EXPANSION OF A PLANE WAVE INTO GAUSSIAN BEAMS

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Summary: *An integral expansion which expresses a plane monochromatic wave as a superposition of Gaussian beams is found. The expansion can be used to solve many wave propagation problems in complicated structures, including laterally inhomogeneous media with curved interfaces.*

1. INTRODUCTION

Gaussian beams play a very important role in various branches of physics; let us mention applications in optics (lasers), radio waves, etc. Many important problems of propagation and scattering of Gaussian beams have been solved (see [12, 13, 15, 22], where other references can be found). Gaussian beams have been found very useful for solving various wave propagation problems.

In this paper, we are not interested in Gaussian beams as in a physical reality. We would like to use them to calculate the wave field in inhomogeneous media, generated by a point source, line source, plane source, etc. To do this, we must first solve the important problem how to expand a plane wave, cylindrical wave, or a spherical wave into Gaussian beams.

A procedure for the computation of seismic wave fields in inhomogeneous, 2-D or3-D media, based on a simulation of the seismic wave field by a system of Gaussian beams, was proposed in [9]. The procedure is asymptotic, valid for high frequencies ω, and gives a good description of the wave field even in singular regions (caustics, critical regions, boundaries of shadow zones, etc.). It is not restricted to the computation of seismic wave fields, but it can also be used to investigate wave propagation problems in other branches of physics (acoustic waves, electromagnetic waves, etc.).

In the procedure [9], the wave field generated by a point or line source is decomposed into contributions, corresponding to individual rays. These contributions, however, are not evaluated by standard ray methods, but by the method of the parabolic wave equation, in which the whole wave field concentrated close to the rays is considered. From a physical point of view, the solutions concentrated close to rays correspond to Gaussian beams.

The parabolic wave equation method is closely connected with the names of Fock and Leontovich. They used the method to solve certain problems in radio wave propagation, see [19, 14]. The method has been applied to many wave propagation problems, in which the waves propagating in certain preferred directions are studied. It was mainly used to investigate radio waves, optical waves and acoustic waves, see the review paper [23], and also seismic waves in seismic prospecting [11]. It was first applied by Babich [1] to the investigation of the solutions of the wave equation which are concentrated close to rays, for more details refer to [2, 3]. The solutions concentrated close to rays were investigated in these papers mostly in relations to the theory of resonators. The same method was applied to elastodynamic equations in [17]. Certain general asymptotic integral representations of the wave field as a superposition of solutions concentrated close to rays were suggested in [4]. It was proposed in [5] to use the solutions concentrated close to rays in the computation of seismic wave fields in laterally inhomogeneous media with curved interfaces and block structures, where other methods usually fail. The first attempts to perform such computations in 2-D media are described in [9, 10]. Popov [20] used the method suggested in [4] and

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derived the asymptotic integral representation of the wave field generated by a point source in a 3-D media and by a line source in a 2-D medium in terms of Gaussian beams. A detailed mathematical treatment of Gaussian beams in a 2-D laterally inhomogeneous medium with several numerical examples of computation is given in [10]. Generally speaking, all the numerical tests performed so far are very promising.

In this paper, an expansion of a plane wave propagating in a 3-D homogeneous medium into Gaussian beams is derived. The paper is to some extent self-contained; it starts with the wave equation and gives a full derivation of the parabolic wave equation and of its solutions, all for a generally 3-D medium. It is believed that the expansion of a plane wave into Gaussian beams will find important applications in many wave propagation problems.

To simplify the procedure, only harmonic solutions are considered. It is, however, not complicated to apply the method also to non-stationary wave fields and to the computation of synthetic seismograms. Three different approaches of doing this are described in [6].

2. PARABOLIC EQUATION

We shall look for the solutions of the wave equation

$$\Delta u = V^{-2}\,\partial^2 u/\partial t^2\,, \tag{1}$$

which are concentrated close to rays. Here $V(x_i)$ is the propagation velocity, t denotes the time, x_i are Cartesian coordinates, $\Delta = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2 + \partial^2/\partial x_3^2$ is the Laplacian operator. The quantity $u(x_i, t)$ may represent various physical quantities in different wave propagation problems.

We select an arbitrary ray Ω and introduce an orthogonal coordinate system (s, q_1, q_2) connected with this ray. The coordinate s measures the arc length along the ray from an arbitrary point, q_1 and q_2 represent length coordinates in the plane perpendicular to Ω at s. This orthogonal coordinate system is called the "ray-centered coordinate system" and was first introduced to seismology by Popov and Pšenčík [21], see also [7, 16]. The basis of the ray-centered coordinate system is formed by three unit vectors $\mathbf{t}$, $\mathbf{e}_1$, $\mathbf{e}_2$. Here $\mathbf{t}$ is the unit tangent to the ray Ω, and the vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ can be formally introduced with the help of the normal vector $\boldsymbol{n}$, binormal vector $\boldsymbol{b}$ and the torsion T of the ray by the formulae

$$\mathbf{e}_1 = \boldsymbol{n}\cos\Theta - \boldsymbol{b}\sin\Theta\,, \quad \mathbf{e}_2 = \boldsymbol{n}\sin\Theta + \boldsymbol{b}\cos\Theta\,,$$

with $\Theta(s) = \int_{s_0}^{s} T(\xi)\,\mathrm{d}\xi + \Theta(s_0)$. The integral is taken along the ray. For the infinitesimal length element $\mathrm{d}r$ in the new coordinate frame we obtain

$$\mathrm{d}r^2 = h^2\,\mathrm{d}s^2 + \mathrm{d}q_1^2 + \mathrm{d}q_2^2\,, \tag{2}$$

where

$$h = 1 + v^{-1}(q_1 v_{,1} + q_2 v_{,2})\,. \tag{3}$$

Formula (2) shows that the ray-centered coordinate system (s, q_1, q_2) is orthogonal, with the scale factors h, 1, 1. In (3), we have used the following notation

$$v = V(s, 0, 0)\,, \quad v_{,i} = \left[\frac{\partial V(s, q_1, q_2)}{\partial q_i}\right]_{q_1 = q_2 = 0}.$$

Thus, v, $v_{,1}$ and $v_{,2}$ are functions of s only, not of q_1 and q_2. Using the symbol $v(s)$ for the velocity measured directly at the ray Ω instead of $V(s, 0, 0)$ we try to simplify the following equations. (It would be, of course, possible to write $V(s, 0, 0)$ instead of $v(s)$ everywhere.)

Note that in case of curved ray Ω the system of planes perpendicular to Ω intersect at certain distances from the ray. In other words, the ray centered coordinate system (s, q_1, q_2) is not regular at large distances from Ω. In the following, we shall consider only the neighbourhood of the ray Ω in which the system (s, q_1, q_2) is regular.

In the ray centered coordinate system (s, q_1, q_2), the wave equation (1) can be rewritten as follows

$$(4) \qquad h(u_{,11} + u_{,22}) + (1/h)\, u_{,ss} - hV^{-2}u_{,tt} + u_{,s}(1/h)_{,s} + \\ + u_{,1}h_{,1} + u_{,2}h_{,2} = 0\,.$$

Here we have used the commonly accepted notation for the derivatives, $u_{,s} = \partial u/\partial s$, $u_{,1} = \partial u/\partial q_1$, $u_{,2} = \partial u/\partial q_2$, etc. Similar notation will be used throughout this paper.

To find the solution of the wave equation when the wave propagates mostly in some preferred direction, it is useful to use the parabolic wave equation method. It is well known that the high-frequency wave field propagates mostly along certain trajectories which can be identified as rays. Therefore, we shall use the method to find the solution of the wave equation concentrated close to Ω.

In the following, we shall consider only time — harmonic solutions, and denote the circular frequency by ω.

The basic step in the parabolic wave equation method is the following substitution

$$(5) \qquad u(s, q_1, q_2, t) = \exp\left\{- \mathrm{i}\omega\left(t - \int_{s0}^{s} v^{-1}(s)\, \mathrm{d}s\right)\right\} U(s, q_1, q_2, \omega)\,.$$

Let us emphasize that the integral is taken along the ray Ω at which $q_1 = q_2 = 0$. Inserting (5) into the wave equation (4) we obtain a new differential equation for U

$$(6) \qquad h^{-1}\{(-\omega^2/v^2 + \mathrm{i}\omega(1/v)_{,s})\, U + 2\mathrm{i}v^{-1}\omega U_{,s} + U_{,ss}\} + h(U_{,11} + U_{,22}) + \\ + hV^{-2}\omega^2 U + (\mathrm{i}\omega v^{-1}U + U_{,s})\,(1/h)_{,s} + U_{,1}h_{,1} + U_{,2}h_{,2} = 0\,.$$

In deriving the parabolic equation, we shall assume that $q_1 = O(\omega^{-\beta})$ and $q_2 = O(\omega^{-\beta})$, with $\beta = \frac{1}{2}$. This assumption expresses the fact that our investigation can be restricted for large ω to a thin "boundary layer" along Ω. The concrete value of $\beta(\beta = \frac{1}{2})$ follows from the investigation of some canonical problems. It is then suitable to introduce new coordinates v_1, v_2 instead of q_1, q_2 by the following relations

$$v_1 = q_1 \sqrt{\omega}\,, \quad v_2 = q_2 \sqrt{\omega}\,. \tag{7}$$

The new coordinates v_1 and v_2 are zero-order quantities. Equation (6) can then be rewritten as follows:

$$\begin{aligned} & h\omega^2(V^{-2} - h^{-2}v^{-2})\, U + \omega\{-\mathrm{i}v^{-2}h^{-1}v_{,s}U + \mathrm{i}v^{-1}(1/h)_{,s}U + \\ & + 2\mathrm{i}v^{-1}h^{-1}U_{,s} + h(U_{,11} + U_{,22})\} + \sqrt{\omega}\{U_{,1}h_{,1} + U_{,2}h_{,2}\} + \\ & + h^{-1}U_{,ss} + U_{,s}(1/h)_{,s} = 0\,. \end{aligned} \tag{8}$$

Equation (8) is still fully equivalent to the wave equation (1). Only now shall we start to solve it asymptotically, for $\omega \to \infty$. In equation (8) we shall only retain the terms of the order ω^γ with $\gamma \geqq 1$ and neglect all terms of the order ω^γ with $\gamma < 1$. We must, however, remember that several coefficients in (8) are functions of ω, too. We find the expansions of these coefficients, again neglecting the terms of lower in ω. We obtain

$$h\omega^2(V^{-2} - h^2v^{-2}) \sim -\omega v^{-3}(v^T\mathbf{V}v)\,,$$

where

$$\mathbf{V} = \begin{pmatrix} v_{,11} & v_{,12} \\ v_{,12} & v_{,22} \end{pmatrix}, \quad v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \quad v_{,ij} = \left[\frac{\partial^2 V(s, q_1, q_2)}{\partial q_i \partial q_j}\right]_{q_1 = q_2 = 0},$$

and where v^T denotes the transpose matrix of v, $v^T = (v_1 v_2)$. In the second term in (8), it is sufficient to consider the following approximations: $h \sim 1$, $(1/h)_{,s} \sim 0$. On inserting the above approximations into (8) and neglecting terms of lower order in ω, we then obtain

$$2\mathrm{i}v^{-1}U_{,s} + U_{,11} + U_{,22} - (v^{-3}v^T\,\mathbf{V}v + \mathrm{i}v^{-2}v_{,s})\, U = 0\,, \tag{9}$$

where $U(s, v_1, v_2)$ denotes in this case the leading term of the corresponding asymptotic series for $U(s, v_1, v_2, \omega)$. Note that the solution $U(s, v_1, v_2)$ depends still on frequency ω, as $v_i = q_i\sqrt{\omega}$.

This is the parabolic wave equation we have been looking for. $U_{,11}$ and $U_{,22}$ denote the second derivatives of U with respect to v_1 and v_2. We can simplify (9) even more by the substitution

$$U(s, v_1, v_2) = (v(s))^{1/2}\, W(s, v_1, v_2)\,. \tag{10}$$

Inserting (10) into (9) we obtain the final form of the parabolic wave equation

$$2iv^{-1}W_{,s} + W_{,11} + W_{,22} - v^{-3}(v^T\mathbf{V}v)\, W = 0\,. \tag{11}$$

We remember that $u(s, q_1, q_2, t)$ is expressed in terms of W as follows

(12) $$u(s, q_1, q_2, t) = (v(s))^{1/2} \exp\left[-i\omega\left(t - \int_{so}^{s} v^{-1}(s)\, ds\right)\right] W(s, v_1, v_2)\,,$$

with $v_1 = q_1\sqrt{\omega}$, $v_2 = q_2\sqrt{\omega}$, where $W(s, v_1, v_2)$ is a solution of (11). Note again that $W(s, v_1, v_2)$ is a function of frequency ω, as $v_i = q_i\sqrt{\omega}$, $i = 1, 2$.

3. THE SOLUTION OF THE PARABOLIC WAVE EQUATION. GAUSSIAN BEAMS

We shall follow [2, 17] and seek the solution of (11) in the following form

(13) $$W(s, v_1, v_2) = A(s) \exp\left(\tfrac{1}{2} i v^T \mathbf{M} v\right),$$

where $\mathbf{M} = \mathbf{M}(s)$ is an unknown 2×2 symmetrical matrix with complex elements. Inserting (13) into (11) yields

(14) $$i\{2v^{-1}A_{,s} + A \operatorname{tr} \mathbf{M}\} - A\{v^T[v^{-1}\mathbf{M}_{,s} + \mathbf{M}^2 + v^{-3}\mathbf{V}]\, v\} = 0\,.$$

We put

(15), (16) $$\mathbf{M}_{,s} + v\mathbf{M}^2 + v^{-2}\mathbf{V} = 0\,, \quad A_{,s} + \tfrac{1}{2} v A \operatorname{tr} \mathbf{M} = 0\,.$$

Equation (14) is then satisfied and the function $W(s, v_1, v_2)$, given by (13), is the solution of the parabolic wave equation (11).

Let us note that the solution (13) is not the only solution of (11). It would be possible to construct an infinite number of other solutions of (11) concentrated close to rays from (13). The solutions will contain Hermite polynomials. These other solutions will be discussed elsewhere, they are not important to the subject of this paper.

We shall now pay attention to the solution of the equation (15). From a mathematical point of view, Eq. (15) is an ordinary non-linear differential equation of the first order of the Riccati type in matrix form that, in general, cannot be solved by elementary analytical methods. We shall rewrite it in the form of a system of linear differential equations. By introducing a new 2×2 matrix $\mathbf{Q}$ by the formula

(17) $$\mathbf{M} = v^{-1}\mathbf{Q}_{,s}\mathbf{Q}^{-1}\,,$$

see [7, 17], we obtain an equivalent system of linear differential equations of second order for $\mathbf{Q}$ from (15),

(18) $$v\,\mathbf{Q}_{,ss} - v_{,s}\mathbf{Q}_{,s} + \mathbf{VQ} = 0\,.$$

This can be rewritten into a new system of linear differential equations of the first order putting $\mathbf{Q}_{,s} = v\mathbf{P}$. We then obtain the system

(19) $$\mathbf{Q}_{,s} = v\mathbf{P}\,, \quad \mathbf{P}_{,s} = -v^{-2}\mathbf{VQ}\,.$$

The formula (17) for $\mathbf{M}$ can be rewritten in the following form

$$\mathbf{M} = \mathbf{P}\mathbf{Q}^{-1} . \tag{20}$$

Note that the systems (15) and (19) are well known from the ray theory of 3-D media, see [21, 16, 7]. The systems serve there for the computation of geometrical spreading. They are called "dynamic ray tracing systems", or "additional systems", see [7, 8, 21]. Eqs. (19) represent a system of eight linear differential equations of the first order for the components of matrices $\mathbf{P}$ and $\mathbf{Q}$. Let us note that the matrices $\mathbf{Q}$ and $\mathbf{P}$ are not generally symmetrical.

The system (19) can be simplified even more. It can be divided into two fully equivalent systems, each consisting of four equations. This will be discussed elsewhere.

To find the solutions of equation (16), it is useful to use the relation $\operatorname{tr} \mathbf{M} = v^{-1} \mathrm{d}(\ln \det \mathbf{Q})/\mathrm{d}s$, which can be proved by direct inspection. We then obtain the solution of (16) in the form

$$A(s) = \Phi/\sqrt{\det \mathbf{Q}(s)} , \tag{21}$$

where Φ is some complex constant.

By inserting (13), (20) and (21) into (12), and returning to the original coordinates q_1, q_2, we find that

$$u(s, q_1, q_2, t) = \Phi \left(\frac{v(s)}{\det \mathbf{Q}(s)} \right)^{1/2} \exp \left\{ -\mathrm{i}\omega \left(t - \int_{s_0}^{s} \frac{\mathrm{d}s}{v(s)} \right) + \frac{\mathrm{i}\omega}{2} \left(q^T \mathbf{P}\mathbf{Q}^{-1} q \right) \right\} , \tag{22}$$

where $q^T = (q_1\ q_2)$, q is the transpose of q^T, $\mathbf{P}$ and $\mathbf{Q}$ are the solutions of the system of linear differential equations (19).

The solutions of the type of (22) are known from the ray theory of 3-D media [7]. In the ray methods, the matrices $\mathbf{Q}$ and $\mathbf{P}$ are real, and the quantity $\det \mathbf{Q}(s)$ may vanish (at caustics). In the case of solutions concentrated close to rays, however, the matrices $\mathbf{Q}$ and $\mathbf{P}$ must be complex valued, with $\operatorname{Im}(\mathbf{P}\mathbf{Q}^{-1}) > 0$, to guarantee the concentration of the solutions close to rays (details will be given later). In other words, the initial conditions for the systems of equations (19) must also be complex-valued.

It is always possible to select the initial complex-valued constants in such a way that they will guarantee the fulfilment of the following three conditions along the whole ray:

a) $\det \mathbf{Q}(s) \neq 0$,

b) $\mathbf{P}\mathbf{Q}^{-1}$ is a symmetric matrix, even though $\mathbf{P}$ and $\mathbf{Q}$ are not symmetrical,

c) $\operatorname{Im}(\mathbf{P}\mathbf{Q}^{-1})$ is a positive-definite matrix.

The condition a) guarantees that the solution (22) is regular along the whole ray (even at caustics). The condition c) guarantees that the solutions are concentrated close to the ray Ω, they decrease exponentially with increasing distance from the ray.

It is easy to see that the exponential decrease is Gaussian. Let us select, e.g., the section $q_2 = 0$. Then

$$\exp\left(\tfrac{1}{2}\mathrm{i}\omega q^T \mathbf{M} q\right) = \exp\left(\tfrac{1}{2}\mathrm{i}\omega q_1^2 \operatorname{Re} M_{11} - \tfrac{1}{2}\omega q_1^2 \operatorname{Im} M_{11}\right).$$

Thus the amplitude profile is bell-shaped ($\sim\exp(-a^2 q_1^2)$) with the maximum at the ray Ω, i.e. for $q_1 = q_2 = 0$. Due to this property, this solution of the wave equation may be called Gaussian beams.

The proper choice of the complex-valued initial constants in the system (19) will be described elsewhere.

Only an inhomogeneous medium with smooth changes of velocity has been considered here. It is not complicated, however, to generalize the above formula (22) for a multiply reflected Gaussian beam in a medium with curved interfaces [20].

4. EXPANSION OF A PLANE WAVE INTO GAUSSIAN BEAMS

In this section, we shall try to expand a plane wave into Gaussian beams. This problem is very important not only from a theoretical point of view, but it can also find many practical applications. Moreover, the wave field generated by more complicated sources can often be expressed in terms of a continuous or a discrete spectrum of plane waves, either approximately or exactly.

Let us now consider the Gaussian beam (22) concentrated close to the ray Ω specified by the ray parameters γ_1, γ_2. We denote $\tau(s) = \int_{s_0}^{s} v^{-1}(s)\,\mathrm{d}s$, omit the factor $\exp(-\mathrm{i}\omega t)$ and put $\Phi = 1$. We can then write, see (22),

$$u_{\gamma_1,\gamma_2}(s, q_1, q_2) = (v(s)/\det \mathbf{Q}(s))^{1/2} \exp\left(\mathrm{i}\omega\tau(s) + \tfrac{1}{2}\mathrm{i}\omega q^T \mathbf{M} q\right), \tag{23}$$

where again $\mathbf{M} = \mathbf{P}\mathbf{Q}^{-1}$. The indices γ_1 and γ_2 specify that the beam is concentrated close to Ω and that the ray-centered coordinate system (s, q_1, q_2) corresponds to the same ray Ω. Of course, we assume that the initial complex-valued constants for the system (19) are selected in such a way that the conditions a, b, c from the preceding section are satisfied.

Now we assume that the medium is homogeneous, with the constant propagation velocity $V = v_0$. We introduce the ray coordinates $(\zeta, \gamma_1, \gamma_2)$. Consider a plane S and introduce a rectangular Cartesian coordiante system γ_1, γ_2 on S, with its origin at an arbitrary point O on S. The coordinates γ_1, γ_2 specify a two-parameter system of rays perpendicular to S. The coordinate ζ specifies the position of a point on the ray; it determines the distance of the point from S.

Now we consider an arbitrary point D with the ray coordinates $\gamma_1 = \gamma_1^0$, $\gamma_2 = \gamma_2^0$, $\zeta = \zeta^0$. We shall study the following integral

$$J(D) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Phi(\gamma_1, \gamma_2)\, u_{\gamma_1,\gamma_2}(s, q_1, q_2)\,\mathrm{d}\gamma_1\,\mathrm{d}\gamma_2\,, \tag{24}$$

where $u_{\gamma_1,\gamma_2}(s, q_1, q_2)$ is given by (23), $\Phi(\gamma_1, \gamma_2)$ is some function of γ_1, γ_2, not yet specified, s, q_1 and q_2 are the ray-centered coordinates of the point D, corresponding to the ray specified by the parameters γ_1, γ_2. Let us emphasize the difference between the coordinates $(\gamma_1^0, \gamma_2^0, \zeta^0)$ and (s, q_1, q_2) of the point D. The ray coordinates of D are $(\gamma_1^0, \gamma_2^0, \zeta^0)$ and the ray centered coordinates of D corresponding to the ray Ω are (s, q_1, q_2). Obviously, $s = s(\gamma_1, \gamma_2, \gamma_1^0, \gamma_2^0, \zeta^0)$, $q_1 = q_1(\gamma_1, \gamma_2, \gamma_1^0, \gamma_2^0, \zeta^0)$, $q_2 = q_2(\gamma_1, \gamma_2, \gamma_1^0, \gamma_2^0, \zeta^0)$.

The integral (24) can be simplified considerably if we choose the ray-centered coordinate systems corresponding to individual rays suitably. In fact, we can choose all of them in the same way. We select the basis vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ the same for all rays, in the direction of the coordinate lines γ_1 and γ_2 in S. The quantities $\gamma_1^0 - \gamma_1$ and $\gamma_2^0 - \gamma_2$ then have the same meaning as the coordinates q_1 and q_2 of the point D in the ray-centered coordinate system connected with the ray specified by γ_1, γ_2. We can also put $\zeta^0 = s$ for all the rays ($s = 0$ corresponds to the plane S). By changing the variables in (24) as follows: $\gamma_1^0 - \gamma_1 = q_1$, $\gamma_2^0 - \gamma_2 = q_2$, (24) yields

$$J(D) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi(\gamma_1^0 - q_1, \gamma_2^0 - q_2)\, u_{\gamma_1{}^0 - q_1, \gamma_2{}^0 - q_2}(\zeta^0, q_1, q_2)\, \mathrm{d}q_1\, \mathrm{d}q_2 \,. \tag{25}$$

Proper choice of the function Φ in (25) can be used to simulate even more complicated wave fields. Here, however, we wish to specify Φ in such a way that (25) would represent a plane wave. The simplest possibility we can try is just to put $\Phi = \Phi_0$, where Φ_0 is some complex-valued constant. Inserting $\Phi = \Phi_0$ and (23) into (25) yields

$$J(D) = \Phi_0 (v_0 / \det \mathbf{Q}(\zeta^0))^{1/2} \exp(\mathrm{i}\omega\tau(\zeta^0))\, \Psi(\zeta^0)\,, \tag{26}$$

where

$$\Psi(\zeta^0) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left(\tfrac{1}{2}\mathrm{i}\omega q^T \mathbf{M} q\right) \mathrm{d}q_1\, \mathrm{d}q_2 \,. \tag{27}$$

To evaluate the integral Ψ, it would be convenient to separate the variables q_1 and q_2 in the exponential term. In fact, we have a quadratic form in the exponent,

$$q^T \mathbf{M} q = q_1^2 M_{11} + q_2^2 M_{22} + 2 q_1 q_2 M_{12} \,.$$

The problem arises with the off-diagonal term M_{12}. For $M_{12} = 0$, integral (27) could be evaluated easily. It would be useful to eliminate the off-diagonal terms by some linear transformation. The matrix $\mathbf{M}$ is, however, complex-valued; it represents two matrices $\mathbf{M} = \mathbf{M}^R + \mathrm{i}\mathbf{M}^I$, where $\mathbf{M}^R = \operatorname{Re} \mathbf{M}$, $\mathbf{M}^I = \operatorname{Im} \mathbf{M}$. We must diagonalize both these matrices by one linear transformation. We know that both $\mathbf{M}^R$ and $\mathbf{M}^I$ are real, symmetric and that $\mathbf{M}^I$ is positive-definite. Thus, both $\mathbf{M}^R$ and $\mathbf{M}^I$ are Hermitian and $\mathbf{M}^I$ has positive eigenvalues.

In this case, both the matrices $\mathbf{M}^R$ and $\mathbf{M}^I$ can be diagonalized by one linear transformation, see [18]. Denote the transformation matrix by $\mathbf{C}$ and the new coordinates we introduce instead of q_1 and q_2 by y_1 and y_2. We can then write

$$q = \mathbf{C}y\,, \quad \text{with} \quad q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}, \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

The quadratic form in the exponent of (27) is then transformed as follows:

$$q^T\mathbf{M}q = y^T\,\mathbf{C}^T\mathbf{M}\mathbf{C}y\,,$$

with

$$\mathbf{C}^T\mathbf{M}\mathbf{C} = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix} + \mathrm{i}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{28}$$

The eigenvalues μ_1, μ_2 and the transformation matrix $\mathbf{C}$ can be determined in the following way:

The eigenvalues μ_1 and μ_2 are the solutions of the characteristic equation of the so-called generalized problem of eigenvalues, $\det(\mathbf{M}^R - \mu\mathbf{M}^I) = 0$, and the transformation matrix $\mathbf{C}$ is formed from the corresponding system of eigenvectors. (For more details see [18].)

We can now evaluate the integral Ψ, see (27). We obtain

$$\Psi = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left(\tfrac{1}{2}\mathrm{i}\omega y^T\mathbf{C}^T\mathbf{M}\mathbf{C}y\right)(\det\mathbf{C})\,\mathrm{d}y_1\,\mathrm{d}y_2\,. \tag{29}$$

The quadratic form in the exponent can be rewritten in the form

$$y^T\mathbf{C}^T\mathbf{M}\mathbf{C}y = y_1^2(\mu_1 + \mathrm{i}) + y_2^2(\mu_2 + \mathrm{i})\,. \tag{30}$$

By inserting this into (29) and taking into account that $\det\mathbf{C}$ depends only on ζ^0, we arrive at

$$\Psi = \det\mathbf{C}\left\{\int_{-\infty}^{\infty} \exp\left[-\tfrac{1}{2}\omega y_1^2(1 - \mathrm{i}\mu_1)\right]\mathrm{d}y_1\right\}\left\{\int_{-\infty}^{\infty} \exp\left[-\tfrac{1}{2}\omega y_2^2(1 - \mathrm{i}\mu_2)\right]\mathrm{d}y_2\right\}.$$

This yields

$$\Psi = 2\pi\omega^{-1}(\det\mathbf{C})\,(1 - \mathrm{i}\mu_1)^{-1/2}\,(1 - \mathrm{i}\mu_2)^{-1/2}\,. \tag{31}$$

This can be rewritten in a more useful form, if we replace the factors containing μ_1 and μ_2 by the determinants of $\mathbf{C}$, $\mathbf{P}$ and $\mathbf{Q}$. Taking the determinants of both the left-hand and right-hand sides of (28), we see that $(\det\mathbf{C})^2\det\mathbf{P}/\det\mathbf{Q} = (\mu_1 + \mathrm{i})(\mu_2 + \mathrm{i})$. Thus, finally for Ψ, we have

$$\Psi = 2\pi\mathrm{i}(\det\mathbf{Q}/\det\mathbf{P})^{1/2}\,, \tag{32}$$

and for $J(D)$,

$$J(D) = \Phi_0\,2\pi\mathrm{i}\omega^{-1}(v_0/\det\mathbf{P})^{1/2}\exp\left(\mathrm{i}\omega\tau(\zeta^0)\right). \tag{33}$$

In a homogeneous medium, the matrix $\mathbf{P}$ does not change along the ray, see (24). Thus, $\det\mathbf{P}(\zeta^0) = \det\mathbf{P}_0$, where $\mathbf{P}_0 = \mathbf{P}(0)$. When we choose

(34) $$\Phi_0 = (\omega/2\pi i)\, v_0^{-1/2} (\det \mathbf{P}_0)^{1/2}\,,$$

we obtain

(35) $$J(D) = \exp(i\omega\tau(\zeta^0))\,.$$

We can finally write

(36) $$\exp(i\omega\tau(\zeta^0)) = \exp(i\omega x_k t_k / v_0) = $$
$$= (\omega/2\pi i)\, v_0^{-1/2} (\det \mathbf{P}_0)^{1/2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u_{\gamma_1,\gamma_2}(s, q_1, q_2)\, d\gamma_1\, d\gamma_2\,,$$

where $u_{\gamma_1,\gamma_2}(s, q_1, q_2)$ are Gaussian beams specified by formula (23), $\tau(\zeta^0) = v_0^{-1}\zeta^0 = x_k t_k / v_0$ ($\mathbf{t}$ is the unit vector in the direction of propagation).

The physical explanation of (36) is as follows. The monochromatic plane wave propagating in a homogeneous medium can be expanded at any $s = 0$ into Gaussian beams using (36). Each beam can then be treated independently, using Eq. (23). Due to the diffusion effects, the width of the individual beams and their amplitudes will change with increasing s. Notwithstanding, the integral superposition (36) of all these beams again yields the same plane wave for any $s = \zeta^0 \neq 0$.

Note that the integral (36) is exact, not asymptotic; it is valid even for small ω.

5. APPLICATIONS

As we have seen, a plane wave in a homogeneous medium can be expanded into Gaussian beams and each beam can be treated independently. It was shown in Sections 2 and 3 that the Gaussian beams can be evaluated for large ω even in arbitrary laterally inhomogeneous medium with curved interfaces asymptotically, without singularities at caustics.

Let us now assume that a plane wave from an homogeneous medium is incident at an arbitrary laterally inhomogeneous structure. To compute the wavefield at any point D, outside or inside the inhomogeneous structure, the following procedure can be used. The plane wave at the homogeneous medium is expanded into Gaussian beams. Each beam is treated independently, no matter whether the structure is homogeneous. The wave field at point D is then calculated as an integral superposition of all the beams with non-vanishing amplitudes at D.

In inhomogeneous media, however, the ray centered coordinate system is regular only in some neighbourhood of the central ray, as was discussed in Section 2. Therefore, the function $u_{\gamma_1,\gamma_2}(s, q_1, q_2)$ in (36) should be multiplied by some windowing (box-car) function which vanishes outside the region of regularity.

Integrals of the type (36) containing the above described windowing function were first introduced and investigated by Babich and Pankratova [4]. They showed that such integrals represent a high frequency asymptotics of the investigated wave field in an inhomogeneous medium.

In practital applications, it would be useful to use a discrete form of formula (36). It is not necessary to consider a large number of Gaussian beams, but only those arriving in some neighbourhood of point D (receiver), as the amplitude of Gaussian beams decrease exponentially in the direction perpendicular to the central ray and the amplitudes of remote Gaussian beams will be negligibly small at the receiver.

It should be emphasized that the integral representation (36) is valid for arbitrary initial complex-valued conditions of the system (19) for the matrices $\mathbf{P}$ and $\mathbf{Q}$, which guarantee the fulfilment of conditions a, b, c from Section 3. It would be very useful to choose these conditions to render the beams in the neighbourhood of the receiver as narrow as possible. This optimization will reduce radically the number of Gaussian beams which must be used to evaluate the wave field with a sufficient accuracy. More details and numerical examples for 2-dimensional structures can be found in [10].

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References

[1] В. М. Бабич: Собственные функции, сосредоточенные в окрестности замкнутой геодетической. Матем. вопросы теории распростр. волн, Т. 9, Наука, Ленинград 1968, 15.

[2] В. М. Бабич, В. С. Булдырев: Асимптотические методы в задачах дифракции коротких волн. Наука, М. 1972.

[3] В. М. Бабич, Н. Я. Кирпичникова: Метод пограничного слоя в задачах дифракции. Изд. Ленинград. унив., Ленинград 1974.

[4] В. М. Бабич, Т. Ф. Панкратова: О разрывах функции Грина смешанной задачи для вольнового уравнения с перемениым коэффициентом. Проблемы мат. физики, Вып. 6, Изд. Ленинград. унив., Ленинград 1973, 9.

[5] V. Červený: Ray Theoretical Seismograms for Laterally Inhomogeneous Structures. J. Geoph., 46 (1979), 335.

[6] В. Червены: Расчет синтетических сейсмограмм для одномерных и двумерных неоднородных сред. Численные методы интерпретации сейсмических данных, Суздал 1980 (in press).

[7] V. Červený, F. Hron: The Ray Series Method and Dynamic Ray Tracing System for Three-Dimensional Inhomogeneous Media. Bull. Seismol. Soc. Am., 70 (1980), 47.

[8] V. Červený, I. A. Molotkov, I. Pšenčík: Ray Method in Seismology. Univ. Karlova, Praha 1977.

[9] V. Červený, M. M. Popov, I. Pšenčík: Computation of Seismic Wave Fields in Laterally Inhomogeneous Crustal Structures-Gaussian Beams Approach. Proc. XVII. Gen. Assembly Europ. Seismol. Commis., Budapest 1980 (in press).

[10] V. Červený, M. M. Popov, I. Pšenčík: Computation of Wave Fields in Inhomogeneous Media — Gaussian Beam Approach. Geoph. J. R. Astr. Soc., 69 (1982) (in press).

[11] J. F. Claerbout: Fundamentals of Geophysical Data Processing. McGraw Hill, New York 1976.

[12] L. B. Felsen: Complex-Source-Point Solutions of the Field Equations and their Relation to the Propagation and Scattering of Gaussian Beams. Ist. Naz. di Alta Matem., Sympos. Math., 18 (1976), 39.

[13] L. B. Felsen, N. Marcuvitz: Radiation and Scattering of Waves. Prentice-Hall, Englewood Cliffs, N. J., 1973.

[14] V. A. Fok: Electromagnetic Diffraction and Propagation Problems. Pergamon Press, N. Y. 1965.

[15] A. C. Green, H. L. Bertoni, L. B. Felsen: Properties of the Shadow Cast by a Half-screen when Illuminated by a Gaussian Beam. J. Opt. Soc. Am., 69 (1979), 1503.

[16] P. Hubral: A Wave Front Curvature Approach to the Computing of Ray Amplitudes in Inhomogeneous Media with Curved Interfaces. Studia geoph. et geod., 23 (1979), 131.

[17] N. J. Kirpichnikova: Construction of Solutions Concentrated Close to Rays for the Equations of Elasticity Theory in an Inhomogeneous Isotropic Space. Matemat. voprosy teorii difrakcii i rasprostranenija voln, T. 1, Nauka, Leningrad 1971 (English Translation by Am. Math. Soc. 1974).

[18] G. A. Korn, T. M. Korn: Mathematical Handbook for Scientists and Engineers. McGraw Hill, New York 1961.

[19] М. А. Леонтович, В. А. Фок: Решение задачи о распространении электромагнитных волн вдол поверхности Земли по методу параболического уравенния. ЖЕТФ, 16 (1946), 557.

[20] M. M. Popov: A New Method of Computation of Wave Fields in the High-Frequency Approximation. Preprint LOMI AN SSSR, Leningrad 1981.

[21] M. M. Popov, I. Pšenčík: Computation of Ray Amplitudes in Inhomogeneous Media with Curved Interfaces. Studia geoph. et geod., 22 (1978), 248.

[22] S. Y. Shin, L. B. Felsen: Gaussian Beam Modes by Multiplies with Complex Source Points. J. Opt. Soc. A., 67 (1977), 699.

[23] F. D. Tappert: The Parabolic Approximation Method. In: Wave Propagation and Underwater Acoustics. Lecture Notes in Physics, Vol. 70. Springer Vlg, Berlin 1977, 224.