider just the *linear* constitutive relations in the Boys–Post representation.

The point constitutive relations without any dispersion can be expressed as (Weiglhofer, 2000, eq. 1.12; 2003, eqs. 57–58)

$$D^i = \varepsilon^{ij} E_j + \alpha^i{}_j B^j \quad , \quad H_i = \beta_i{}^j E_j + \mu^{-1}_{ij} B^j \quad . \tag{5}$$

We insert constitutive relations (5) into Maxwell equations (3),

$$\varepsilon^{ijk}(\beta_k{}^l E_l + \mu^{-1}_{kl} B^l)_{,j} - (\varepsilon^{ij} E_j + \alpha^i{}_j B^j)_{,4} = J^i \quad , \tag{6}$$

$$(\varepsilon^{ij} E_j + \alpha^i{}_j B^j)_{,i} = J^4 \quad . \tag{7}$$

We insert expressions (2) into Maxwell equations (6) and (7),

$$\varepsilon^{ijk}[\beta_k{}^l(A_{4,l} - A_{l,4}) + \mu^{-1}_{kl}\varepsilon^{lmn} A_{n,m}]_{,j} - [\varepsilon^{ij}(A_{4,j} - A_{j,4}) + \alpha^i{}_j \varepsilon^{jlm} A_{m,l}]_{,4} = J^i \quad , \tag{8}$$

$$[\varepsilon^{ij}(A_{4,j} - A_{j,4}) + \alpha^i{}_j \varepsilon^{jlm} A_{m,l}]_{,i} = J^4 \quad . \tag{9}$$

We define *constitutive tensor* $\chi^{\alpha\beta\gamma\delta}$ by relations

$$\chi^{4i4j} = -\chi^{i44j} = -\chi^{4ij4} = \chi^{i4j4} = -\varepsilon^{ik} \quad , \tag{10}$$

$$\chi^{ij4k} = -\chi^{ijk4} = \varepsilon^{ijr}\beta_r{}^k \quad , \tag{11}$$

$$\chi^{4ikl} = -\chi^{i4kl} = -\alpha^i{}_s \varepsilon^{skl} \quad , \tag{12}$$

$$\chi^{ijkl} = \varepsilon^{ijr}\mu^{-1}_{rs}\varepsilon^{skl} \quad . \tag{13}$$

The constitutive tensor is skew with respect to its first pair of superscripts and its last pair of superscripts,

$$\chi^{\alpha\beta\gamma\delta} = -\chi^{\beta\alpha\gamma\delta} \quad , \qquad \chi^{\alpha\beta\gamma\delta} = -\chi^{\alpha\beta\delta\gamma} \quad , \tag{14}$$

and has 36 distinct components.

Maxwell equations (8) and (9) with linear constitutive relations in the Boys–Post representation then read

$$(\chi^{\alpha\beta\gamma\delta} A_{\delta,\gamma})_{,\beta} = J^\alpha \tag{15}$$

(Post, 2003, eq. 26). Differentiating these Maxwell equations, we obtain the continuity equation

$$J^\alpha_{,\alpha} = 0 \quad . \tag{16}$$

If the initial conditions for the Maxwell equations satisfy the fourth Maxwell equation and the continuity equation is satisfied, the fourth Maxwell equation follows from the first three Maxwell equations. We can thus replace the fourth Maxwell equation by its initial conditions and by the continuity equation for the source terms. For the electromagnetic wave propagation, we then need just the first three of four Maxwell equations (15),

$$(\chi^{i\beta\gamma\delta} A_{\delta,\gamma})_{,\beta} = J^i \quad . \tag{17}$$

In our coordinate system, we choose the Weyl gauge condition

$$A_4 = 0 \quad . \tag{18}$$

Maxwell equations (17) then simplify to

$$(\chi^{i\beta\gamma l} A_{l,\gamma})_{,\beta} = J^i \quad . \tag{19}$$

Within Weyl gauge condition (18), electric field strength $E_j = E_j(x^m, x^4)$ reads

$$E_k = -A_{k,4} \quad . \tag{20}$$

The magnetic induction is given by relation (2).

B. Fourier Transform

For the sake of simplicity, we assume that the structure is time–independent (static) in our coordinate system,

$$\chi^{\alpha\beta\gamma\delta} = \chi^{\alpha\beta\gamma\delta}(x^m) \quad . \tag{21}$$

Our coordinate system and the Weyl gauge condition (18) are thus related to the static property of the medium.

In a static medium, we can efficiently work in the frequency domain. We define the Fourier transform

$$A_i(x^m, \omega) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{+\infty} \mathrm{d}t \; A_i(x^m, x^4) \exp(\mathrm{i}\omega x^4) \quad . \tag{22}$$

of the magnetic vector potential. Note that coefficient $(2\pi)^{-\frac{1}{2}}$ can arbitrarily be modified.

C. Frequency–domain Maxwell Equations with Constitutive Relations

In frequency domain, Maxwell equations (19) with linear constitutive relations in the Boys–Post representation for $A_i = A_i(x^m, \omega)$ read

$$(\chi^{ijkl} A_{l,k})_{,j} - \mathrm{i}\omega(\chi^{ij4l} A_l)_{,j} - \mathrm{i}\omega\chi^{i4kl} A_{l,k} - \omega^2 \chi^{i44l} A_l = J^i \; . \tag{23}$$

Here electric current density J^i represents the source term. Outside the source, $J^i = 0$. Within Weyl gauge condition (18), electric field strength $E_j = E_j(x^m, \omega)$ in frequency domain is a simple multiple of the magnetic vector potential,

$$E_k = \mathrm{i}\omega \, A_k \quad . \tag{24}$$

III. Ray Theory in the Frequency Domain

A. Standard Ray Series

We express the frequency–domain magnetic vector potential $A_j = A_j(x^m, \omega)$ in terms of its vectorial amplitude $a_i = a_i(x^m, \omega)$ and travel time $\tau = \tau(x^m)$ as

$$A_i = a_i \exp(\mathrm{i}\omega\tau) \quad . \tag{25}$$

We express the vectorial amplitude in the form of asymptotic series

$$a_i = \sum_{n=0}^{\infty} (\mathrm{i}\omega)^{-n} \, a_i^{[n]} \quad , \tag{26}$$

where $a_i^{[n]} = a_i^{[n]}(x^m, \omega)$ is the n–th order vectorial amplitude.

The electric field strength is given by relation (24). Magnetic induction $B^j = B^j(x^m, \omega)$ reads

$$B^i = \mathrm{i}\omega\varepsilon^{ijk}\tau_{,j}A_k + \varepsilon^{ijk}a_{k,j}\exp(\mathrm{i}\omega\tau) \quad , \qquad (27)$$

see relation (2) with (25). If we neglect the term of order ω^{-1}, the magnetic induction may be approximated by

$$B^i \approx \varepsilon^{ijk}\tau_{,j}E_k \quad , \qquad (28)$$

where electric field strength $E_k = E_k(x^m, \omega)$ is given by (24).

B. Ray–theory Maxwell Equations

We insert expression (25) into Maxwell equations (23) and obtain ray–theory Maxwell equations

$$(\mathrm{i}\omega)^2 N^i(a_m, \tau_{,n}) + \mathrm{i}\omega M^i(a_m, \tau_{,n}) + L^i(a_m) = 0 \quad . \qquad (29)$$

The linear operators in ray–theory Maxwell equations (29) read

$$N^i(a_m, \tau_{,n}) = \Gamma^{il}(x^m, \tau_{,n}, -1)\, a_l \quad , \qquad (30)$$

where matrix function

$$\Gamma^{il}(x^m, p_n, p_4) = \chi^{i\beta\gamma l}(x^m) p_\beta p_\gamma \qquad (31)$$

represents the 3×3 *Kelvin–Christoffel matrix*,

$$M^i(a_m, \tau_{,n}) = \chi^{ijkl}\tau_{,j}a_{l,k} + (\chi^{ijkl}\tau_{,k}a_l)_{,j} - (\chi^{ij4l}a_l)_{,j} - \chi^{i4jl}a_{l,j} \quad , \qquad (32)$$

$$L^i(a_m) = (\chi^{ijkl}a_{l,k})_{,j} \quad . \qquad (33)$$

Inserting series (26) into ray–theory Maxwell equations (29) and sorting the terms according to the order of $\mathrm{i}\omega$, we obtain the system of equations

$$N^i\big(a_k^{[n]}, \tau_{,l}\big) + M^i\big(a_k^{[n-1]}, \tau_{,l}\big) + L^i\big(a_k^{[n-2]}\big) = 0 \qquad (34)$$

for each order $n = 0, 1, 2, ...$ Here $a_k^{[-1]} = 0$ and $a_k^{[-2]} = 0$, i.e., operator M^i is missing in equation (34) for $n = 0$ and operator L^i is missing in equation (34) for $n = 0, 1$.

C. Kelvin–Christoffel Equation

Equation (34) for $n = 0$ constitutes the matrix Kelvin–Christoffel equation

$$\Gamma^{il}(x^m, \tau_{,n}, -1)\, a_l^{[0]} = 0 \quad . \qquad (35)$$

In order to satisfy Kelvin–Christoffel equation (35), the 3×3 Kelvin–Christoffel matrix (31) must be singular,

$$\det\big[\Gamma^{ad}(x^\mu, \tau_{,n}, -1)\big] = 0 \quad , \qquad (36)$$

which represents the first–order partial differential equations for several branches of travel time $\tau = \tau(x^m)$. We shall refer to each of these first–order partial differential equations as the *Eikonal equation*.

We select one of the branches of travel time $\tau = \tau(x^m)$ satisfying characteristic equation (36) and denote the corresponding zero eigenvalue by G, the corresponding right–hand unit eigenvector by g_i, and the corresponding left–hand unit eigenvector by $\vec{g}_i$. The zero–order vectorial amplitude then reads

$$a_i^{[0]} = a^{[0]}\, g_i \quad , \qquad (37)$$

where the zero–order ray–theory amplitude $a^{[0]}$ is determined by the *transport equation*.

D. Eikonal Equation and the Hamiltonian Function

We define phase–space functions $p_4 = p_4(x^m, p_n)$ as the solutions of Characteristic equation

$$\det\big[\Gamma^{ad}(x^m, p_n, p_4)\big] = 0 \qquad (38)$$

for given coordinates x^m and slowness vector p_n. Functions $p_4 = p_4(x^m, p_n)$ are homogeneous functions of the first degree with respect to slowness vector p_n.

Sixth–order Characteristic equation (38) has two zero solutions $p_4 = 0$. The other four solutions $p_4 = p_4(x^m, p_n)$ are the solutions of a fourth–order polynomial equation, which has two solutions with negative real parts and two solutions with positive real parts.

In order to identify parameter ω with circular frequency, we need $p_4 = -1$. We thus consider two solutions with negative real parts only. We choose one of them, and rescale the slowness vector

$$p_n \longrightarrow p_n/(-p_4) \qquad (39)$$

in order to obtain

$$p_4 \longrightarrow -1 \quad . \qquad (40)$$

The Hamilton–Jacobi equation for τ then reads

$$p_4(x^m, \tau_{,n}) = -1 \quad , \qquad (41)$$

where $p_4(x^m, p_n)$ is the homogeneous Hamiltonian function of the first degree with respect to slowness vector p_n. Since the perturbation expansions of travel time are most accurate for homogeneous Hamiltonian functions of the minus first degree with respect to slowness vector p_n, we prefer Hamiltonian function

$$H(x^m, p_n) = [p_4(x^m, p_n)]^{-1} \qquad (42)$$

which is a homogeneous function of the minus first degree with respect to slowness vector p_n, and express the Hamilton–Jacobi equation for travel time τ as

$$H(x^m, \tau_{,n}) = -1 \quad . \qquad (43)$$

The methods for solving of the Hamilton–Jacobi equation are already well developed (Hamilton, 1837; Červený, 1972; Klimeš, 2002; 2010).

E. Hamilton's Equations of Rays

The corresponding rays satisfy Hamilton's equations

$$\frac{\mathrm{d}x^i}{\mathrm{d}\gamma} = \frac{\partial H}{\partial p_i}(x^m, p_n) \quad , \quad \frac{\mathrm{d}p_i}{\mathrm{d}\gamma} = -\frac{\partial H}{\partial x^i}(x^m, p_n) \quad . \qquad (44)$$

For our homogeneous Hamiltonian function with respect to p_n, independent parameter γ along rays coincides with travel time τ, and

$$V^i(x^m) = \frac{\partial H}{\partial p_i}\big(x^m, \tau_{,n}(x^r)\big) \qquad (45)$$

represents the ray–velocity vector.

Differentiating the Kelvin–Christoffel equation, we obtain expressions

$$\frac{\partial H}{\partial x^i} = -\frac{1}{2\,\varrho}\, \vec{g}_a \chi^{a\beta\gamma d}_{,i} p_\beta p_\gamma g_d \qquad (46)$$

and

$$\frac{\partial H}{\partial p_i} = -\frac{1}{2\,\varrho}\, \vec{g}_a \big(\chi^{ai\gamma d} + \chi^{a\gamma id}\big) p_\gamma g_d \qquad (47)$$

with

$$\varrho = -\tfrac{1}{2}\, \vec{g}_a \big(\chi^{a4\gamma d} + \chi^{a\gamma 4d}\big) p_\gamma g_d \qquad (48)$$

for the phase–space derivatives of the Hamiltonian function. Note that here $p_4 = -1$.

F. Vectorial Amplitudes

Analogously to the zero–order vectorial amplitude (37), we define the amplitude components with respect to the three right–hand eigenvectors of the Kelvin–Christoffel matrix: eigenvector g_i corresponding to the selected eigenvalue $G=0$ and other two eigenvectors $g_i^\perp$ corresponding to other two eigenvalues $G^\perp$. We decompose each vectorial amplitude $a_i^{[n]}$ into principal component $a^{[n]}$ and two additional components $a^{\perp[n]}$,

$$a_i^{[n]} = a_i^{[n]} g_i + \sum_\perp a^{\perp[n]} g_i^\perp \quad . \tag{49}$$

Considering expression (37), we assume that $a^{\perp[0]} = 0$.

We multiply equation (34) for $n > 0$ by two left–hand eigenvectors $\vec{g}_i^\perp$. Since

$$N^i(a_m^{[n]}, \tau_{,n}) = \sum_\perp a^{\perp[n]} G^\perp g_i^\perp \quad , \tag{50}$$

we immediately obtain two additional components

$$a^{\perp[n]} = -\left[\vec{g}_i^\perp M^i(a_k^{[n-1]}, \tau_{,n}) + \vec{g}_i^\perp L^i(a_k^{[n-2]})\right] (G^\perp)^{-1} \quad . \tag{51}$$

To obtain the transport equation, we multiply ray–theory Maxwell equations (34) by left–hand eigenvector $\vec{g}_i$, consider (50), and obtain transport equation

$$\vec{g}_i M^i(a_k^{[n]}, \tau_{,n}) + \vec{g}_i L^i(a_k^{[n-1]}) = 0 \quad . \tag{52}$$

We decompose the amplitude argument of linear operator M^i according to decomposition (49) and arrive at equation

$$\vec{g}_i M^i(a^{[n]} g_k, \tau_{,n}) = -\sum_\perp \vec{g}_i M^i(a^{\perp[n]} g_k^\perp, \tau_{,n}) - \vec{g}_i L^i(a_k^{[n-1]}) \, . \tag{53}$$

The left–hand side of equation (53) can be expressed in terms of ray velocity vector (45) given by (47) as

$$\vec{g}_i M^i(a^{[n]} g_m, \tau_{,n}) = -2\varrho V^j a_{,j}^{[n]} - (\varrho V^j)_{,j} a^{[n]} + 2\varrho\, S\, a^{[n]} \, , \tag{54}$$

where quantity (Klimeš, 2016, eq. 115)

$$S = \sum_\perp \frac{1}{4\varrho G^\perp}\left(\vec{g}_k \frac{\partial \Gamma^{kl}}{\partial x^j} g_l^\perp \vec{g}_r^\perp \frac{\partial \Gamma^{rs}}{\partial p_j} g_s - \vec{g}_k \frac{\partial \Gamma^{kl}}{\partial p_j} g_l^\perp \vec{g}_r^\perp \frac{\partial \Gamma^{rs}}{\partial x^j} g_s\right) + \frac{\vec{g}_i}{4\varrho}\left[\left(\chi^{ijkl} - \chi^{ikjl}\right)_{,j} \tau_{,k} - \left(\chi^{ij4l} - \chi^{i4jl}\right)_{,j}\right] g_l - \vec{g}_i \frac{\mathrm{d} g_i}{\mathrm{d}\gamma} \tag{55}$$

vanishes for a constitutive tensor symmetric with respect to the first and second pairs of indices. For a non–symmetric constitutive tensor, quantity S vanishes in a homogeneous medium. The last term $\vec{g}_i \frac{\mathrm{d} g_i}{\mathrm{d}\gamma}$ in expression (55) represents just the correction of principal amplitude $U^{[n]}$ in decomposition (49) due to the undefined length of right–hand eigenvector g_i, and may be put to zero without a loss of generality. Expression (55) for quantity S may be singular at slowness–surface singularities, but is regular at spatial caustics.

For orders $n > 0$, we define quantities

$$Z^{[n-1]} = -\frac{1}{2\sqrt{\varrho}}\left[\sum_\perp \vec{g}_i M^i\left(a^{\perp[n]} g_k^\perp, \tau_{,n}\right) + \vec{g}_i L^i\left(a_k^{[n-1]}\right)\right] , \tag{56}$$

where additional amplitude components $a^{\perp[n]}$ are given by expression (51). Quantity (56) is thus determined by the amplitudes up to the $(n-1)^{\text{th}}$ order.

The n^{th}–order transport equation then reads

$$\sqrt{\varrho}\, V^j a_{,j}^{[n]} + \tfrac{1}{2\sqrt{\varrho}} (\varrho V^j)_{,j} a^{[n]} = \sqrt{\varrho}\, S\, a^{[n]} + Z^{[n-1]} \, . \tag{57}$$

For the zero order, term $Z^{[n-1]}$ vanishes and the solution of this transport equation is (Klimeš, 2006)

$$a^{[0]} = a_0^{[0]} \, (\varrho_0 \, J_0)^{\frac{1}{2}} \, (\varrho \, J)^{-\frac{1}{2}} \, \exp\left(\textstyle\int_{\tau_0}^{\tau} \mathrm{d}\gamma \, S\right) \quad , \tag{58}$$

where subscript $_0$ denotes the initial conditions. Squared geometrical spreading J (Červený, 2001, eq. 3.10.9) represents the Jacobian $J = \det\left(\frac{\partial x^i}{\partial \gamma^a}\right)$ of transformation from ray coordinates $\gamma^1, \gamma^2, \gamma^3 = \gamma$ to spatial coordinates x^i.

The solution of transport equation (57) for higher orders reads (Červený, 2001, eq. 5.7.30)

$$a^{[n]} = a^{[0]}\left[\frac{a_0^{[n]}}{a_0^{[0]}} + \int_{\tau_0}^{\tau} \mathrm{d}\gamma \, \frac{Z^{[n-1]}}{a^{[0]} \sqrt{\varrho}}\right] \quad . \tag{59}$$

Refer to Klimeš (2016) for a more detailed derivation.

Acknowledgements

The research has been supported by the Grant Agency of the Czech Republic under contract 16-01312S, and by the members of the consortium "Seismic Waves in Complex 3–D Structures" (see "http://sw3d.cz").

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