*Geophys. J. Int.* (1991) **106**, 1–10

# Perturbation from isotropic to anisotropic heterogeneous media in the ray approximation

Robert L. Nowack[1] and Ivan Pšenčík[2]

[1] *Department of Earth and Atmospheric Sciences, Purdue University, West Lafayette, IN 47907, USA*
[2] *Geophysical Institute, Czechoslovak Academy of Sciences, Boční II, 191 31 Praha 4, Czechoslovakia*

Accepted 1991 January 2. Received 1990 December 11; in original form 1989 October 31

## SUMMARY

In this paper, perturbation methods are used to obtain ray results in slightly anisotropic, heterogeneous media from a nearby isotropic medium. The advantage of this approach is that rays computed by available 2-D isotropic ray-tracing algorithms can be used to approximately compute, by perturbation, ray theoretical results in more general 3-D slightly anisotropic, heterogeneous media. The perturbation approach also makes possible an efficient computation of ray sensitivity operators necessary in seismic tomography studies. In order to compute approximate ray results for quasi-shear waves, degenerate perturbation theory must be used in order to split a single shear wave in an isotropic medium into two quasi-shear waves in a nearby anisotropic medium. The perturbation approach is tested on an effectively anisotropic model of aligned cracks within a linear, vertical velocity gradient. Perturbed traveltimes, ray paths and particle motions of quasi-compressional and quasi-shear waves for such an anisotropic medium are compared with exact ray results. Good agreement between the exact and perturbed results is found.

**Key words:** ray method, ray perturbations, slight anisotropy, traveltime perturbations.

## 1 INTRODUCTION

The effects of material anisotropy on the propagation of seismic waves have been investigated for the past several decades. Seismic anisotropy from body waves has been detected in the crust and mantle (Hess 1964; Stephen 1985; Shearer & Orcutt 1985, 1986; Fuchs 1983) and laboratory measurements imply anisotropy must be widespread in both crystalline and sedimentary rocks (Babuška 1984; Christensen & Salisbury 1979). A recent review is also given by Crampin, Chesnokov & Hipkin (1984).

There are fundamental differences in the propagation of seismic waves in isotropic and anisotropic media. In anisotropic media, there are two quasi-shear waves which can propagate at different phase velocities giving rise to shear wave splitting. In rapidly varying velocity regions, where both quasi-shear waves propagate with nearly the same phase velocity, strong coupling of these waves and polarization anomalies occur. In anisotropic media, the particle motion need not be normal (for $qP$-waves) or tangent (for $qS$-waves) to the wavefront, and the phase velocity is in general different in different directions. Phase and group velocities in anisotropic media can also diverge (see Musgrave 1970; Auld 1973; Crampin 1981).

In order to understand the practical aspects of anisotropy, computational methods must be developed to model these effects. In this paper, a perturbation approach based on the ray method is used. The ray method for heterogeneous anisotropic media is well known (see Červený 1972). Recently 3-D complete computer algorithms have also appeared (see Petrashen & Kashtan 1984; Gajewski & Pšenčík 1987). Although the codes based on these algorithms are much faster than those based on other methods for numerical modelling of seismic body wave propagation in anisotropic media, even faster, though approximate, approaches are desirable. Perturbation approaches seem to fit these requirements. Especially attractive are the approaches in which an isotropic medium is used as an initial unperturbed medium. Such perturbation approaches are justified for slight anisotropy which has most commonly been observed *in situ*. The density-normalized elastic parameters are in this case assumed to be of the form

$$a_{ijkl} = c_{ijkl}/\rho = a^{\mathrm{I}}_{ijkl} + \Delta a_{ijkl}$$

where $a^{\mathrm{I}}_{ijkl}$ are isotropic elastic parameters and $\Delta a_{ijkl}$ small anisotropic perturbations. An advantage of such approaches is that available ray codes for 2-D isotropic heterogeneous media can be used. The two-point ray tracing problem which usually complicates 3-D computations can be, in this way, reduced to 2-D two-point ray tracing, for which efficient algorithms and codes exist. On the other hand, the considered anisotropy must be strong enough to avoid $qS$-wave coupling. Since in this paper we concentrate on traveltime and ray path perturbations in slowly varying media this problem is not as serious.

For seismological applications of perturbation approaches, important contributions are those of Backus (1965), Romanov (1972), Červený (1982) and Hanyga (1982), who suggested linearized formulae for the determination of phase velocities and traveltimes of seismic body waves propagating in anisotropic media. Their formulae are universally applicable to all waves if the unperturbed medium is anisotropic. If the unperturbed medium is isotropic, their application to $qS$-waves is complicated by coincidence of the phase velocities of shear waves in isotropic media. Examples of applications of the above mentioned formulae can be found, e.g., in Červený & Jech (1982), Červený & Firbas (1984) and Firbas (1984). Perturbation approach for determination of rays and ray amplitudes in media with hexagonal symmetry was recently suggested by Farra (1989, 1990). In this case, as an initial unperturbed medium, a medium with elliptical anisotropy is required.

In most of the previously suggested perturbation approaches, either just $qP$-waves can be considered or an anisotropic medium must be used as the unperturbed medium. Exceptions are the results of Kravtsov & Orlov (1990) in electromagnetic wave propagation and Jech & Pšenčík (1989) in seismology. These results for slightly anisotropic media are obtained from results for nearby isotropic media.

For perturbations from isotropic to anisotropic media to be universally applicable for all waves, degenerate perturbation theory must be used (see Landau & Lifshitz 1974; Jech & Pšenčík 1989) which splits a single shear wave in the isotropic medium into two quasi-shear waves in the nearby anisotropic medium.

In this paper, an overview of perturbation formulae for a non-degenerate and degenerate eigenvalue problem are given in the appendices. Perturbation formulae for traveltime, ray paths and particle motion for $qP$- and $qS$-waves from isotropic to anisotropic media are then developed, again for both non-degenerate $qP$ and degenerate $qS$ cases. The perturbation approach is tested on an effectively anisotropic model of aligned cracks within a vertical velocity gradient, which was recently used by Shearer & Chapman (1988, 1989). Perturbed traveltimes, ray paths and particle motions of $qP$- and $qS$-waves computed for this anisotropic model from a nearby isotropic medium are compared with exact ray results.

Future applications of perturbation approach include forward modelling in slightly anisotropic media, including calculations of the geometric spreading, and thus, ray amplitudes, which can be obtained from perturbations of ray paths. Another application is the calculation of the sensitivity operators for tomographic studies in heterogeneous anisotropic media (see Hanyga 1982; Hirahara & Ishikawa 1984; Hirahara 1988; Jech 1988).

Most equations are written in component notation. Lower case indices take values 1, 2, 3 and upper case indices take values 1, 2. The Einstein summation convention is applied for indices other than $m$, $n$, $M$, $N$.

## 2 FORMULATION

### 2.1 Traveltime perturbation

In this section, the first-order perturbation of the traveltime in a heterogeneous medium resulting from the perturbation $\Delta a_{ijkl}$ of density normalized elastic parameters $a_{ijkl}$ is derived. The ray Hamiltonian can be written as $H = 1/2(G-1)$. For this choice of Hamiltonian, the integration parameter along the ray, $\tau$, is time $\tau = T$. The traveltime can be written as

$$T(\mathbf{x}) = \int_{\tau_0}^{\tau} L(x_i, \dot{x}_i, a_{ijkl})\, d\tau, \tag{1}$$

where the Lagrangian can be written as $L(x_i, \dot{x}_i, a_{ijkl}) = (p_i\dot{x}_i - H)$. $L$ is a function of $x_i$, $\dot{x}_i = dx_i/dt$, and the components of the normalized elastic parameters. Since $H = 0$ along the ray, the traveltime can be evaluated as $T(\mathbf{x}) = \int_{\mathbf{x}_0}^{\mathbf{x}} p_i\, dx_i$.

The first-order perturbation of the traveltime can be written $\Delta T = \int_{\tau_0}^{\tau} \Delta L\, d\tau$. From the ray equations (A1.7), this can be rewritten as

$$\Delta T = p_i\, \Delta x_i \,\big|_{\tau_0}^{\tau} - \int_{\tau_0}^{\tau} \Delta H\, d\tau. \tag{2}$$

For fixed endpoints, this can be written as

$$\Delta T = -\frac{1}{2}\int_{\tau_0}^{\tau} \Delta G\, d\tau. \tag{3}$$

For the non-degenerate, anisotropic case, equation (A2.1a) can be used to obtain $\Delta G$ in equation (3) (see Červený 1982; Hanyga 1982). Thus,

$$\Delta T = -\frac{1}{2}\int_{\tau_0}^{\tau} \Delta a_{ijkl} p_i p_l g_j^{(m)} g_k^{(m)}\, d\tau. \tag{4}$$

Equation (4) can even be used for the evaluation of the traveltime perturbation in degenerate cases, e.g. for $qS$-waves from an unperturbed, isotropic medium. In that case, the specially chosen vectors, $e_j^{(1)}$ and $e_j^{(2)}$ situated in the plane perpendicular to $g_i^{(3)}$, and chosen so that $D_{12} = 0$ [see equation (A2.4) for $D_{MN}$], must be used as eigenvectors in equation (4) [see discussion after equation (A2.4)]. An alternative procedure is to use in equation (3) the perturbation $\Delta G$ given by equation (A2.3). In this case, the mutually perpendicular vectors $e_j^{(1)}$, $e_j^{(2)}$ can be chosen arbitrarily in the plane perpendicular to $g_j^{(3)}$. Then, along the path of the unperturbed ray we get for the traveltime perturbation

$$\Delta T = -\frac{1}{4}\int_{\tau_0}^{\tau} \{(D_{11} + D_{22}) \pm [(D_{11} - D_{22})^2 + 4D_{12}^2]^{1/2}\}\, d\tau. \tag{5}$$

See Jech & Pšenčík (1989) as well as an analogous formula for electromagnetic waves in Kravtsov & Orlov (1990).

Equations (4) and (5) are first-order corrections to the traveltime evaluated along the unperturbed ray trajectories.

Higher order perturbations to the traveltime require corrections to the form of the ray trajectory itself.

### 2.2 Ray path and particle motion perturbations

In this section, perturbations of the ray trajectories and particle motions from an isotropic, heterogeneous initial medium to a heterogeneous medium with arbitrary anisotropy are performed. The perturbed ray trajectories due to either isotropic or anisotropic perturbation $\Delta a_{ijkl}$ can be written

$$\frac{d}{d\tau}\mathbf{X}(\tau)=\mathbf{A}(\tau)\mathbf{X}(\tau)+\Delta\mathbf{S}(\tau), \tag{6}$$

where

$$\mathbf{A}=\begin{pmatrix} H_{,p_ix_j} & H_{,p_ip_j} \\ -H_{,x_ix_j} & -H_{,x_ip_j}\end{pmatrix}, \qquad \Delta\mathbf{S}=\begin{pmatrix}\Delta H_{,p_i} \\ -\Delta H_{,x_i}\end{pmatrix},$$

$$\mathbf{X}(\tau)=\begin{pmatrix}\Delta x_i \\ \Delta p_i\end{pmatrix},$$

where $H$ denotes the ray Hamiltonian [for the isotropic case, in reduced notation, see Farra & Madariaga (1987); Nowack & Lutter (1988); Nowack & Lyslo (1989); Nowack (1990)]. $\Delta x_i$ and $\Delta p_i$ are perturbations of the ray trajectory and slowness vector. Here we use the non-reduced 6-dimensional form of the perturbed ray equations in Cartesian spatial and slowness coordinates. Explicit expressions for the components of the matrix $\mathbf{A}$ for an anisotropic case are given by Červený (1972) [for alternative forms see also Kendall & Thompson (1989) or Gajewski & Pšenčík (1990)].

For the case of an isotropic, unperturbed medium, which is studied in this paper, the explicit expressions for the components of the matrix $\mathbf{A}$ are

$$H_{,p_ip_j}=V^2\delta_{ij}, \qquad H_{,p_ix_j}=2VV_{,j}p_i,$$

$$H_{,x_ix_j}=\frac{1}{V^2}\frac{\partial V}{\partial x_i}\frac{\partial V}{\partial x_j}+\frac{1}{V}\frac{\partial^2 V}{\partial x_i\,\partial x_j}.$$

Since $H=0$ along the ray (see equation A1.6), the perturbations $\Delta x_i$ and $\Delta p_i$ of ray path and slowness must satisfy

$$H_{,x_i}\Delta x_i+H_{,p_i}\Delta p_i+\Delta H=0, \tag{7}$$

which is the perturbation of the eikonal equation. This constraint can be used to check the numerical accuracy of the solution of equations (6) along the ray. If the perturbation term $\Delta\mathbf{S}$ is zero, equations (6) reduce to the equations for paraxial rays, i.e., they give ray path perturbations about an existing ray trajectory in the unperturbed medium. A non-zero perturbation term $\Delta\mathbf{S}$ must be treated separately for the non-degenerate and degenerate cases.

In the non-degenerate case $\Delta H=1/2\,\Delta G$, where $\Delta G$ is given by equation (A2.1a). The derivatives $\Delta H_{,p_i}=1/2\,\partial B_{mm}/\partial p_i$ and $\Delta H_{,x_i}=1/2\,\partial B_{mm}/\partial x_i$ for the $m$th wave can then be expressed as follows:

$$\Delta H_{,p_s}=\Delta a_{sjkl}p_lg_j^{(m)}g_k^{(m)}+\Delta a_{ijkl}p_ip_l\frac{\partial g_j^{(m)}}{\partial p_s}g_k^{(m)}, \tag{8a}$$

$$\Delta H_{,x_s}=\frac{1}{2}\frac{\partial\,\Delta a_{ijkl}}{\partial x_s}p_ip_lg_j^{(m)}g_k^{(m)}+\Delta a_{ijkl}p_ip_l\frac{\partial g_j^{(m)}}{\partial x_s}g_k^{(m)}. \tag{8b}$$

This can be rewritten as

$$\Delta H_{,p_i}=\Delta a_{ijkl}p_lg_j^{(m)}g_k^{(m)}+a_{ijkl}p_l\times(\Delta g_j^{(m)}g_k^{(m)}+\Delta g_k^{(m)}g_j^{(m)}), \tag{9a}$$

$$\Delta H_{,x_i}=\frac{1}{2}\frac{\partial\,\Delta a_{sjkl}}{\partial x_i}p_sp_lg_j^{(m)}g_k^{(m)}+\frac{\partial a_{sjkl}}{\partial x_i}p_sp_l\,\Delta g_j^{(m)}g_k^{(m)}. \tag{9b}$$

To derive equations (9a) and (9b), we used the relation

$$\frac{\partial g_j^{(m)}}{\partial z_s}=\sum_{n=1}^{3(n\neq m)}(G_m-G_n)^{-1}\frac{\partial\Gamma_{il}}{\partial z_s}g_l^{(m)}g_i^{(n)}g_j^{(n)}, \tag{10}$$

(see Gajewski & Pšenčík 1990), where $z_s$ may be either $x_s$ or $p_s$. Further, we used equation (A2.1b) for perturbations $\Delta g_j^{(m)}$ of the polarization vectors.

In the degenerate case $\Delta H=1/2\,\Delta G$, where $\Delta G$ is given by equation (A2.3). In this case

$$\Delta H_{,z_s}=\frac{1}{4}\left(\frac{\partial D_{11}}{\partial z_s}+\frac{\partial D_{22}}{\partial z_s}\right)\pm\frac{1}{4}\times\frac{(D_{11}-D_{22})\left(\frac{\partial D_{11}}{\partial z_s}-\frac{\partial D_{22}}{\partial z_s}\right)+4D_{12}\frac{\partial D_{12}}{\partial z_s}}{[(D_{11}-D_{22})^2+4D_{12}^2]^{1/2}}, \tag{11}$$

where, again, $z_s$ stands either for $x_s$ or $p_s$. Differentiating equation (A2.4) with respect to $p_s$ and $x_s$ we get

$$\frac{\partial D_{mn}}{\partial p_s}=(\Delta a_{sjkl}p_l+\Delta a_{ijks}p_i)e_j^{(m)}e_k^{(n)}+\Delta a_{ijkl}p_ip_l\times\left(\frac{\partial e_j^{(m)}}{\partial p_s}e_k^{(n)}+\frac{\partial e_k^{(n)}}{\partial p_s}e_j^{(m)}\right),$$

$$\frac{\partial D_{mn}}{\partial x_s}=\frac{\partial\,\Delta a_{ijkl}}{\partial x_s}p_ip_le_j^{(m)}e_k^{(n)}+\Delta a_{ijkl}p_ip_l\times\left(\frac{\partial e_j^{(m)}}{\partial x_s}e_k^{(n)}+e_j^{(m)}\frac{\partial e_k^{(n)}}{\partial x_s}\right), \tag{12}$$

where, using indices $m$, $n$ instead of $M$, $N$, we include also derivatives of $D_{33}$ with $e_j^{(3)}=g_j^{(3)}$.

Several approaches can be used to evaluate equation (11). In the approach given here, the specially chosen vectors $e_j^{(1)}$ and $e_j^{(2)}$ such that $D_{12}=0$ are used. Equation (11) then reduces to $\Delta H_{,z_s}=1/2\,\partial D_{11}/\partial z_s$ or $\Delta H_{,z_s}=1/2\,\partial D_{22}/\partial z_s$ and this can be expressed as

$$\Delta H_{,p_s}=\Delta a_{sjkl}p_le_j^{(m)}e_k^{(m)}+\Delta a_{ijkl}p_ip_l\frac{\partial e_j^{(m)}}{\partial p_s}e_k^{(m)}, \tag{13a}$$

$$\Delta H_{,x_s}=\frac{1}{2}\frac{\partial\,\Delta a_{ijkl}}{\partial x_s}p_ip_le_j^{(m)}e_k^{(m)}+\Delta a_{ijkl}p_ip_l\frac{\partial e_j^{(m)}}{\partial x_s}e_k^{(m)}, \tag{13b}$$

which are the same as equations (8a) and (8b) only instead of $g_j^{(m)}$, specially chosen vectors $e_j^{(m)}$ are used. This can then be further reduced to a form equivalent to equations (9a) and (9b) with the perturbations of the eigenvectors, $\Delta g_j^{(m)}$, given by equation (A2.5). For $m = 1$, this is shown by substituting $\partial e_j^{(1)}/\partial z_s = \alpha e_j^{(2)} + \beta e_j^{(3)}$ into (13), and noting that $D_{12} = \Delta a_{ijkl} p_i p_l e_j^{(1)} e_k^{(2)} = 0$ and $\beta = (G_1 - G_3)^{-1} \partial\Gamma_{il}/\partial z_s e_l^{(3)} e_i^{(1)}$. For the specially chosen vectors $e_j^{(1)}$ and $e_j^{(2)}$ such that $D_{12} = 0$, this has the advantage of using just $\Delta g_j^{(m)}$ from (A2.5) and not explicitly requiring the derivatives of $e_j^{(m)}$ with respect to $x_s$ or $p_s$.

Once the derivatives, $\Delta H_{,z_s}$, are specified, equation (6) can be solved along the unperturbed ray path. It is very convenient to use the propagator formalism for the solution of (6). Let $\mathbf{X} = (\Delta x_i, \Delta p_i)^{\mathrm{T}}$ be the perturbation of the reference ray; then it can be determined from

$$\mathbf{X}(\tau) = \mathbf{P}_0(\tau, \tau_0)\mathbf{X}(\tau_0) + \int_{\tau_0}^{\tau} \mathbf{P}_0(\tau, \tau') \Delta\mathbf{S}(\tau')\, d\tau, \tag{14a}$$

or

$$\mathbf{X}(\tau) = \mathbf{P}_0(\tau, \tau_0)\mathbf{X}(\tau_0) + \mathbf{X}^{\mathrm{Int}}(\tau), \tag{14b}$$

where $\mathbf{P}_0(\tau, \tau_0)$ is the $6 \times 6$ isotropic ray propagator corresponding to equation (6) without the source term. $\mathbf{X}^{\mathrm{Int}}(\tau)$ represents the integral in (14a). The perturbation vector $\mathbf{X}(\tau) = [\Delta x_i(\tau), \Delta p_i(\tau)]^{\mathrm{T}}$ must satisfy equation (7) along the ray.

Equations (14b) can be used for solving different initial or boundary-value problems (see Farra 1989). For example, they can be used to approximately solve two-point ray tracing for perturbed rays. Here we consider the case of identical 'source' points for the initial unperturbed rays, i.e. we require

$$\Delta x_i(\tau_0) = 0, \tag{15a}$$

where $\tau_0$ corresponds to the source point of the unperturbed ray.

The next relation for the two-point ray ensures that the perturbed ray passes through the receiver point. We must take into account the fact that the value of the integration parameter $\tau$ corresponding to the point of incidence of the perturbed ray at the receiver surface may not coincide with the value of $\tau$ corresponding to the unperturbed ray at the receiver surface. This is a problem which also arises whenever a perturbed ray intersects an interface (see Farra, Virieux & Madariaga 1989; Farra 1989). We follow the approach taken by Farra (1989). Let us denote by $dX_i$ the distance from the prescribed receiver point of the perturbed ray to the intersection of the unperturbed ray with the receiver surface. Further, let us denote by $n_i$ the unit normal to the receiver surface at the point of incidence of the unperturbed ray, and by $s_i$ the derivative $dx_i/d\tau$ taken at the same point. We can then write to first order,

$$dX_i \simeq \Delta x_i(\tau) - \frac{n_j \Delta x_j(\tau)}{n_k s_k(\tau)} s_i(\tau). \tag{15b}$$

The case $dX_i = 0$ would result for coincident receiver points of the unperturbed and perturbed rays. Alternatively, $dX_i$ can be specified to find the perturbed ray to a different receiver point along the surface specified by local normal, $n_i$. Equations (15a, b) and (7) can be used along with (14b) to obtain the perturbed slowness at the source, where $\mathbf{X}(\tau_0) = [0, \Delta p_i(\tau_0)]^{\mathrm{T}}$. Equations (14b) can then be used to find the approximate two-point ray.

The perturbed particle motion vectors finally need to be computed along the perturbed ray trajectory. The perturbed particle motion vectors given by equation (A2.5) are for positions, $x_i^0$, and directions along the initial, isotropic ray path. In order to compute the particle motion vectors along the new perturbed ray trajectory, the following expansion will be used

$$g_i(x_i, p_i, a_{ijkl}) \simeq g_i(x_i^0, p_i^0, a_{ijkl}^0) + \Delta g_i, \tag{16}$$

where $\Delta g_i$ is given by equation (A2.5), in which

$$\Delta\Gamma_{jk} = (\partial a_{ijkl}^0/\partial x_m\, \Delta x_m + \Delta a_{ijkl}) p_i p_l + 2a_{ijkl}^0\, \Delta p_i p_l,$$

where the partial derivatives of the elastic parameters with respect to $x_m$ are taken at $x_i^0$. $\Delta x_i$ and $\Delta p_i$ are obtained from equations (14) with appropriate boundary conditions for the approximate two-point ray. $a_{ijkl}^0$ represents the normalized elastic parameters for the initial unperturbed medium. The particle motions in (16) along the approximate two-point ray in the perturbed medium are thus approximately evaluated along the initial, unperturbed ray trajectory.

In a way similar to obtaining equations (14), we can obtain the solution for perturbed paraxial rays (see e.g., Farra *et al.* 1989; Farra 1989) which are important for evaluation of ray amplitudes along the perturbed reference ray. The equations for perturbed paraxial rays read

$$\begin{pmatrix} \delta x_i \\ \delta p_i \end{pmatrix} \approx \int_{\tau_0}^{\tau} \mathbf{P}_0(\tau, \tau')\, \boldsymbol{\delta}\mathbf{A}(\tau')\mathbf{P}_0(\tau', \tau_0)\, d\tau',$$

where $\boldsymbol{\delta}\mathbf{A}$ is the change in the linearized $\mathbf{A}$ matrix given in equation (6). An alternative procedure is to compute the paraxial rays directly along the new ray trajectories once the perturbed reference rays have been computed.

## 3 EXAMPLES

In this section, several examples are given in order to test the above perturbation formulation from isotropic to anisotropic media. The model used is an effectively anisotropic model of aligned cracks of Hudson (1980). Traveltimes, ray trajectories and particle motions in an anisotropic solid with a linear, vertical velocity gradient are computed by perturbation of an isotropic solid and compared with exact ray calculations.

The specific model is taken from Shearer & Chapman (1988) with elastic parameters of the background rock at the reference depth [see equation (4) of Shearer & Chapman (1988)] of $V_p = 4.5\ \mathrm{km\,s^{-1}}$, $V_s = 2.53\ \mathrm{km\,s^{-1}}$, and $\rho = 2.8\ \mathrm{g\,cm^{-3}}$ and aligned, water-filled cracks with aspect ratio of $d = 0.001$ and crack density of $\varepsilon = 0.1$. The resulting density-normalized elastic parameters for the reference depth are $a_{1111} = 20.04$, $a_{2222} = 20.22$, $a_{1212} = 5.10$, $a_{2323} = 6.38$ and $a_{1122} = 7.41\ \mathrm{km^2\,s^{-2}}$ with a (100) hexagonal symmetry axis.

Figure 1 shows the slowness cross-section in the symmetry plane of the above anisotropic solid rotated 30° about the (010) axis. The sections are taken at the reference depth of 4.405 km. The initial, isotropic model is chosen to minimize in a least-squares sense the difference between the

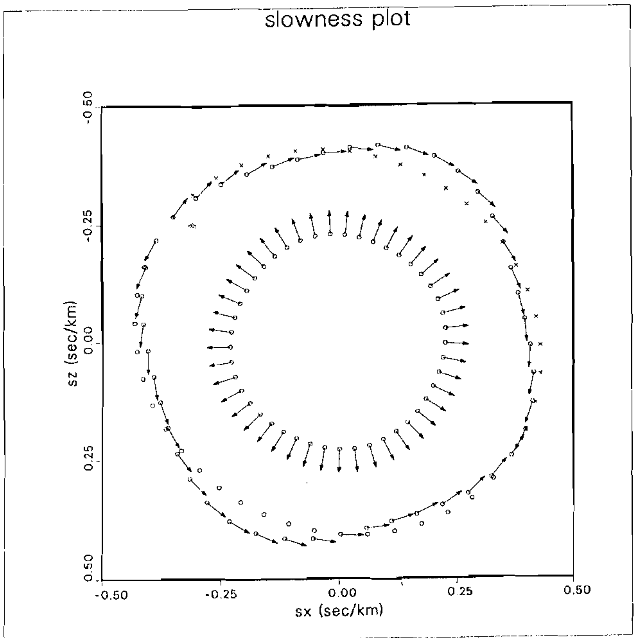


**Figure 1.** Slowness sheet cross-sections in the symmetry plane and particle motions. The anisotropic medium is an aligned crack model of Hudson (1980) with water-filled cracks with aspect ratio $d = 0.001$ and crack density, $\varepsilon = 0.1$. The host rock has $\alpha = 4.5\,\mathrm{km\,s^{-1}}$, $\beta = 2.53\,\mathrm{km\,s^{-1}}$, $\rho = 2.8\,\mathrm{g\,cm^{-3}}$. This is then rotated 30° about the (010) axis. The particle motions are projected onto the $sx$–$sz$ plane with an 'x' indicating particle motion into the page and a circle indicating particle motion out of the page.

unperturbed and the perturbed elastic constants in reduced form (i.e. the 21 independent values). However, other methods of choosing the initial model are possible. Using the notation of Crampin (1981), the $qSR$-wave is the wave with out-of-plane particle motion and the $qSP$-wave is the wave with in-plane particle motion in Fig. 1. The symmetry axis is 30° from the horizontal $x_1$ axis, and a kiss singularity for the $qS$-waves occurs along the symmetry axis. Also, for several directions, the $qS$-waves intersect, but with the particle motion changing continuously. Along the symmetry axis, the $qS$-wave particle motions change sign abruptly, with the $qSR$ particle motions rotating about the symmetry axis and the $qSP$ particle motions emanating or converging upon the symmetry axis [see fig. 12 of Chapman & Shearer (1989)]. The symmetry axis acts as a centre of rotation for $qSR$ particle motion and a point of convergence/divergence for the $qSP$. The $qP$-wave anisotropy [defined by $(V_{max} - V_{min})/V_{avg}$] is 3.5 per cent and the $qSP$- and $qSR$-wave anisotropy is 11.2 per cent (see Shearer & Chapman 1989). The approximate slowness cross-section and particle motions found using the degenerate perturbation theory overlay the exact values.

For a model consisting of a linear, vertical velocity gradient in an anisotropic medium, Shearer & Chapman (1988) observed that rays map out the same shape in vertical depth-section as the slowness cross-section rotated by 90°. This gives physical insight, as well as allowing for the tracing of rays in anisotropic media with linear velocity gradients to be performed by the calculation of the slowness cross-section, and two additional integrals for the out-of-plane ray deviation and traveltime. However, in the following examples, the exact rays are found by direct numerical integration of equations (A1.7).

For the following examples, the vertical velocity gradient is approximately $1.0\,\mathrm{s^{-1}}$ and the reference depth is 4.405 km. The same anisotropy as in Fig. 1 is considered. The rays are traced in the symmetry plane shown in Fig. 1 and the exact and perturbed ray trajectories are shown in Fig. 2. Their ray shapes can be seen in the 90° rotated slowness diagrams in Fig. 1. For this case no out-of-plane propagation occurs.

The exactly traced ray trajectories are then compared with the perturbed ray trajectories using the formulation of Section 2. Fig. 2(a) shows the ray trajectories and particle motions for the $qSP$-wave. The solid line is the exactly traced ray in the anisotropic, linear gradient. The dotted line is the initial unperturbed ray traced in the nearby isotropic, linear gradient, and the dashed line is the perturbed ray trajectory derived from the isotropic ray. The perturbed ray is computed using equation (14a) specified to perform approximate two-point ray tracing. The perturbed particle motions are computed using equation (16). Figs 2(b) and (c) show similar results for the $qSR$- and $qP$-waves. For each of these cases, there is very good agreement between the exact and perturbed ray trajectories and particle motions.

In Fig. 3, the exact and perturbed traveltimes are

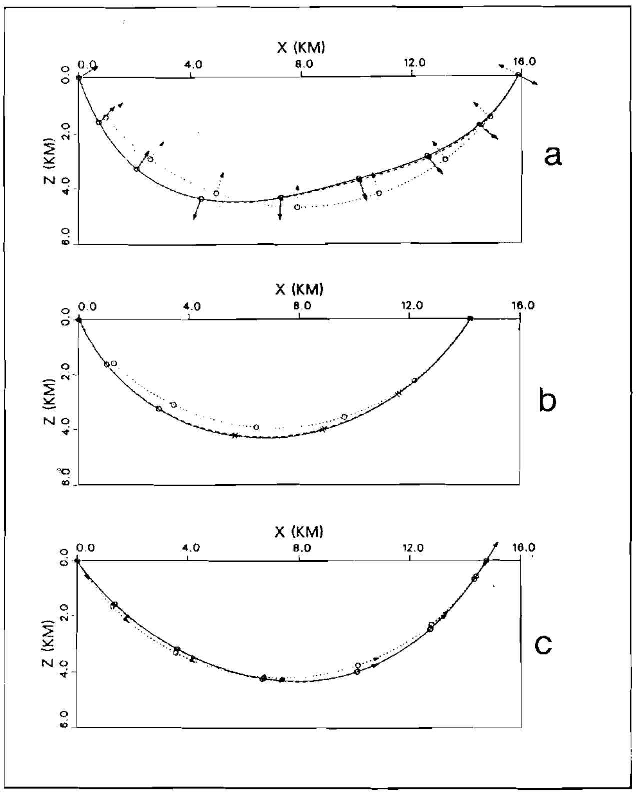


**Figure 2.** Ray trajectories for an anisotropic medium with vertical, linear velocity gradient with the anisotropy of Fig. 1. (a) $qSP$-wave, (b) $qSR$-wave, (c) $qP$-wave. For each case, the solid line is the exactly computed ray in the anisotropic medium with constant vertical gradient. The dotted line is the initial ray in the nearby isotropic, linear gradient and the dashed line is the perturbed ray. Particle motions are projected onto the $x$–$z$ planes with an 'x' indicating particle motion into the page and a circle indicating particle motion out of the page.

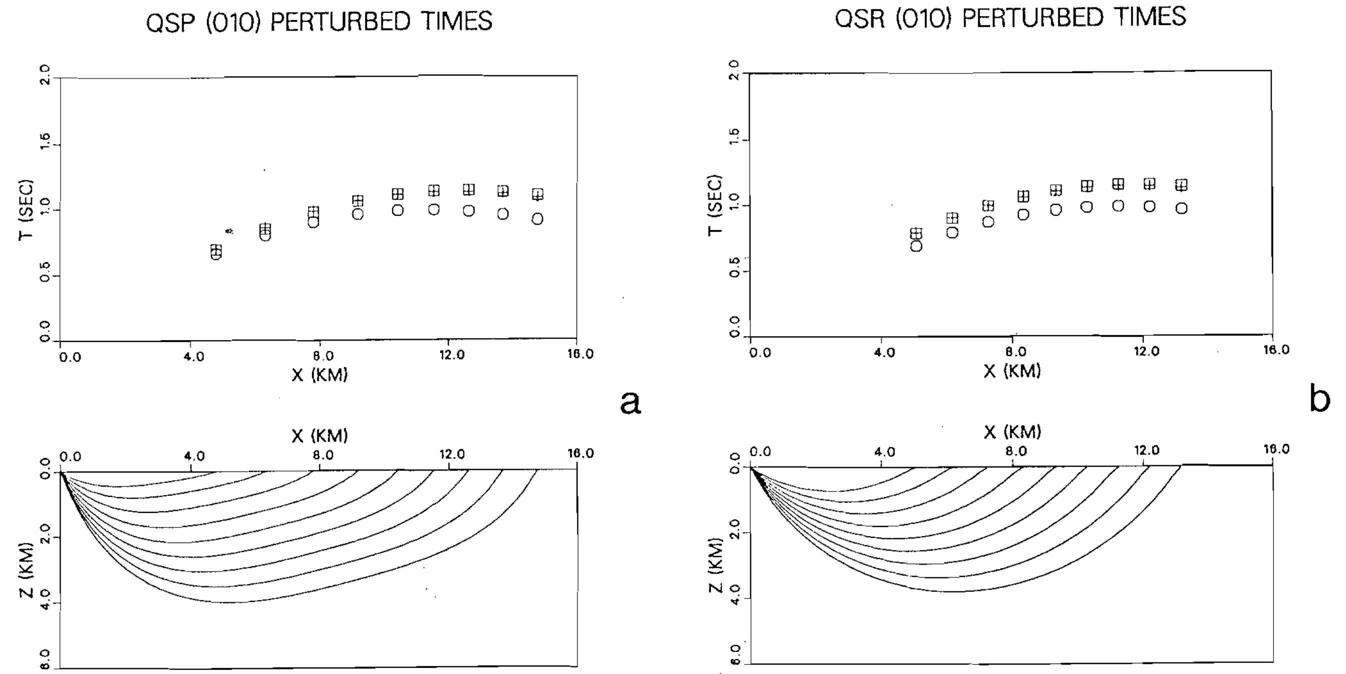


**Figure 3.** Traveltimes for an anisotropic medium with constant vertical, linear velocity gradient with the anisotropy of Fig. 1. (a) $qSP$-wave traveltimes reduced by $4\,\mathrm{km\,s^{-1}}$, (b) $qSR$-wave traveltimes reduced by $4\,\mathrm{km\,s^{-1}}$. For each case, the perturbed rays are shown below. The squares are the exactly computed traveltimes. The circles are the initial isotropic traveltimes, and the plus symbols are the perturbed traveltimes.

compared. The perturbed ray trajectories are shown below each traveltime frame. The first-order perturbed traveltimes are computed using equation (4) with the $e_j^{(m)}$ specially chosen such that $D_{12}=0$. Fig. 3(a) shows the traveltimes reduced by $4\,\mathrm{km\,s^{-1}}$ for the $qSP$-wave in a vertical velocity gradient with anisotropy shown in Fig. 1. The squares are the exactly computed traveltimes. The circles are the initial traveltimes for the nearby isotropic linear gradient, and the plus symbols are the perturbed traveltimes. Fig. 3(b) shows the traveltime reduced by $4\,\mathrm{km\,s^{-1}}$ for the $qSR$-wave. In both cases, the relative errors in the perturbation with respect to the difference between the exact and isotropic traveltimes are less than 9 per cent. Further work is required in computing perturbed traveltime estimates higher than first order.

In the final example, the aligned crack model is rotated 30° about the (001) axis giving a cross-section which is no longer in the symmetry plane. The constant vertical velocity gradient above is again assumed. The slowness cross-section at the reference depth of 4.405 km is shown in Fig. 4. The perturbed slowness cross-section and particle motions, from an initial unperturbed isotropic model, overlay the exact slowness cross-section and particle motions for this case.

Figure 5 shows the ray trajectories in the vertical velocity gradient in the anisotropic medium shown in Fig. 4 at the reference depth. For this case, out-of-plane ray propagation occurs. Fig. 5(a) shows the ray trajectory of the $qS$-wave which is associated with the $qSP$-wave branch from the symmetry plane. Both a map view, showing out-of-plane propagation, and a vertical cross-section are given. The solid line is the exactly computed ray trajectory. The dotted line is the initial unperturbed ray for a nearby isotropic, linear velocity gradient, and the dashed line is the perturbed ray trajectory. Fig. 5(b) shows similar results for the $qS$-wave associated with the $qSR$-wave branch from the symmetry

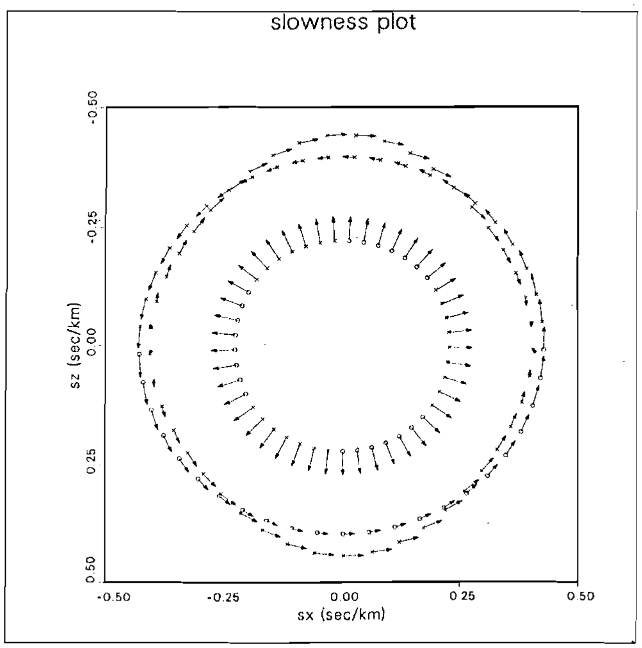


**Figure 4.** (a) Slowness sheet cross-sections and particle motions for aligned crack model rotated 30° about (001). The particle motions are projected onto the $sx$–$sz$ plane with an 'x' indicating particle motion into the page and a circle indicating particle motion out of the page.

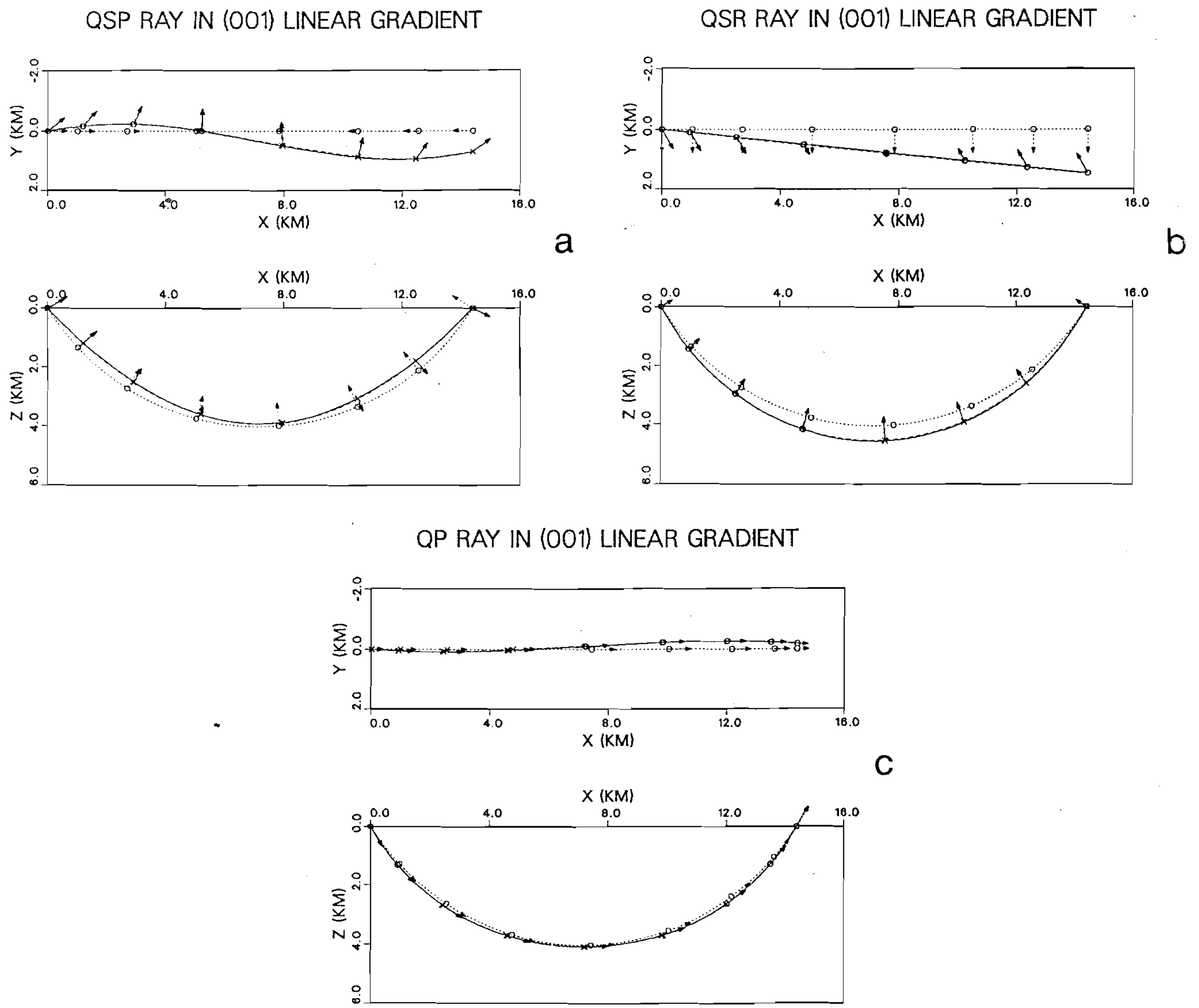


**Figure 5.** Ray trajectories for an anisotropic medium with vertical, linear velocity gradient with the anisotropy of Fig. 4. (a) *qS*-wave associated with the *qSP*-wave from the symmetry plane, (b) *qS*-wave associated with the *qSR*-wave from the symmetry plane, (c) *qP*-wave. For each case, a map-view is given, showing out-of-plane propagation, and a vertical cross-section. The solid lines are the exactly computed rays, the dotted lines are the rays for the initial isotropic, linear gradient, and the dashed lines are the perturbed rays. Particle motions are projected onto the $x$–$z$ and $x$–$y$ planes with an 'x' indicating particle motion into the page and a circle indicating particle motion out of the page.

plane, and Fig. 5(c) shows results for the *qP*-wave. For these cases, the perturbed ray is computed to terminate at the endpoints of the exact anisotropic rays. Excellent agreement is found between the exact and perturbed ray trajectories, for both the in-plane and out-of-plane propagation, as well as for the particle motions.

In these several examples, only first-order perturbations to the ray trajectories were used. Higher order corrections could be directly incorporated within the perturbation formulation. For larger perturbations, an iterative approach could be used. For the simple anisotropic media described here, excellent agreement has been found between the exact and perturbed ray trajectories and particle motions. This provides the possibility of using available 2-D isotropic ray tracing codes to obtain more general 3-D ray tracing results in slightly anisotropic media by perturbation.

## 4 CONCLUSIONS

In this paper, perturbation methods are used to obtain ray theoretical results specifically, traveltimes, ray paths and particle motions, in anisotropic media from nearby isotropic media. The advantage of our approach is that rays computed by available 2-D isotropic ray-tracing algorithms can be used to approximately compute, by perturbation, ray theoretical results in more general 3-D, slightly anisotropic media. The sensitivity of the traveltime and ray path in anisotropy can also be obtained from the calculation of the anisotropic partial derivatives. In order to compute approximate ray results for quasi-shear waves, degenerate perturbation theory was used in order to separate the isotropic *S*-wave components.

In order to test the perturbation results, an anisotropic medium with constant vertical gradient is assumed. An exact 3-Day ray numerical calculation is first compared with the analytical results of Shearer & Chapman (1988) for a cracked, anisotropic solid. The perturbed ray trajectories are obtained by a propagator approach with an equivalent source related to the degree of anisotropy from an initial isotropic model. Good agreement between the exact and perturbed ray trajectories and traveltimes are found for the simple anisotropic media described here. However, further work is required to analyse higher order corrections for both traveltimes and ray trajectories.

The results given in this paper apply to perturbation methods within the ray approximation. Their extension to amplitude and waveform computation will, therefore suffer from the well-known limitations of the ray method. An important limitation of the ray method in anisotropic media is its failure to describe properly quasi-shear wave coupling. Extension of the results presented in this paper will, therefore, be applicable provided the anisotropy is strong enough to avoid $qS$-wave coupling. Recently, Chapman & Shearer (1989) presented connection formulae for $qS$-wave coupling to correct the ray approximation at finite frequencies. Further work is required to apply this theory to the perturbation formulation given in this paper.

## ACKNOWLEDGMENTS

One of the authors (RLN) was partially supported during this work as a visiting scientist at the Institut de Physique du Globe, Paris France, as well as by NSF Grant No. EAR-8904169. The second author (IP) was partially supported during this work as a visiting scientist at the Center for Computational Seismology, Lawrence Berkeley Laboratory, University of California, CA. The authors wish to thank the two reviewers for useful and constructive comments on the original manuscript.

## REFERENCES

Aki, K. & Richards, P., 1980. *Quantitative Seismology, Theory and Methods*, Freeman, San Francisco.

Auld, B. A., 1973. *Acoustic Fields and Waves in Solids*, Wiley, New York.

Babuška, V., 1984. *P*-wave anisotropy in crystalline rocks, *Geophys. J.R. astr. Soc.*, **76,** 113–119.

Backus, G. E., 1965. Possible forms of seismic anisotropy of the upper most mantle under oceans, *J. geophys. Res.*, **70,** 3429–3439.

Backus, G. E., 1982. Reply: Limits of validity of first-order perturbation theory for quasi-*P* velocity in weakly anisotropic media, *J. geophys. Res.*, **87,** 4641–4644.

Červený, V., 1972. Seismic rays and ray intensities in inhomogeneous anisotropic media, *Geophys. J.R. astr. Soc.*, **29,** 1–13.

Červený, V., 1982. Direct and inverse kinematic problem for inhomogeneous anisotropic media—linearization approach, *Contr. Geophys. Inst. Slov. Acad. Sci.*, **13,** 127–133.

Červený, V. & Firbas, P., 1984. Numerical modelling and inversion of traveltimes of seismic body waves in inhomogeneous anisotropic media, *Geophys. J.R. astr. Soc.*, **76,** 41–51.

Červený, V. & Jech, J., 1982. Linearized solutions of kinematic problems of seismic body waves in inhomogeneous slightly anisotropic media, *J. Geophys.*, **51,** 96–104.

Čereny, V., Molotkov, I. A. & Pšenčík, I., 1977. *Ray Method in Seismology*, Charles University Press, Prague.

Chapman, C. H. & Shearer, P. M., 1989. Ray tracing in azimuthally anisotropic media—II. Quasi-shear wave coupling, *Geophys. J.*, **96,** 65–83.

Christensen, N. I. & Salisbury, M. H., 1979. Seismic anisotropy in the oceanic upper mantle: evidence from the Bay of Islands ophiolite complex, *J. geophys. Res.*, **84,** 4601–4610.

Crampin, S., 1981. A review of wave motion in anisotropic and cracked elastic media, *Wave Motion*, **3,** 343–391.

Crampin, S., 1982. Comments on 'Possible forms of anisotropy of the uppermost mantle under oceans' by George E. Backus, *J. geophys. Res.*, **87,** 4636–4640.

Crampin, S. & Yedlin, M., 1981. Shear-wave singularities of wave propagation in anisotropic media, *J. Geophys.*, **49,** 43–46.

Crampin, S., Chesnokov, E. M. & Hipkin, R. A., 1984. Seismic anisotropy—state of the art: II, *Geophys. J.R. astr. Soc.*, **76,** 1–16.

Farra, V., 1989. Ray perturbation theory for heterogeneous hexagonal anisotropic medium, *Geophys. J.*, **99,** 723–738.

Farra, V., 1990. Amplitude computation in heterogeneous media by ray perturbation theory: a finite element approach, *Geophys. J. Int.*, **103,** 341–354.

Farra, V. & Madariaga, R., 1987. Seismic waveform modeling in heterogeneous media by ray perturbation theory, *J. geophys. Res.*, **92,** 2697–2712.

Farra, V., Virieux, J. & Madariaga, R., 1989. Ray perturbation theory for interfaces, *Geophys. J.*, **99,** 377–390.

Firbas, P., 1984. Traveltime curves for complex inhomogeneous slightly anisotropic media, *Studia geoph. geod.*, **28,** 393–406.

Fuchs, K., 1983. Recently formed elastic anisotropy and petrologic models for the continental subcrustal lithosphere in Southern Germany, *Phys. Earth planet. Inter.*, **31,** 93–118.

Gajewski, D. & Pšenčík, I., 1987. Computation of high-frequency seismic wavefields in 3-D lateraly inhomogeneous anisotropic media, *Geophys. J.R. astr. Soc.*, **91,** 383–411.

Gajewski, D. & Pšenčík, I., 1990. Vertical seismic profile synthetics by dynamic ray tracing in laterally varying layered anisotropic structures, *J. geophys. Res.*, **95,** 11 301–315.

Hanyga, A., 1982. The kinematic inverse problem for weakly laterally inhomogeneous anisotropic media, *Tectonophysics*, **90,** 253–262.

Hess, H. H., 1964. Seismic anisotropy of the uppermost mantle under oceans, *Nature*, **203,** 629–631.

Hirahara, K., 1988. Detection of three-dimensional velocity anisotropy, *Phys. Earth planet. Inter.*, **51,** 71–85.

Hirahara, K. & Ishikawa, Y., 1984. Travel-time inversion for three-dimensional *P*-wave velocity anisotropy, *J. Phys. Earth*, **32,** 197–218.

Hudson, J. A., 1980. Overall properties of a cracked solid, *Math. Proc. Camb. Phil. Soc.*, **88,** 371–384.

Jech, J., 1988. Three-dimensional inverse problem for inhomogeneous transversely isotropic media, *Studia geoph. geod.*, **32,** 136–143.

Jech, J. & Pšenčík, I., 1989. First-order perturbation method for anisotropic media, *Geophys. J.*, **99,** 369–376.

Kendall, J. M. & Thomson, C. J., 1989. A comment on the form of the geometrical spreading equations, with some numerical examples of seismic ray tracing in inhomogeneous anisotropic media, *Geophys. J.*, **99,** 401–414.

Kravtsov, Yu. A. & Orlov, Yu. I., 1990. *Geometrical Optics of Inhomogeneous Media*, Springer-Verlag, Berlin.

Landau, L. D. & Lifshitz, E. M., 1974. *Quantum Mechanics—Nonrelativistic theory*, Pergamon Press, Oxford.

Mathews, J. & Walker, R. L., 1970. *Mathematical Methods of Physics*, Benjamin/Cummings, Menlo Park, CA.

Messiah, A., 1962. *Quantum Mechanics*, Wiley, New York.

Musgrave, M. J. P., 1970. *Crystal Acoustics*, Holden-Day, London.

Nowack, R. L., 1990. Perturbation methods for rays and beams, in *Computational Acoustics*, vol. 2, pp. 167–180, eds Lee, D., Cakmak, A. & Vichnevetsky, R., North-Holland, Amsterdam.
Nowack, R. L. & Lutter, W. J., 1988. Linearized rays, amplitude and inversion, *Pure appl. Geophys.*, **128,** 401–421.
Nowack, R. L. & Lyslo, J., 1989. Frechet derivatives for curved interfaces in the ray approximation, *Geophys. J.*, **97,** 497–509.
Petrashen, G. I. & Kashtan, B. M., 1984. Theory of body wave propagation in inhomogeneous anisotropic media, *Geophys. J.R. astr. Soc.*, **76,** 29–39.
Romanov, V. V., 1972. *Some Inverse Problems for Hyperbolic Equations,* Nauka, Novosibirsk (in Russian).
Schiff, L. I., 1968. *Quantum Mechanics*, McGraw-Hill, New York.
Shearer, P. M. & Orcutt, J. A., 1985. Anisotropy in the oceanic lithosphere—theory and observations from the Ngendei seismic refraction experiment in southwest Pacific, *Geophys. J.R. astr. Soc.*, **80,** 493–526.
Shearer, P. M. & Orcutt, J. A., 1986. Compressional and shear wave anisotropy in the oceanic lithosphere—the Ngendei seismic refraction experiment, *Geophys. J.R. astr. Soc.*, **87,** 967–1003.
Shearer, P. M. & Chapman, C. H., 1988. Ray tracing in anisotropic media with a linear gradient, *Geophys. J.*, **94,** 575–580.
Shearer, P. M. & Chapman, C. H., 1989. Ray tracing in azimuthally anisotropic media—I. Results for models of aligned cracks in the upper crust, *Geophys. J.*, **96,** 51–64.
Stephen, R. A., 1985. Seismic anisotropy in the upper oceanic crust, *J. geophys. Res.*, **90,** 11 383–11 396.

## APPENDIX A

In this appendix, the basic ray equations for an anisotropic medium are given. The elastic equation of motion can be written (see Aki & Richards 1980, equation 2.17)

$$\rho\frac{\partial^2 u_i}{\partial t^2}=\frac{\partial}{\partial x_j}\sigma_{ij}+f_i. \tag{A1.1}$$

For a linear anisotropic solid, $\sigma_{ij}=c_{ijkl}e_{kl}$, where $\sigma_{ij}$ is the stress tensor, $e_{kl}$ is the strain tensor and $c_{ijkl}$ are the elastic parameters. For infinitesimal strain, $e_{kl}=1/2(u_{k,l}+u_{l,k})$. Using symmetry of the stress and strain and energetic considerations, it is possible to show that $c_{ijkl}(x_j)$ contains 21 independent elastic parameters at each point for a general, linear anisotropic solid. The body force $f_i$ is assumed to act in a limited source region.

We seek an approximate solution of equation (A1.1) outside the source region in the form of a leading term of a ray series (see Červený, Molotkov & Pšenčík 1977):

$$u_i(x_j,t)=A(x_j)g_i(x_j)e^{i\omega[t-\tau(x_j)]}, \tag{A1.2}$$

where $A(x_j)$ is a scalar amplitude factor, $\tau(x_j)$ is the phase function and $g_i$ is a unit particle motion vector. Substituting this solution form into (A1.1) results in the basic system of equations of the ray method for inhomogeneous anisotropic media (see Červený 1972; Gajewski & Pšenčík 1987).

Computations of traveltimes and ray paths are controlled by the first equation of the basic system,

$$(\Gamma_{jk}-\delta_{jk})g_j=0, \tag{A1.3}$$

where $\Gamma_{jk}=a_{ijkl}p_ip_l$, $a_{ijkl}=c_{ijkl}/\rho$, and $p_i=\partial\tau/\partial x_i$. Given the direction of the normal to the wavefront, $N_i$, (i.e. specifying the direction of the slowness vector $p_i=N_i/V$, where $V$ is the phase velocity), the particle motion vectors $g_i$ and the phase speeds $V$ of the three waves propagating in an anisotropic solid can be obtained by solving (A1.3). One of the three waves is quasi-compressional, the two remaining are quasi-shear. Equation (A1.3) closely resembles the characteristic equation for the eigenvalues $G$ and the eigenvectors $g_i$ of the matrix $\Gamma_{jk}$,

$$(\Gamma_{jk}-G\delta_{jk})g_j=0. \tag{A1.4}$$

This equation is called the Christoffel equation. The matrix $\Gamma_{jk}$ has three eigenvalues corresponding to the three waves which can propagate in an anisotropic medium. By comparing (A1.3) and (A1.4), we can see that the system (A1.3) has non-trivial solution only if the eigenvalue corresponding to the considered wave is equal to one,

$$G(x_j,p_i)=1, \tag{A1.5}$$

and the particle motion vectors $g_i$ correspond to the eigenvectors. The particle motion vectors are not in general parallel or perpendicular to the corresponding slowness vectors $p_i$. Equation (A1.5) is the eikonal equation for the considered wave.

If the three eigenvalues of the matrix $\Gamma_{jk}$ are mutually different, the corresponding eigenvectors can be uniquely determined. Such a case is referred to as non-degenerate. In case of the coincidence of two eigenvalues corresponding to quasi-shear waves, which we refer to as the degenerate case, the eigenvectors corresponding to the coinciding eigenvalues cannot be determined uniquely. They can be chosen as any two mutually perpendicular vectors situated in the plane perpendicular to the third eigenvector (corresponding to the $qP$-wave). In anisotropic media, such a situation occurs only for some special, singular directions, in which the wavesheets of the corresponding waves have common points (see Crampin & Yedlin 1981). In isotropic media, this situation is universal and thus the isotropic medium is universally degenerate.

The ray Hamiltonian, $H$, satisfies, due to equation (A1.5),

$$H=1/2(G-1)=0. \tag{A1.6}$$

For an anisotropic medium, this reads

$$1/2(a_{ijkl}p_ip_lg_jg_k-1)=0,$$

where $p_i$ and $g_i$ are the slowness vector and particle motion vector corresponding to the considered wave. For an isotropic medium, $a_{ijkl}=\lambda/\rho\delta_{ij}\delta_{kl}+\mu/\rho(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})$, and equation (A1.6) reads

$$1/2(V^2p_ip_i-1)=0,$$

where $V=V_p=\sqrt{(\lambda+2\mu)/\rho}$ for $P$-waves and $V=V_s=\sqrt{\mu/\rho}$ for $S$-waves. Using the Hamiltonian, $H$, the ray equations can be written as

$$\frac{dx_i}{d\tau}=H_{,p_i},\qquad \frac{dp_i}{d\tau}=-H_{,x_i}, \tag{A1.7}$$

where $\tau$ is the traveltime along the ray. Note that other choices of integration parameter and ray Hamiltonians can be used (see Farra 1989). For an anisotropic medium, the ray equations can be written

$$\frac{dx_i}{d\tau} = a_{ijkl} p_l g_j g_k,$$

$$\frac{dp_i}{d\tau} = -\frac{1}{2}\frac{\partial a_{sjkl}}{\partial x_i} p_s p_l g_j g_k.$$

For an isotropic medium, the above ray tracing equations reduce to the well-known ray tracing system

$$\frac{dx_i}{d\tau} = V^2 p_i, \qquad \frac{dp_i}{d\tau} = -1/V \frac{\partial V}{\partial x_i}.$$

## APPENDIX B

In this appendix, the perturbation of the Christoffel equation for the particle motion and phase speed in a homogeneous, anisotropic medium is presented following Jech & Pšenčík (1989). Given the phase speed and particle motion for one anisotropic medium, an approximate solution for a 'nearby' anisotropic medium can be obtained using perturbation theory (see Messiah 1962; Schiff 1968; Mathews & Walker 1970; Landau & Lifshitz 1974). This was first applied to anisotropic media by Backus (1965). The regions of validity for the first-order perturbation theory for $P$-waves has been given by Backus (1982) (see also Crampin 1982).

Assuming an eigenvalue problem for three eigenvalues, $G_n$, and corresponding eigenvectors, $g_j^{(n)}$, in the unperturbed medium, then

$$(\Gamma_{jk} - G_n \delta_{jk}) g_j^{(n)} = 0.$$

We want to solve a perturbed problem of the form

$$(\Gamma_{jk} + \Delta\Gamma_{jk} - \bar{G}_n \delta_{jk}) \bar{g}_j^{(n)} = 0,$$

for perturbed eigenvalues, $\bar{G}_n$, and eigenvectors, $\bar{g}_j^{(n)}$. In other words, we wish to find perturbations of the eigenvalues $G_n$ and eigenvectors $g_j^{(n)}$ due to the perturbation $\Delta\Gamma_{jk}$ of the matrix $\Gamma_{jk}$. The eigenvalues $\bar{G}_n$ and the eigenvectors $\bar{g}_j^{(n)}$ can be expanded into a series

$$\bar{G}_n = G_n + \Delta G_n + \cdots,$$

and

$$\bar{g}_j^{(n)} = g_j^{(n)} + \Delta g_j^{(n)} + \cdots,$$

in which the $\Delta$ terms are the same order as the perturbation $\Delta\Gamma_{jk}$ and the following terms, which are not shown, are of higher orders. The determination of the individual terms in the series can be done in the way described by Landau & Lifshitz (1974) and Jech & Pšenčík (1989).

For the purposes of this paper, it is sufficient to present expressions for the first-order perturbations of the eigenvalues and eigenvectors. For an unperturbed anisotropic medium, for which all the three eigenvalues differ (non-degenerate case), the first-order perturbations are

$$\Delta G_n = B_{nn}, \tag{A2.1a}$$

$$\Delta g_j^{(n)} = \sum_{m=1}^{3(m\neq n)} \frac{B_{mn}}{G_n - G_m} g_j^{(m)}, \tag{A2.1b}$$

where

$$B_{mn} = \Delta\Gamma_{jk} g_j^{(m)} g_k^{(n)}. \tag{A2.2}$$

The symbol $\sum_{m=1}^{3(m\neq n)}$ denotes the summation in which the term $m = n$ is not included. For the above expansions to converge, the coefficients for the higher order terms should successively decrease. From equation (A2.1b), it then follows that

$$|B_{mn}| \ll |G_n - G_m|.$$

For the case of a degenerate unperturbed medium, the approach and expressions discussed above need some modification. The modified perturbation formulae for eigenvalues and eigenvectors for the case of a degenerate unperturbed medium with coinciding eigenvalues $G_1 = G_2$ of the quasi-shear waves are given by Jech & Pšenčík (1989). The first-order perturbations $\Delta G_1$ and $\Delta G_2$ are given by the formula

$$\Delta G_{1,2} = \tfrac{1}{2}\{(D_{11} + D_{22}) \pm [(D_{11} - D_{22})^2 + 4D_{12}^2]^{1/2}\}, \tag{A2.3}$$

where one of the two signs corresponds to $\Delta G_1$ and the other to $\Delta G_2$. The symbols $D_{mn}$ have a similar meaning as $B_{mn}$ in (A2.2) only the eigenvectors $g_j^{(1)}$ and $g_j^{(2)}$ are substituted by two mutually perpendicular vectors $e_j^{(1)}$ and $e_j^{(2)}$ arbitrarily chosen in the plane perpendicular to $g_j^{(3)}$,

$$D_{MN} = \Delta\Gamma_{jk} e_j^{(M)} e_k^{(N)}. \tag{A2.4}$$

The freedom in the choice of vectors $e_j^{(1)}$ and $e_j^{(2)}$ is a consequence of the coincidence of the corresponding two eigenvalues of $qS$-waves. It is shown by Landau & Lifshitz (1974) (see also Jech & Pšenčík 1989) that this non-uniqueness is removed when the changes of eigenvectors due to the small perturbations $\Delta\Gamma_{jk}$ are required to be small. For such a special choice of vectors $e_j^{(1)}$ and $e_j^{(2)}$, then $D_{12} = 0$ and (A2.3) reduces to the same form as in (A2.1a), with $\Delta G_{1,2}$ equal to either $D_{11}$ or $D_{22}$. Thus, alternatively, the first-order perturbation of eigenvalues in the form shown in (A2.1a) can be used even in a degenerate case with the unperturbed eigenvectors equal to the specially chosen vectors $e_j^{(1)}$ and $e_j^{(2)}$ for which $D_{12} = 0$.

The first-order perturbations $\Delta g_j^{(n)}$ of the eigenvectors in the degenerate case $G_1 = G_2$ are given by

$$\Delta g_j^{(n)} = \sum_{m=1}^{3} c_{nm} e_j^{(m)}, \tag{A2.5}$$

where

$$c_{n3} = \frac{B_{n3}}{G_n - G_3} \qquad \text{for } n \neq 3,$$

$$c_{12} = \frac{B_{13}B_{23}}{(G_1 - G_3)(\Delta G_1 - \Delta G_2)},$$

$c_{mn} = -c_{nm}$ and $c_{nn} = 0$. In (A2.5), $e_j^{(1)}$ and $e_j^{(2)}$ are the specially chosen vectors such that $D_{12} = 0$, see above.

The case in which (A2.3) yields $\Delta G_1 = \Delta G_2$, indicates a case in which both the initial unperturbed and perturbed states are degenerate. This may happen in directions where the final $qS$-wave surfaces coincide, e.g. intersection or kiss singularities. To avoid such a situation in computing rays, a possible solution is to slightly perturb the initial ray parameters. However, further work is required for applications of the perturbation approach in these directions. Currently, a check for these directions is made in the computer code and computation is terminated if such a situation occurs.