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Green's functions for inhomogeneous weakly anisotropic media

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SUMMARY

Formulae for the zeroth-order principal term plus the first-order additional term of the *qP*- and *qS*-wave Green's functions in the so-called quasi-isotropic (QI) approximation are derived for an unbounded inhomogeneous weakly anisotropic medium. The basic idea of this approximation is to seek the asymptotic solution of the elastodynamic equation as an expansion with respect to two small parameters of the same order: the small parameter used in the standard ray method and a parameter characterizing differences of tensors of elastic parameters of a weakly anisotropic medium and of a nearby 'background' isotropic medium. As a result, the procedure of constructing the Green's functions is split into two steps: (1) calculation of rays, traveltimes, the geometrical spreading and polarization vectors in the background isotropic medium; (2) calculation of corrections of traveltimes, amplitudes and polarization vectors due to the deviation of the weakly anisotropic medium from the isotropic background.

Application of the QI approximation to *qP*-wave propagation leads to useful simplified formulae found earlier by the application of the perturbation methods. Application of the QI approximation to *qS*-wave propagation leads to formulae of basic importance. The zeroth-order QI approximation removes the well-known problems of the standard ray method for anisotropic media and gives regular solutions for regions in which the differences between the phase velocities of *qS* waves in the direction of propagation are small. This is the case for weakly anisotropic media as well as for singular regions of *qS* waves such as in the vicinities of kiss and intersection singularities, for example. In such situations, frequency-dependent amplitudes of the *qS* waves in the zeroth-order QI approximation are obtained by a numerical solution of two coupled first-order ordinary differential equations along a ray in the background isotropic medium. When a medium is strongly anisotropic and/or high frequencies are considered, approximate closed-form solutions of the two coupled differential equations have the form of the ray solutions describing two decoupled *qS* waves. The standard ray method for anisotropic media can substitute the QI approximation in such regions. On the other hand, in the limit of infinitely weak anisotropy, the formulae for the QI approximation smoothly converge to formulae for isotropic media. Thus the QI approximation represents a link between ray formulae for anisotropic and isotropic media. The formulae for the zeroth-order QI approximation are regular everywhere except for singular regions of the ray method for isotropic media. The accuracy of the QI approximation can be increased by considering the first-order additional terms of the QI approximation.

The two coupled differential equations are equivalent to the equations of the coupling ray theory (CRT) based on a simplification of a coupling volume integral. Use of a vectorial framework along rays in the background medium that is different in the QI approximation from that used in the CRT avoids some problems of the CRT approach. The QI approximation including the first-order additional terms is expected to yield results of comparable quality to or better quality than those of the CRT. The equivalence of the zeroth-order QI approximation to the CRT promises acceptable results of the QI approximation not only in weakly anisotropic media but also in singular regions of *qS* waves.

Key words: Anisotropy, body waves, Green's function, inhomogeneous media, mode coupling, shear-wave splitting.

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1 INTRODUCTION

It is well known that the application of the standard ray method (Červený 1972; Červený, Molotkov & Pšenčík 1977) to the propagation of seismic waves in inhomogeneous anisotropic media is connected with certain difficulties unknown in studies of wave propagation in isotropic media. These problems are connected with the propagation of qS waves; the computation of the qP waves can be performed without any additional difficulties to those known from isotropic media. The difficulties with qS waves arise *locally* in the vicinity of the qS-wave singularities such as intersection or kiss singularities, for example, or *globally* in media with weak anisotropy. The cause of the difficulties in both cases is the close proximity or even coincidence of the phase velocities of the two qS waves. This leads to problems with the application of ray tracing and dynamic ray tracing equations (see Shearer & Chapman 1989; Gajewski & Pšenčík 1990; Bakker 1996).

The problems mentioned above are also well known from studies of the propagation of electromagnetic waves in inhomogeneous anisotropic media. Kravtsov & Orlov (1980) explain that the difficulties are caused by the implicit assumption of the standard ray method that it describes *independent* waves. In regions, in which phase velocities of the two qS waves nearly coincide, the two waves are indistinguishable and cannot be considered as two independent waves any more. In other words, when searching for an asymptotic solution of the equation of motion, two small parameters must be taken into account in regions of proximity of the qS-wave phase velocities. In addition to the small parameter $\epsilon_1 \sim c/(\omega L)$, where ω is the circular frequency, c is the phase velocity and L is the characteristic length (the shortest of the distances, on which the quantities such as slowness, amplitude, etc. change by an amount comparable to their size), another small parameter, $\epsilon_2 \sim |\Delta a_{ijkl}|/||a_{ijkl}||(\sim \Delta c/c)$, which characterizes the strength of anisotropy (or the proximity of the phase velocities of qS waves along the considered ray), must be considered. The symbol a_{ijkl} denotes the tensor of density-normalized elastic parameters. For $\epsilon_2 \gg \epsilon_1$ the qS waves can be dealt with separately and the standard ray method can be used. For $\epsilon_2 \leq \epsilon_1$, a modified approach must be used. For the latter situation, Kravtsov (1968) proposed the so-called *quasi-isotropic approximation* or, in brief, the QI approximation. In this approach, the elastodynamic equation is solved asymptotically with ϵ_1 and ϵ_2 being considered as two small parameters of the same order (Kravtsov 1968; Kravtsov & Orlov 1980; Kravtsov, Naida & Fuki 1996). In the actual application, we assume, formally, that Δa_{ijkl} is a quantity of the order of ω^{-1} (Kravtsov & Orlov 1980). Note that a similar procedure has been successfully applied by Moczo, Bard & Pšenčík (1987) and Gajewski & Pšenčík (1992) to problems concerning wave propagation in media with weak absorption.

The QI approximation was modified for the elastodynamic case by several authors (see Naida 1977; Sharafutdinov 1994; Zillmer, Kashtan & Gajewski 1998). In the zeroth-order approximation, the QI approximation yields two coupled ordinary differential equations for the determination of the amplitudes of the qS wave along a ray in the nearby background isotropic medium. The equations must be solved numerically along each ray considered. The solutions of the coupled equations are regular everywhere, including in regions of qS-wave singularities. In fact, the coupled equations can even be used in regions of qS-wave singularities in media that cannot be considered weakly anisotropic. In the limit of infinitely weak anisotropy, the solutions smoothly converge to the solutions for isotropic media.

The problems of qS-wave singularities and qS-wave propagation in weakly anisotropic media were studied thoroughly by Chapman & Shearer (1989) and Coates & Chapman (1990), respectively. In the former case, the authors developed the so-called *connection* formulae, applicable in qS-wave singular regions. In front of and behind these regions, the standard ray method for inhomogeneous anisotropic media can be used. Using a simplification of a coupling volume integral, Coates & Chapman (1990) developed a *coupling ray theory* (CRT) for inhomogeneous weakly anisotropic media. Like the QI approximation, the CRT is based on the solution of two coupled ordinary differential equations, which must be solved along rays in the background medium, and which yield quantities necessary for the determination of qS-wave amplitudes. Coates & Chapman (1990) showed that CRT yields acceptable results not only in weakly anisotropic media but also in qS-wave singular regions.

In this paper, the basic equations of the QI approximation for the elastodynamic case are first reviewed in Section 2. It is shown that the QI approximation can also be used for the study of qS-wave propagation in isolated regions in inhomogeneous strongly anisotropic media, in which the qS waves propagate with nearly the same phase velocity. In Section 3, the application of the QI approximation is illustrated for the simple case of qP-wave propagation in an unbounded inhomogeneous arbitrary but weakly anisotropic medium. The sum of the zeroth-order principal term and the first-order additional term, the $(0+1A)$ term, of the qP-wave Green's function in the QI approximation is presented. In Section 4.1, the QI approximation is applied to the qS waves in an unbounded inhomogeneous weakly anisotropic medium of arbitrary symmetry. Two coupled ordinary differential equations found earlier by the authors mentioned above using the QI approximation are rederived. It is shown that when the characteristic length is large, anisotropy becomes stronger and frequencies higher, then solutions of the two coupled equations *decouple* and become frequency-independent. They can be written, approximately, in the form of simple *closed-form* expressions similar to that derived for the qP wave. The accuracy of the zeroth-order QI approximation can be increased by considering the first-order additional terms. In this way, the $(0+1A)$ term of the qS-wave Green's function is obtained. In Section 4.2, it is shown that the zeroth-order QI approximation is equivalent to the CRT of Coates & Chapman (1990) (see also Zillmer *et al.* 1998). Since Coates & Chapman (1990) made several numerical tests illustrating the applicability of the CRT to qS-wave propagation in weakly anisotropic media as well as to qS-wave singular regions, the results of Section 4.2 have important consequences for the applicability of the formulae presented in this paper. In Section 5, various aspects of the results obtained in this paper are discussed.

Component notation is used consistently throughout the paper. The lower-case indices can have values of 1, 2 and 3; the upper-case indices can only have values of 1 and 2. The Einstein summation convention is used for repeated indices. The following notation is used for partial derivatives with respect to spatial coordinates: $u_{i,jk} = \partial^2 u_i / \partial x_j \partial x_k$.

GJI000 14/9/98 13:14:53 3B2 version 5.20
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2 QUASI-ISOTROPIC RAY APPROXIMATION

In the frequency domain, the homogeneous elastodynamic equation (without a source term) for an inhomogeneous anisotropic medium reads

$$(\hat{c}_{ijkl}u_{k,l})_{,j}+\hat{\rho}\omega^2 u_i=0\,. \tag{1}$$

Here $\hat{c}_{ijkl}=\hat{c}_{ijkl}(x_m)$ is a tensor of the fourth rank called the *tensor of elastic parameters*. It has the following symmetry properties:

$$\hat{c}_{ijkl}=\hat{c}_{jikl}=\hat{c}_{ijlk}=\hat{c}_{klij}\,. \tag{2}$$

The symbols $\hat{\rho}=\hat{\rho}(x_m)$ and ω denote the density and the circular frequency, respectively.

We seek the solution of the elastodynamic equation (1) in the form of an asymptotic high-frequency ray series:

$$u_k(x_j,\,\omega)=\mathrm{e}^{i\omega\tau(x_j)}\sum_{n=0}^{\infty}\frac{U_k^{(n)}(x_j)}{(-i\omega)^n}\,. \tag{3}$$

Here i is the imaginary unit, the scalar real-valued function $\tau(x_j)$ is the *eikonal* or *phase function* and the vectorial, generally complex-valued functions $U_i^{(n)}(x_j)$ are the *amplitude coefficients* of the ray series. The phase function and the amplitude coefficients are the quantities to be determined. Inserting (3) into (1), we get

$$\sum_{n=0}^{\infty}(-i\omega)^{-n+2}[\hat{N}_i(U_k^{(n)})-\hat{M}_i(U_k^{(n-1)})+\hat{L}_i(U_k^{(n-2)})]=0\,. \tag{4}$$

In (4), we introduced formally

$$U_k^{(-1)}=0,\qquad U_k^{(-2)}=0\,. \tag{5}$$

The differential vectorial operators $\hat{N}_i$, $\hat{M}_i$ and $\hat{L}_i$ are given by the relations

$$\hat{N}_i(U_k^{(n)})=\hat{c}_{ijkl}U_k^{(n)}\tau_{,l}\tau_{,j}-\hat{\rho}\,U_i^{(n)}\,, \tag{6a}$$

$$\hat{M}_i(U_k^{(n)})=(\hat{c}_{ijkl}U_k^{(n)}\tau_{,l})_{,j}+\hat{c}_{ijkl}U_{k,l}^{(n)}\tau_{,j}\,, \tag{6b}$$

$$\hat{L}_i(U_k^{(n)})=(\hat{c}_{ijkl}U_{k,l}^{(n)})_{,j} \tag{6c}$$

(see e.g. Červený *et al.* 1977).

Formally, the tensor of the elastic parameters and the density can be written in the following form:

$$\hat{c}_{ijkl}=c_{ijkl}+\Delta c_{ijkl}\,,\qquad \hat{\rho}=\rho+\Delta\rho\,. \tag{7}$$

In (7), c_{ijkl} denotes the tensor of the elastic parameters of a background *isotropic* medium. The tensor c_{ijkl} is given by the formula

$$c_{ijkl}=\lambda\delta_{ij}\delta_{kl}+\mu(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})\,. \tag{8}$$

The symbols λ, μ and ρ denote Lamé's parameters and the density of the background isotropic medium. The quantities Δc_{ijkl} and $\Delta\rho$ in (7) denote small deviations of elastic parameters from isotropy and of the density from its background value. We assume that $\epsilon_2\sim|\Delta a_{ijkl}|/||a_{ijkl}||\sim|\Delta\rho|/\rho\ll 1$, where $a_{ijkl}=\rho^{-1}c_{ijkl}$.

In the following, we consider the QI approximation assuming that ϵ_2 is of the same order as $\epsilon_1\sim c/(\omega L)$. The same results can be obtained if we assume, formally, that the quantities Δa_{ijkl} and $\Delta\rho$ are of the order ω^{-1} (see Kravtsov & Orlov 1980). Inserting (7) into (4), taking into account the above assumptions and setting the coefficients of ω^2 and ω to zero, we obtain the following:

$$N_i(U_k^{(0)})=0\,, \tag{9a}$$

$$N_i(U_k^{(1)})-M_i(U_k^{(0)})-i\omega\Delta c_{ijkl}U_k^{(0)}\tau_{,l}\tau_{,j}+i\omega\Delta\rho\,U_i^{(0)}=0\,, \tag{9b}$$

where

$$N_i(U_k^{(n)})=(\lambda+\mu)\tau_{,j}\tau_{,i}U_j^{(n)}+\mu\tau_{,j}\tau_{,j}U_i^{(n)}-\rho\,U_i^{(n)}\,, \tag{10a}$$

$$\begin{aligned}M_i(U_k^{(0)})=&(\lambda+\mu)(\tau_{,j}U_{j,i}^{(0)}+\tau_{,i}U_{j,j}^{(0)}+\tau_{,ij}U_j^{(0)})\\&+\mu(2\tau_{,j}U_{i,j}^{(0)}+\tau_{,jj}U_i^{(0)})+\lambda_{,i}\tau_{,j}U_j^{(0)}\\&+\mu_{,j}\tau_{,j}U_i^{(0)}+\mu_{,j}\tau_{,i}U_j^{(0)}\,.\end{aligned} \tag{10b}$$

The vectorial equation (9a) yields the eikonal equations for the determination of the phase functions $\tau(x_j)$ of the two wave modes propagating in the *isotropic* medium a_{ijkl}. The eikonal equations have the form

$$\tau_{,i}\tau_{,i}=\frac{1}{c^2}\,, \tag{11}$$

with

$$c=\alpha=\sqrt{\frac{\lambda+2\mu}{\rho}}\quad\text{or}\qquad c=\beta=\sqrt{\frac{\mu}{\rho}}, \tag{12}$$

where α is the P-wave velocity and β is the S-wave velocity in the isotropic medium described by (8). The eikonal equations can be solved with the use of the ray tracing equations for isotropic media (see e.g. Červený *et al.* 1977):

$$\frac{dx_i}{d\tau}=c^2\tau_{,i}\,,\qquad \frac{d\tau_{,i}}{d\tau}=-\frac{1}{c}\,\frac{\partial c}{\partial x_i}\,. \tag{13}$$

The quantity τ is the traveltime along a ray and $\tau_{,i}$ is the slowness vector. We can see that the ray trajectories are affected only by the distribution of elastic parameters in the background isotropic medium. The weak anisotropy has *no effect* on the rays.

Note that eq. (9a) also specifies the orientation of the polarization of waves in the zeroth-order QI approximation. It indicates that the polarization of the qP waves in a weakly anisotropic medium in the zeroth-order QI approximation is the same as the polarization of P waves in the background isotropic medium; that is, the qP-wave polarization vectors are tangent to the ray. In the case of qS waves, the polarization vectors in a weakly anisotropic medium are confined to the plane perpendicular to the ray in the background isotropic medium (like the polarization vectors of S waves).

In the following, we will work with three orthogonal vectorial bases connected with rays in background isotropic media. The vectorial basis of the ray-centred coordinate system q_i (see e.g. Popov & Pšenčík 1978; Červený *et al.* 1977; Červený 1985) proved to be extremely useful in many applications in isotropic media. The basis consists of one unit vector $e_i^{(3)}$ tangent to a ray and two mutually perpendicular unit vectors $e_i^{(1)}$ and $e_i^{(2)}$ situated in a plane perpendicular to the ray. The vector $e_i^{(3)}$ can be determined from the slowness

GJI000 14/9/98 13:15:18 3B2 version 5.20
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vector $\tau_{,i}$,

$$e_i^{(3)} = c\tau_{,i}\,. \tag{14a}$$

The symbol c denotes either *P*- or *S*-wave velocity in the isotropic medium. The vectors $e_i^{(I)}$ (I = 1 or 2) can be determined from the equations

$$\frac{de_i^{(I)}}{d\tau} = (c_{,k}e_k^{(I)})e_i^{(3)} \tag{14b}$$

(see e.g. Popov & Pšenčík 1978; Červený 1985). The vectors $e_i^{(I)}$ represent the polarization vectors of an *S* wave. In contrast to the following vectorial bases, the vectors $e_i^{(I)}$ can be chosen arbitrarily at any point on the ray. From that point they change according to eq. (14b).

The next two vector bases to be considered are closely related to the previous one. The vector $e_i^{(3)}$ is common to all the three bases. In the second basis, the vectors $e_i^{(I)}$ are substituted by the normal n_i and the binormal b_i to the ray. The vectors n_i and b_i are related to $e_i^{(I)}$ as follows:

$$n_i = e_i^{(1)}\cos\theta + e_i^{(2)}\sin\theta\,, \qquad b_i = -e_i^{(1)}\sin\theta + e_i^{(2)}\cos\theta\,, \tag{15a}$$

where

$$d\theta/d\tau = cT\,. \tag{15b}$$

The symbol T in (15b) denotes the torsion of the ray.

In the third basis, the vectors $e_i^{(I)}$ are substituted by two mutually perpendicular unit vectors $g_i^{(I)}$, obtained as approximate *qS*-wave polarization vectors from the first-order perturbation method for anisotropic media (see Jech & Pšenčík 1989). The relation of the vectors $g_i^{(I)}$ and $e_i^{(I)}$ is given in eqs (49).

In all of the bases mentioned, the amplitude coefficient $U_i^{(n)}$ can be expressed as follows:

$$U_i^{(n)} = U_1^{f^{(n)}} f_i^{(1)} + U_2^{f^{(n)}} f_i^{(2)} + U_3^{f^{(n)}} e_i^{(3)}\,, \tag{16}$$

where $f_i^{(I)}$ are two mutually perpendicular unit vectors situated in the plane perpendicular to the ray. They can belong to any of the vectorial bases discussed above. The variation of the vectors $f_i^{(I)}$ along rays can thus be arbitrary. The superscript f indicates which of the bases mentioned above is used for the expansion of $U_i^{(n)}$. In the following we adopt the notation used in the standard ray method for isotropic media. For *qP* waves, the component $U_3^{f^{(n)}} e_i^{(3)}$ is called the *principal component* and $U_1^{f^{(n)}} f_i^{(1)} + U_2^{f^{(n)}} f_i^{(2)}$ the *additional component*. For *qS* waves, the *principal component* is $U_1^{f^{(n)}} f_i^{(1)} + U_2^{f^{(n)}} f_i^{(2)}$ and the *additional component* is $U_3^{f^{(n)}} e_i^{(3)}$. In this way, eq. (9a) indicates that in the zeroth-order QI approximation only principal components of *qP* and *qS* waves are non-zero. The zeroth-order additional components are zero. To simplify the notation, we sometimes also use $f_i^{(3)} = e_i^{(3)}$ in the following.

In order to satisfy eq. (9b), we require

$$[N_i(U_k^{(1)}) - M_i(U_k^{(0)}) - i\omega\Delta c_{ijkl}U_k^{(0)}\tau_{,l}\tau_{,j} + i\omega\Delta\rho U_i^{(0)}]f_i^{(m)} = 0 \tag{17}$$

for any m. Substitution of $U_k^{(1)}$ specified by (16) into $N_i(U_k^{(1)})$ given by (10a) and multiplication of $N_i(U_k^{(1)})$ by the vector $f_i^{(m)}$ yields

$$N_i(U_k^{(1)})f_i^{(m)} = \frac{\alpha^2 - c^2}{c^2}\,\rho U_3^{f^{(1)}} e_i^{(3)} f_i^{(m)} + \frac{\beta^2 - c^2}{c^2}\,\rho U_J^{f^{(1)}}\delta_{Jm}\,. \tag{18}$$

Inserting eq. (18) into eq. (17), we obtain for *qP* waves ($c = \alpha$)

$$M_i(U_k^{(0)})e_i^{(3)} = -i\omega U_3^{f^{(0)}}(\alpha^{-2}\Delta c_{ijkl}e_i^{(3)}e_j^{(3)}e_l^{(3)}e_k^{(3)} - \Delta\rho)$$
$$= -i\omega\rho\alpha^{-2}B_{33}U_3^{f^{(0)}}\,, \tag{19a}$$

$$U_J^{f^{(1)}} = \frac{\alpha^2}{\rho(\beta^2 - \alpha^2)}\,[M_i(U_k^{(0)})f_i^{(J)} + i\omega\rho\alpha^{-2}B_{3J}U_3^{f^{(0)}}]\,. \tag{19b}$$

In a similar way, for *qS* waves ($c = \beta$) we obtain

$$M_i(U_k^{(0)})f_i^{(K)} = -i\omega\rho\beta^{-2}B_{JK}U_J^{f^{(0)}}\,, \quad K = 1, 2\,, \tag{20a}$$

$$U_3^{f^{(1)}} = \frac{\beta^2}{\rho(\alpha^2 - \beta^2)}\,[M_i(U_k^{(0)})e_i^{(3)} + i\omega\rho\beta^{-2}B_{J3}U_J^{f^{(0)}}]\,. \tag{20b}$$

The symbols B_{mn} denote the elements of an important matrix, the *weak anisotropy* (WA) matrix, which appears in formulae related to weakly anisotropic media (see e.g. Backus 1965; Jech & Pšenčík 1989; Pšenčík & Gajewski 1998):

$$B_{mn} = \Delta a_{ijkl}f_i^{(m)}e_j^{(3)}e_l^{(3)}f_k^{(n)}\,. \tag{21}$$

Eqs (19a) and (20a) are the transport equations for the zeroth-order principal terms; eqs (19b) and (20b) yield the first-order additional terms. Comparing them with the corresponding equations for isotropic media, we can see that the differences are caused by the presence of terms containing the elements of the matrix B_{mn}. Eqs (19) and (20) were obtained under the assumption that Δa_{ijkl} and $\Delta\rho$ are of the order of ω^{-1}. We see that the same equations could be obtained under the assumption that B_{mn} is of the order of ω^{-1}. Small values of B_{mn} indicate the proximity of the phase velocities to the values of velocities in the background isotropic medium, and in case of *qS* waves the mutual proximity of the phase velocities of these two waves (see Jech & Pšenčík 1989). In this way, the assumption of small values of B_{mn} extends the region of applicability of the QI approximation to regions of small differences between phase velocities of the *qS* waves in the direction of propagation (*qS*-wave singular regions) in otherwise strongly anisotropic media.

3 *qP* WAVES

In this section we assume that $f_i^{(I)} = e_i^{(I)}$; that is, we consider the vectors $f_i^{(I)}$ to be the basis vectors of the ray-centred coordinate system. Components of the vectorial ray amplitude in this frame are thus denoted by e instead of f. We determine the zeroth-order principal component of the *qP* wave:

$$U_i^{(0)} = \mathscr{A}e_i^{(3)}\,, \tag{22}$$

where $\mathscr{A} = U_3^{e^{(0)}}$ denotes the scalar amplitude in the zeroth-order QI approximation. The amplitude is generally complex-valued. Inserting (22) into (10b) and the result into (19a) yields, after some algebra, the transport equation

$$2\rho\,\frac{d\mathscr{A}}{d\tau} + \rho\alpha^2\mathscr{A}\tau_{,jj} + (\rho\alpha^2)_{,i}\tau_{,i}\mathscr{A} + i\omega\rho\alpha^{-2}B_{33}^e\mathscr{A} = 0\,. \tag{23}$$

The superscript e in the term B_{33}^e indicates that $f_i^{(I)}$ are substituted by $e_i^{(I)}$ in (21). Eq. (23) differs from the transport equation for isotropic media by the presence of the last term on the left-hand side. It can be further rewritten in the form

$$\frac{d}{d\tau}\,(\mathscr{A}\sqrt{\rho\alpha\mathscr{J}}) + \frac{1}{2}\,i\omega\alpha^{-2}B_{33}^e\mathscr{A}\sqrt{\rho\alpha\mathscr{J}} = 0\,. \tag{24}$$

GJI000 14/9/98 13:15:53 3B2 version 5.20
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Here the symbol $\mathscr{J}$ denotes the Jacobian of the transformation from ray to Cartesian coordinates:

$$\mathscr{J} = \frac{\partial(x_1, x_2, x_3)}{\partial(\gamma_1, \gamma_2, s)} . \tag{25}$$

The Jacobian is evaluated in the background *isotropic* medium and thus weak anisotropy has *no effect* on it. The ray coordinates γ_I specify a ray, and the ray coordinate s specifies the length along the ray. The elements of the Jacobian (25) can be obtained as a solution of dynamic ray tracing (see e.g. Červený 1972; Gajewski & Pšenčík 1990).

The solution of eq. (24) can be written in the form

$$\mathscr{A}(\tau) = \frac{\psi(\gamma_1, \gamma_2, \tau_0)}{\sqrt{\rho(\tau)\alpha(\tau)\Omega_M(\tau)}} \exp\left[-\frac{i\omega}{2}\int_{\tau_0}^{\tau} \alpha^{-2} B^e_{33}(\xi) d\xi\right] . \tag{26}$$

From (26) we can see that, in contrast to the standard ray method, the scalar amplitude coefficient $\mathscr{A}$ depends on frequency in the QI approximation. The function $\psi(\gamma_1, \gamma_2, \tau_0)$ is constant along the ray and depends on the type of source considered. The quantity Ω_M is related to the Jacobian $\mathscr{J}$ and to the relative geometrical spreading obtained from dynamic ray tracing with specially chosen initial conditions (see eq. (A.5) of Pšenčík & Teles 1996). Here and in the following, the points on a selected ray are parametrized by the traveltimes calculated along the ray path in the background isotropic medium.

Let us now consider the wavefield generated by a point force f_k acting at the point x_{0m}. By comparing formula (26) with the expression for the ray amplitude of a wave generated by a point force in an anisotropic medium (see Pšenčík & Teles 1996), we find that in the zeroth-order QI approximation the function $\psi(\gamma_1, \gamma_2, \tau_0)$ reads

$$\psi(\gamma_1, \gamma_2, \tau_0) = \frac{e^{(3)}_k(\tau_0) f_k}{4\pi\sqrt{\rho(\tau_0)\alpha(\tau_0)}} . \tag{27}$$

Inserting (27) into (26), we get

$$\mathscr{A}(\tau) = \mathscr{A}_0(\tau) \exp(i\omega\Delta\tau) , \tag{28}$$

where

$$\mathscr{A}_0(\tau) = \frac{e^{(3)}_k(\tau_0) f_k}{4\pi\sqrt{\rho(\tau_0)\alpha(\tau_0)\rho(\tau)\alpha(\tau)\Omega_M(\tau)}} \tag{29a}$$

and

$$\Delta\tau = -\frac{1}{2}\int_{\tau_0}^{\tau} \alpha^{-2}(\xi) B^e_{33}(\xi) d\xi . \tag{29b}$$

Note that the time-shift $\Delta\tau$ was obtained independently by Červený (1982) and Hanyga (1982) (see also Červený & Jech 1982) using the perturbation method.

If we specify the force f_k as a unit vector oriented successively along all three coordinate axes x_n, i.e. $f_k = \delta_{kn}$, eqs (3), (22) and (26)–(29) yield the zeroth-order QI approximation $G^{(0)}_{\mathrm{in}}$ of the qP-wave Green's function for an unbounded inhomogeneous weakly anisotropic medium as follows:

$$G^{(0)}_{\mathrm{in}}(\tau, \tau_0, \omega) = \frac{e^{(3)}_n(\tau_0) e^{(3)}_i(\tau)}{4\pi\sqrt{\rho(\tau_0)\alpha(\tau_0)\rho(\tau)\alpha(\tau)\Omega_M(\tau)}} \exp[i\omega(\tau + \Delta\tau)] . \tag{30}$$

Here $\tau = \tau(x_m)$, $\tau_0 = \tau(x_{0m})$ and $\Delta\tau$ is given by eq. (29b). We see that formula (30) only differs from the corresponding formula for the isotropic background (see e.g. Eisner & Pšenčík 1996) by the time-shift $\Delta\tau$. Thus, in the zeroth-order QI approximation, the deviation Δa_{ijkl} of the weakly anisotropic medium from the isotropic background a_{ijkl} affects only the arrival time of the qP wave. The amplitude and the polarization are not affected. The polarization of the Green's function in the zeroth-order QI approximation is thus *linear*. Due to the reciprocity of the relative geometrical spreading function Ω_M and the element B^e_{33} of the matrix B^e_{mn}, the qP-wave Green's function $G^{(0)}_{\mathrm{in}}$ is *reciprocal*.

Eisner & Pšenčík (1996) showed that the accuracy of the ray Green's function in a homogeneous isotropic medium considerably increases if the sum of the zeroth-order principal term and the first-order additional term of the ray series, the $(0+1A)$ term, is considered. Since we assume that the medium differs only slightly from an isotropic medium and that the variations of its parameters are small, we can expect similar behaviour of the Green's function in the QI approximation. Of course, numerical tests are necessary to confirm this hypothesis.

The $(0+1A)$ term $G^{(L)}_{\mathrm{in}}$ of the qP-wave Green's function in the QI approximation in an inhomogeneous weakly anisotropic medium, following on from eqs (30) and (19b), is

$$G^{(L)}_{\mathrm{in}}(\tau, \tau_0, \omega) = \left[\mathscr{A}_{n0} e^{(3)}_i + \frac{\mathscr{B}^1_{n1} e^{(1)}_i}{(-i\omega)} + \frac{\mathscr{B}^1_{n2} e^{(2)}_i}{(-i\omega)}\right] \times \exp\left[i\omega\left(\tau - \frac{1}{2}\int_{\tau_0}^{\tau} \alpha^{-2}(\xi) B^e_{33}(\xi) d\xi\right)\right] . \tag{31}$$

We denoted the first-order additional components as $\mathscr{B}^1_{nI}$, where $\mathscr{B}^1_{nI} = U^{e^{(1)}}_I$. Inserting (10b) into (19b) and after some algebra, we obtain the following for $\mathscr{B}^1_{nI}$:

$$\mathscr{B}^1_{nI} = \left[-\alpha\left(\frac{\partial\mathscr{A}_{n0}}{\partial q_I} + i\omega\mathscr{A}_{n0}\frac{\partial\Delta\tau}{\partial q_I}\right) + \frac{\mathscr{A}_{n0} e^{(I)}_i}{\beta^2 - \alpha^2} X_i + \frac{i\omega}{\beta^2 - \alpha^2}\mathscr{A}_{n0} B^e_{I3}\right] \exp(i\omega\Delta\tau) , \tag{32}$$

where

$$X_i = (\alpha^2 - \beta^2)\alpha_{,i} - 4\alpha\beta\beta_{,i} + \alpha(\alpha^2 - 2\beta^2)\rho^{-1}\rho_{,i} . \tag{33}$$

Here $\mathscr{A}_{n0}$ equals $\mathscr{A}_0$ given in eq. (29a) with $f_k = \delta_{kn}$. All the quantities in (32) and (33), with the exception of the derivatives with respect to q_I can be evaluated without any problems. The derivatives with respect to q_I, can be evaluated approximately using the procedure described by Eisner & Pšenčík (1996). Inspecting eq. (32), we see that the weak anisotropy is responsible for the following correction Δg_i of the qP-wave polarization vector $e^{(3)}_i$ in the zeroth-order QI approximation:

$$\Delta g_i = \alpha\frac{\partial\Delta\tau}{\partial q_I} e^{(I)}_i + \frac{B^e_{13} e^{(1)}_i}{\alpha^2 - \beta^2} + \frac{B^e_{23} e^{(2)}_i}{\alpha^2 - \beta^2} . \tag{34}$$

The first term on the right-hand side of eq. (34) represents a correction due to the perturbation of the phase front. It gives the deviation, due to the perturbation Δa_{ijkl}, of the slowness vector at the receiver from its orientation in the background isotropic medium. This term thus represents a way of determining the perturbed slowness vector that is an alternative to

GJI000 14/9/98 13:16:18 3B2 version 5.20
The Charlesworth Group, Huddersfield 01484 517077

the standard method based on integration along the ray in the background isotropic medium (see e.g. Farra & Le Bégat 1995).

The second and third terms on the right-hand side of eq. (34) represent corrections of the polarization vector due to the perturbation Δa_{ijkl} for the fixed slowness vector. They can also be obtained using the perturbation method (see Jech & Pšenčík 1989; Kiselev 1994; Pšenčík & Gajewski 1998).

Note that all terms in (34) lead to the deviation of the polarization vector from the direction $e_i^{(3)}$, but the polarization remains linear. The terms are non-zero even in homogeneous media. The remaining terms in (32), if non-zero, lead to elliptical polarization of the *qP*-ray Green's function in the $(0+1A)$ QI approximation. Elliptical polarization is thus caused by inhomogeneity of the medium or non-symmetry of the source, not by anisotropy.

4 *qS* WAVES

As indicated below eq. (16), the principal component of a *qS* wave has the form

$$U_i^{(0)} = \mathscr{B} f_i^{(1)} + \mathscr{C} f_i^{(2)} . \tag{35}$$

Here $\mathscr{B} = U_1^{f^{(0)}}$ and $\mathscr{C} = U_2^{f^{(0)}}$ denote scalar amplitudes in the zeroth-order QI approximation with respect to the vectors $f_i^{(I)}$. The amplitudes are generally complex-valued. Inserting (35) into (10b) and the result into (20a), we obtain, after some algebra, two coupled transport equations for $\mathscr{B}$ and $\mathscr{C}$:

$$\begin{aligned}
&2\rho \frac{d\mathscr{B}}{d\tau} + \rho\beta^2 \tau_{,jj}\mathscr{B} + (\rho\beta^2)_{,i}\tau_{,i}\mathscr{B} \\
&\quad + 2\rho \frac{df_j^{(2)}}{d\tau} f_j^{(1)}\mathscr{C} + i\omega\rho\beta^{-2}(B_{11}\mathscr{B} + B_{12}\mathscr{C}) = 0 , \\
&2\rho \frac{d\mathscr{C}}{d\tau} + \rho\beta^2 \tau_{,jj}\mathscr{C} + (\rho\beta^2)_{,i}\tau_{,i}\mathscr{C} \\
&\quad + 2\rho \frac{df_j^{(1)}}{d\tau} f_j^{(2)}\mathscr{B} + i\omega\rho\beta^{-2}(B_{12}\mathscr{B} + B_{22}\mathscr{C}) = 0 .
\end{aligned} \tag{36}$$

The elements of the matrix B_{mn} are defined in eq. (21). Eqs (36) must be solved along the ray of the *S* wave in the background isotropic medium.

If the vectors $f_i^{(I)}$ in eq. (35) are the normal and the binormal to the considered ray, then the coupled system of two equations (36) is an elastodynamic form of the QI equations of Kravtsov (1968) and Kravtsov & Orlov (1980) (see also Naida 1977). If the vectors $f_i^{(I)}$ are the approximate polarization vectors $g_i^{(I)}$ of the *qS* waves, resulting from the first-order perturbation method (Jech & Pšenčík 1989), the system (36) leads to the equations of Coates & Chapman (1990); see Section 4.2. If we consider the vectors $f_i^{(I)}$ to be the basis vectors of the ray-centred coordinate system, i.e. $f_i^{(I)} = e_i^{(I)}$, then we get the system of Zillmer *et al.* (1998).

In the following, we investigate eqs (36) expressed first with respect to the ray-centred vectorial basis $e_i^{(I)}$ (Section 4.1) and then with respect to the basis formed by the approximate polarization vectors $g_i^{(I)}$ of the *qS* waves (Section 4.2).

4.1 Vectorial basis $f_i^{(I)} = e_i^{(I)}$

We consider $f_i^{(I)} = e_i^{(I)}$ and, therefore, we have $\mathscr{B} = U_1^{e^{(0)}}$ and $\mathscr{C} = U_2^{e^{(0)}}$. The coupling terms in eqs (36) containing the scalar products $df_j^{(2)}/d\tau\, f_j^{(1)}$ and $df_j^{(1)}/d\tau\, f_j^{(2)}$ vanish due to eq. (14b). By analogy with the derivation of the formulae for the *qP* waves, we look for a solution of eqs (36) in the form

$$\mathscr{B}(\tau) = \frac{\mathscr{B}_0(\tau)}{\sqrt{\rho(\tau)\beta(\tau)\Omega_M(\tau)}} , \qquad \mathscr{C}(\tau) = \frac{\mathscr{C}_0(\tau)}{\sqrt{\rho(\tau)\beta(\tau)\Omega_M(\tau)}} . \tag{37}$$

The quantity $\Omega_M(\tau)$ has the same meaning as in eq. (26), here, of course, for the *S* wave in the background isotropic medium. Inserting (37) into (36) yields

$$\begin{aligned}
\frac{d\mathscr{B}_0}{d\tau} &= -\frac{1}{2} i\omega\beta^{-2}(B_{11}^e\mathscr{B}_0 + B_{12}^e\mathscr{C}_0) , \\
\frac{d\mathscr{C}_0}{d\tau} &= -\frac{1}{2} i\omega\beta^{-2}(B_{12}^e\mathscr{B}_0 + B_{22}^e\mathscr{C}_0) .
\end{aligned} \tag{38}$$

Similarly to the case of *qP* waves, we can see that amplitude coefficients $\mathscr{B}$ and $\mathscr{C}$ are frequency-dependent in the QI approximation. The superscript e in B_{mn}^e indicates that the vectors $f_i^{(I)}$ are substituted by $e_i^{(I)}$ in (21). Eqs (38) represent two coupled linear differential equations for $\mathscr{B}_0$ and $\mathscr{C}_0$. If the wavefield is generated by a point force f_k, then comparison of eqs (37) with the corresponding expressions of Pšenčík & Teles (1996) yields initial conditions for (38) in the form

$$\mathscr{B}_0(\tau_0) = \frac{e_k^{(1)}(\tau_0) f_k}{4\pi\sqrt{\rho(\tau_0)\beta(\tau_0)}} , \qquad \mathscr{C}_0(\tau_0) = \frac{e_k^{(2)}(\tau_0) f_k}{4\pi\sqrt{\rho(\tau_0)\beta(\tau_0)}} . \tag{39}$$

If we specify the force f_k as a unit vector oriented successively along all coordinate axes x_n, i.e. $f_k = \delta_{kn}$, the zeroth-order QI approximation of the *qS*-wave Green's function in unbounded inhomogeneous weakly anisotropic media is

$$G_{\mathrm{in}}(\tau, \tau_0, \omega) = (\mathscr{B}_n(\tau) e_i^{(1)}(\tau) + \mathscr{C}_n(\tau) e_i^{(2)}(\tau))\, e^{i\omega\tau} . \tag{40}$$

The functions $\mathscr{B}_n(\tau)$ and $\mathscr{C}_n(\tau)$ are given by the relations (37), in which the functions $\mathscr{B}_0(\tau)$ and $\mathscr{C}_0(\tau)$ are obtained by solving eqs (38) with initial conditions (39) with $f_k = \delta_{kn}$.

From the form of the zeroth-order QI approximation of the *qS*-wave Green's function in eq. (40), we immediately see that its polarization is generally *elliptical*.

Similarly to the case of *qP* waves, we can derive the $(0+1A)$ term $G_{\mathrm{in}}^{(L)}$ of the *qS*-wave Green's function in the QI approximation:

$$G_{\mathrm{in}}^{(L)}(\tau, \tau_0, \omega) = e^{i\omega\tau}\left(\mathscr{B}_n e_i^{(1)} + \mathscr{C}_n e_i^{(2)} + \frac{\mathscr{A}_n^1 e_i^{(3)}}{(-i\omega)}\right) , \tag{41}$$

where $\mathscr{A}_n^1$ denotes the first-order additional component, $\mathscr{A}_n^1 = U_3^{e^{(1)}}$. Inserting (10b) into (20b), after some algebra we obtain for $\mathscr{A}_n^1$

$$\begin{aligned}
\mathscr{A}_n^1 = &\left[\beta \frac{\partial\mathscr{B}_n}{\partial q_1} + \frac{\mathscr{B}_n e_i^{(1)}}{\alpha^2 - \beta^2} Y_i + \frac{i\omega\mathscr{B}_n}{\alpha^2 - \beta^2} B_{13}^e\right] \\
&+ \left[\beta \frac{\partial\mathscr{C}_n}{\partial q_2} + \frac{\mathscr{C}_n e_i^{(2)}}{\alpha^2 - \beta^2} Y_i + \frac{i\omega\mathscr{C}_n}{\alpha^2 - \beta^2} B_{23}^e\right] .
\end{aligned} \tag{42}$$

In (42), Y_i is as follows:

$$Y_i = (\alpha^2 + 3\beta^2)\beta_{,i} + \beta^2\rho^{-1}\rho_{,i} . \tag{43}$$

As in the case of *qP* waves, we see that weak anisotropy is responsible for the corrections of directions of the polarization vectors $g_i^{(I)}$. These corrections again consist of the correction of

GJI000 14/9/98 13:16:33 3B2 version 5.20
The Charlesworth Group, Huddersfield 01484 517077

the orientation of the slowness vector and the term

$$\frac{B^e_{I3} e^{(3)}_i}{\beta^2-\alpha^2}\,, \tag{44}$$

describing the deviation of the polarization for a fixed slowness vector. The latter term has also been obtained by use of the perturbation theory (see Jech & Pšenčík 1989). Due to the direction corrections, the qS wave is polarized in a plane that is not perpendicular to the ray in the background isotropic medium. These effects remain non-zero even in homogeneous media.

The coupled differential equations (38) indicate that the two qS waves are coupled together and cannot be investigated separately. Eqs (38) must be solved numerically along the ray of an S wave in the background isotropic medium.

In the limiting case of an isotropic medium, eqs (38) yield $\mathscr{B}_0=$constant, $\mathscr{C}_0=$constant, and solutions (37) of eqs (36) thus give the ray amplitudes of an S wave in the isotropic medium.

For stronger anisotropy and/or higher frequencies, eqs (36) yield solutions that have the structure of the standard ray method solutions for anisotropic media. This means that eqs (36) describe two approximately decoupled qS waves. Let us seek solutions of eqs (36) in the form of an approximate zeroth-order ray solution:

$$\mathscr{B}(\tau)=\mathscr{V}\,e^{i\omega\Delta\tau}\,,\qquad \mathscr{C}(\tau)=\mathscr{W}\,e^{i\omega\Delta\tau}\,. \tag{45}$$

Here $\mathscr{V}$ and $\mathscr{W}$ are the approximations of the ray amplitudes and the $\Delta\tau$ are the traveltime perturbations corresponding to $qS1$ and $qS2$ waves. If $\mathscr{V}$ and $\mathscr{W}$ satisfy corresponding transport equations, then inserting (45) into (36) yields two frequency-independent equations:

$$\begin{aligned}&\frac{d\Delta\tau}{d\tau}\mathscr{V}+\frac{1}{2}\beta^{-2}(B^e_{11}\mathscr{V}+B^e_{12}\mathscr{W})=0\,,\\&\frac{d\Delta\tau}{d\tau}\mathscr{W}+\frac{1}{2}\beta^{-2}(B^e_{12}\mathscr{V}+B^e_{22}\mathscr{W})=0\,.\end{aligned} \tag{46}$$

As is shown below, eqs (46) yield independent expressions for both of the qS waves considered; that is, they describe two decoupled qS waves.

We can estimate roughly the conditions under which eqs (36) reduce to (46). The terms that form the transport equations in (36) are of the order c/L and the remaining terms, which also appear in (46), are of the order $\omega\Delta c/c$. Here c denotes the phase velocity, Δc is the difference of qS-wave phase velocities along the ray in the background isotropic medium and L is the characteristic length. Thus the condition under which eqs (36) reduce to eqs (46) is

$$\frac{c}{L}\ll\omega\,\frac{\Delta c}{c}\qquad\text{or}\qquad \epsilon_1\ll\epsilon_2\,, \tag{47}$$

if we recall the definitions of the small quantities ϵ_1 and ϵ_2. If the characteristic length L can be approximated as $L\sim c/|\nabla c|$, then the inequality (47) can be rewritten as

$$\omega^{-1}\ll\frac{\Delta c}{c|\nabla c|}\,. \tag{48}$$

The inequality (47) was also obtained by Kravtsov & Orlov (1980) by a different consideration; for (48) see Gajewski & Pšenčík (1990).

Eqs (46) represent a system of equations for two eigenvalues $-d\Delta\tau/d\tau$ and corresponding eigenvectors $g^{(1)}_i$, $g^{(2)}_i$ of the matrix $\frac{1}{2}\beta^{-2}B^e_{MN}$. The vectors $g^{(I)}_i$ are the approximate polarization vectors of the two qS waves; see also Jech & Pšenčík (1989) and Kiselev (1994). They are unit vectors, mutually orthogonal and are situated in the plane specified by the vectors $e^{(I)}_i$; that is, in the plane perpendicular to the ray of the S wave in the background isotropic medium:

$$\begin{aligned}&g^{(1)}_i(\tau)=e^{(1)}_i(\tau)\cos\phi(\tau)+e^{(2)}_i(\tau)\sin\phi(\tau)\,,\\&g^{(2)}_i(\tau)=-e^{(1)}_i(\tau)\sin\phi(\tau)+e^{(2)}_i(\tau)\cos\phi(\tau)\,.\end{aligned} \tag{49}$$

The angle $\phi(\tau)$ can be determined from eqs (46) in the form of eqs (20) of Jech & Pšenčík (1989). Here we present a corrected form of these equations (see also Thomson, Kendall & Guest 1992, eq. A13):

$$\begin{aligned}&\cos\phi(\tau)=\left[\frac{1}{2}\left(1+\frac{B^e_{11}(\tau)-B^e_{22}(\tau)}{D(\tau)}\right)\right]^{1/2}\,,\\&\sin\phi(\tau)=\mathrm{sgn}(B^e_{12}(\tau))\left[\frac{1}{2}\left(1-\frac{B^e_{11}(\tau)-B^e_{22}(\tau)}{D(\tau)}\right)\right]^{1/2}\,,\end{aligned} \tag{50}$$

where

$$D=[(B^e_{11}-B^e_{22})^2+4(B^e_{12})^2]^{1/2}\,. \tag{51}$$

An alternative formula for ϕ is

$$\tan 2\phi=\frac{2B^e_{12}}{B^e_{11}-B^e_{22}}\,. \tag{52}$$

The eigenvalues of the matrix $(1/2)\beta^{-2}B^e_{mn}$ yield

$$\Delta\tau(\tau)=-\frac{1}{4}\int_{\tau_0}^{\tau}\beta^{-2}(\xi)[(B^e_{11}(\xi)+B^e_{22}(\xi))\pm D(\xi)]d\xi\,. \tag{53}$$

Eq. (53) gives the traveltime (phase) shifts of the two qS waves with respect to the S wave in the isotropic background. The shift is caused by the deviation of the weakly anisotropic medium from the background isotropic medium. The angle ϕ in eqs (49) and (50) or (52) was chosen so that the vector $g^{(1)}_i$ corresponds to the time-shift with a positive sign in front of $D(\xi)$ in (53); that is, to the faster of the qS waves. We call it the $qS1$ wave and denote its time-shift by $\Delta\tau_{qS1}$. The vector $g^{(2)}_i$ corresponds to the slower qS wave, which we call the $qS2$ wave. It is related to the time-shift with a negative sign in front of $D(\xi)$ in (53); we denote it by $\Delta\tau_{qS2}$.

For the principal components of the decoupled qS waves we can write two independent approximate relations:

$$\begin{aligned}&U^{qS1(0)}_i(\tau)=\mathscr{V}(\tau)g^{(1)}_i(\tau)\exp[i\omega(\tau+\Delta\tau_{qS1})]\,,\\&U^{qS2(0)}_i(\tau)=\mathscr{W}(\tau)g^{(2)}_i(\tau)\exp[i\omega(\tau+\Delta\tau_{qS2})]\,.\end{aligned} \tag{54}$$

The amplitude terms $\mathscr{V}$ and $\mathscr{W}$ can be determined at a point τ of the ray in the background isotropic medium at which the condition (47) is satisfied by comparing the solutions (40) and (54). Such a comparison yields

$$\begin{aligned}&\mathscr{V}(\tau)=(\mathscr{B}_n(\tau)\cos\phi(\tau)+\mathscr{C}_n(\tau)\sin\phi(\tau))\exp(-i\omega\Delta\tau_{qS1})\,,\\&\mathscr{W}(\tau)=(-\mathscr{B}_n(\tau)\sin\phi(\tau)+\mathscr{C}_n(\tau)\cos\phi(\tau))\exp(-i\omega\Delta\tau_{qS2})\,.\end{aligned} \tag{55}$$

From the point τ, the standard ray method for inhomogeneous anisotropic media can be used. Necessary initial conditions can be determined from (54) with (55), (49) and (53).

GJI000 14/9/98 13:17:08 3B2 version 5.20
The Charlesworth Group, Huddersfield 01484 517077

Thus in media satisfying the condition (47), eqs (36) yield two *uncoupled qS* waves. We can thus call the condition (47) the *decoupling condition*. Considering (48), we see that the qS waves decouple in regions in which 'anisotropy is stronger than inhomogeneity'.

Uncoupled qS waves can also be obtained in media in which $B^e_{12}(\tau)=0$ along the ray in the isotropic background; that is, when the vectors $g^{(I)}_i$ coincide with the vectors $e^{(I)}_i$. This situation occurs when the symmetry axis of a transversely isotropic medium is situated in the plane of propagation, for example. In such a case, decoupling occurs even if condition (47) is not satisfied.

The formulae derived above are regular everywhere except in singular regions of the ray method for isotropic media. The formulae are valid both in weakly anisotropic media and in isotropic media. In the limit of infinitely weak anisotropy, the formulae reduce to the formulae for the zeroth-order ray approximation of the S-wave Green's function in the isotropic medium. The formulae are also applicable to media containing smooth transitions from isotropy to weak anisotropy and vice versa; see the discussion of this topic in Section 5. When condition (47) is satisfied, the qS waves decouple and can be described by independent explicit formulae.

4.2 Vectorial basis $f^{(I)}_i = g^{(I)}_i$

Let us specify eqs (36) in the frame formed by the approximate polarization vectors $g^{(I)}_i$ of the qS waves, i.e. $f^{(I)}_i = g^{(I)}_i$. The quantities $\mathscr{B}$ and $\mathscr{C}$ in eq. (35) are now defined as $\mathscr{B}=U^{g(0)}_1$ and $\mathscr{C}=U^{g(0)}_2$. As in the previous section, we express $\mathscr{B}$ and $\mathscr{C}$ as in (37). Inserting this into eq. (36), we obtain the following two coupled linear differential equations:

$$\begin{aligned} &\rho\,\frac{d\mathscr{B}_0}{d\tau}+\rho\,\frac{dg^{(2)}_j}{d\tau}\,g^{(1)}_j\mathscr{C}_0+\frac{1}{2}\,i\omega\rho\beta^{-2}B^g_{11}\mathscr{B}_0=0\,,\\ &\rho\,\frac{d\mathscr{C}_0}{d\tau}+\rho\,\frac{dg^{(1)}_j}{d\tau}\,g^{(2)}_j\mathscr{B}_0+\frac{1}{2}\,i\omega\rho\beta^{-2}B^g_{22}\mathscr{B}_0=0\,. \end{aligned} \tag{56}$$

In deriving eqs (56), we took into account an important identity:

$$B^g_{12}=0\,. \tag{57}$$

The superscripts g in B^g_{mn} indicate that the vectors $f^{(I)}_i$ are substituted by $g^{(I)}_i$ in (21). For a proof of eq. (57), see e.g. eqs (19) of Jech & Pšenčík (1989).

We can simplify equations (56) by making use of the results obtained in the previous section. Let us express $\mathscr{B}_0$ and $\mathscr{C}_0$ as follows:

$$\mathscr{B}_0(\tau)=r_1(\tau)\exp(i\omega\Delta\tau_1)\,,\quad \mathscr{C}_0(\tau)=r_2(\tau)\exp(i\omega\Delta\tau_2)\,. \tag{58}$$

The traveltime-shifts are given by

$$\begin{aligned} \Delta\tau_1(\tau)&=-\frac{1}{2}\int_{\tau_0}^{\tau}\beta^{-2}(\xi)B^g_{11}(\xi)d\xi\,,\\ \Delta\tau_2(\tau)&=-\frac{1}{2}\int_{\tau_0}^{\tau}\beta^{-2}(\xi)B^g_{22}(\xi)d\xi\,, \end{aligned} \tag{59}$$

which follows from eq. (53) taking into account eq. (57). By inspecting the formula (53), we easily find that the time-shifts it yields are independent of the choice of vector frame along the ray (this is a consequence of the fact that $-d\Delta\tau/d\tau$ are eigenvalues of the matrix $1/2\beta^{-2}B^e_{MN}$). This means that the quantities $\Delta\tau_1(\tau)$ and $\Delta\tau_2(\tau)$ in eq. (59) are equal to $\Delta\tau_{qS1}(\tau)$ and $\Delta\tau_{qS2}(\tau)$ from (53). If $B^g_{11} > B^g_{22}$, then $\Delta\tau_1(\tau)=\Delta\tau_{qS1}(\tau)$ and $\Delta\tau_2(\tau)=\Delta\tau_{qS2}(\tau)$ and vice versa for $B^g_{11} < B^g_{22}$.

Inserting (58) into (56), we obtain the following two coupled equations:

$$\begin{aligned} &\frac{dr_1(\tau)}{d\tau}\,e^{i\omega\Delta\tau_1}+r_2(\tau)\,\frac{dg^{(2)}_j}{d\tau}\,g^{(1)}_j\,e^{i\omega\Delta\tau_2}=0\,,\\ &\frac{dr_2(\tau)}{d\tau}\,e^{i\omega\Delta\tau_2}+r_1(\tau)\,\frac{dg^{(1)}_j}{d\tau}\,g^{(2)}_j\,e^{i\omega\Delta\tau_1}=0\,. \end{aligned} \tag{60}$$

Due to the perpendicular nature of the vectors $g^{(1)}_j$ and $g^{(2)}_j$, we have

$$\frac{dg^{(1)}_j}{d\tau}\,g^{(2)}_j=-\,\frac{dg^{(2)}_j}{d\tau}\,g^{(1)}_j=\gamma_{12}\,. \tag{61}$$

If we insert the vectors $g^{(I)}_i$ specified by (49) and (50) into the above relation and take into account eq. (14b), we find (see also Coates & Chapman 1990) that

$$\gamma_{12}=\frac{d\phi}{d\tau}\,, \tag{62}$$

where ϕ is the angle introduced in (50) and (52). Thus the quantity γ_{12} describes the rate of change of the angle ϕ by which the polarization vectors of qS waves rotate with respect to the vectorial basis of the ray-centred coordinate system. If γ_{12} is small, the polarization vectors of qS waves behave like the vectors $e^{(I)}_i$ and the qS waves decouple. From eqs (60), we can draw the same conclusions concerning the decoupling as in the preceding section.

Let us further denote

$$\delta\tau_{12}=\Delta\tau_1-\Delta\tau_2\sim\tau_1-\tau_2\,,\qquad \delta\tau_{21}\sim\tau_2-\tau_1\,. \tag{63}$$

Here τ_1 and τ_2 are the approximate traveltimes of the qS waves related to $\Delta\tau_1$ and $\Delta\tau_2$, respectively. With the above notation, we can rewrite eqs (60) into the final form

$$\begin{aligned} \frac{dr_1(\tau)}{d\tau}&=\gamma_{12}r_2(\tau)\,\exp(-i\omega\delta\tau_{12})\,,\\ \frac{dr_2(\tau)}{d\tau}&=-\gamma_{12}r_1(\tau)\,\exp(-i\omega\delta\tau_{21})\,. \end{aligned} \tag{64}$$

Quantities r_I are obviously functions of frequency ω. If the initial conditions for solving (64) have the forms

$$r_1(\tau_0)=\frac{g^{(1)}_n(\tau_0)}{4\pi\sqrt{\rho(\tau_0)\beta(\tau_0)}}\,,\qquad r_2(\tau_0)=\frac{g^{(2)}_n(\tau_0)}{4\pi\sqrt{\rho(\tau_0)\beta(\tau_0)}}\,, \tag{65}$$

we can construct from the solutions of (64) Green's functions similar to those in eqs (40) and (41).

Eqs (64) are identical to eqs (30) of Coates & Chapman (1990). Coates & Chapman obtained the above equations by a completely independent approach, based on the use of a coupling volume integral. We note that the equations of Coates & Chapman are a specification of a more general system whose applicability is much broader than that of the equations presented in this paper. On the other hand, Coates & Chapman (1990) observe that the coupling coefficient (62) often changes rapidly through a large value (near singularities), which makes the numerical evaluation of eqs (64) difficult. To avoid these problems, Coates & Chapman (1990) suggest several modifications of eqs (64). From what was shown in the preceding section, we see that these complications could

GJI000 14/9/98 13:17:44 3B2 version 5.20
The Charlesworth Group, Huddersfield 01484 517077

probably be avoided if instead of vectors $g_i^{(I)}$, the vectors $e_i^{(I)}$ are used as a vectorial frame along the ray. Besides the removal of the problems mentioned, the procedure described in the previous section requires knowledge of neither the approximate polarization vectors $g_i^{(I)}$ nor the time-shifts $\Delta\tau_I$ along the ray of the *S* wave in the background medium.

5 DISCUSSION AND CONCLUSIONS

Formulae for the calculation of the $(0+1A)$ terms of the Green's functions for *qP* and *qS* waves in the quasi-isotropic approximation have been derived. The formulae are applicable to laterally inhomogeneous media of arbitrary but weak anisotropy. In the limit of infinitely weak anisotropy, the formulae reduce to the formulae for $(0+1A)$ terms of the *P*- and *S*-wave ray Green's functions for inhomogeneous isotropic media. When anisotropy becomes stronger and/or frequency higher and/or the characteristic length larger, so that condition (47) is fulfilled, the results of the QI approximation are approximately decoupled waves, which can be further studied separately by the standard ray method for anisotropic media. Except in singular regions of the ray method for isotropic media, for example in the vicinity of a caustic, the zeroth-order QI approximation is regular everywhere. It is expected to yield regular results not only in regions of weak anisotropy but also in regions of *qS*-wave singularities such as intersection or kiss singularities, for example.

The procedure for constructing the Green's functions in the QI approximation can be divided into two steps: (1) calculation of rays, traveltimes, the geometrical spreading and polarization vectors in the background isotropic medium; (2) calculation of corrections of traveltimes, amplitudes and polarization vectors due to the deviation of the weakly anisotropic medium from the isotropic background at termination points of rays of the wave. The corrections for *qP* waves are given by explicit frequency-dependent expressions. The corrections for *qS* waves can be obtained either by solving numerically a system of two simple linear ordinary differential equations or from approximate explicit frequency-dependent formulae. In the latter case, the formulae describe uncoupled *qS* waves.

The QI formulae were presented here in the frequency domain. Their transformation into the time domain is straightforward. The formulae can easily be incorporated in programs for the computation of seismic wavefields in laterally varying layered isotropic media such as the 'Complete ray tracing' package (Červený, Klimeš & Pšenčík 1988) or the ANRAY package (Gajewski & Pšenčík 1990). The coupled differential equations for *qS* waves must then be solved along rays in the isotropic background repeatedly for all the frequencies considered using the fast frequency response approach. Alternatively, calculations in the time domain, similar to those performed by Coates & Chapman (1990), can be used.

In the zeroth-order QI approximation, the formulae presented for the *qS* waves are equivalent to the coupling ray theory (CRT) formulae of Coates & Chapman (1990). This means that the QI approximation is applicable everywhere the CRT works. The QI approximation is expected to give results of comparable or even better quality (when, in addition to the zeroth-order terms, the first-order additional terms of the QI approximation are considered). The equivalence with the CRT guarantees that the QI approximation yields acceptable results in inhomogeneous weakly anisotropic media with anisotropy up to about 10 per cent, as well as in *qS*-wave singular regions in strongly anisotropic media (see Coates & Chapman 1990).

As mentioned above, in regions of stronger anisotropy, the QI approximation yields results that can be matched with the results of the standard ray method for anisotropic media. The combination of the two methods should thus cause no difficulties. It is interesting to note that the use of such a combination in a singular region in an anisotropic medium of arbitrary strength will lead to the generation of both *qS* waves behind the singularity.

There are no problems in applying the formulae in regions of a smooth transition from an isotropic to an anisotropic medium and vice versa. This problem was investigated recently by Thomson *et al.* (1992) using a generalization of the standard ray method. A single *S* wave propagating from an isotropic medium to an anisotropic medium will split into two waves calculated along a common ray specified in the background isotropic medium. When anisotropy becomes stronger and the two *qS* waves separate, the initial values for the continuation of calculations with the standard formulae of the ray method for inhomogeneous anisotropic media (see e.g. Červený 1972; Gajewski & Pšenčík 1990), can be found from (54). From the point of applicability of eqs (54), rays of two separate waves will be calculated. The slowness vectors of the rays of both waves at the point of matching will have the same direction. On transition from a weakly anisotropic medium to an isotropic medium, two separate waves with a time delay developed in the anisotropic region will propagate along a common ray.

The procedure presented has, of course, certain limitations, and its application raises new questions. The procedure has only a limited region of applicability. It is evident that the region of applicability of the quasi-isotropic approximation differs from the region of applicability of the standard ray method. This is a consequence of the appearance of the new small parameter $\epsilon_2 \sim |\Delta a_{ijkl}|/||a_{ijkl}||$ in the QI approximation [in addition to the small parameter $\epsilon_1 \sim c/(\omega L)$]. The consequences of this are seen in the calculation of the traveltime. In the standard ray method, the traveltime calculated along the ray is exact and does not need any additional corrections. It is used in all higher-order approximations. In the QI approximation, the traveltime is evaluated only approximately as a sum of the 'isotropic' traveltime along the ray in the background isotropic medium and a correction $\Delta\tau$ due to the deviation Δa_{ijkl} from isotropy. The exact ray traveltime is thus expanded into a series in powers of ϵ_2, from which only the linear term is kept. An upper estimate of the error of the QI solutions is given by Naida (1977), who also proposes a generalization of the QI approximation extending the region of its applicability. Neglecting the second- and higher-order terms in the traveltime expansion in the QI approximation leads to the limitation of applicability of the QI formulae for smaller distances. This fact probably explains the low accuracy of the perturbation formulae for the calculation of traveltimes of *qP* and *qS* waves reported by various authors. Further investigation of the accuracy of the formulae presented either by studying the behaviour of the higher-order terms of the QI approximation or by comparison with other methods of modelling seismic wavefields in inhomogeneous anisotropic media is, therefore, necessary. The effects of the choice of

GJI000 14/9/98 13:18:09 3B2 version 5.20
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background isotropic medium on the QI approximation must also be studied. Special attention must be devoted to the application of the QI approximation in singular regions of qS waves.

The procedure described suffers from all the limitations of the standard ray method: it is approximate and it fails at caustic regions and at transitions from illuminated to shadow regions.

The formulae also offer extensions and generalizations. The most straightforward extension would be the extension to slightly absorbing, weakly anisotropic media in a way similar to that used by Gajewski & Pšenčík (1992). Extensions to *layered* weakly anisotropic media seem straightforward.

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GJI000 14/9/98 13:18:33 3B2 version 5.20
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