Notes on summation of Gaussian beams and packets

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Summary

A Gaussian beam is a high–frequency asymptotic time–harmonic solution of the wave equation, with an approximately Gaussian profile perpendicularly to the central ray. A Gaussian packet is a high–frequency asymptotic space–time solution of the wave equation. The envelope of a Gaussian packet at any given time is nearly Gaussian function. The accuracy of a Gaussian beam or of a Gaussian packet depends on its shape. The shape of a Gaussian beam or packet should thus be optimized for the propagation between a given source and a given receiver.

Gaussian beams and packets may serve as building blocks of a wavefield. The summation of Gaussian beams and packets overcomes the problems of the standard ray theory with caustics. The summation of Gaussian beams and packets is considerably comprehensive and flexible. It may be formulated in many ways and depends primarily on the specification of the wavefield. The summation of Gaussian beams and packets includes two–parametric summation of Gaussian beams forming a time–harmonic wavefield specified asymptotically in terms of the amplitude and travel time, three–parametric summation of Gaussian packets forming a time–harmonic wavefield also specified asymptotically in terms of the amplitude and travel time, the Maslov method and its various generalizations, coherent–state transform methods, four–parametric decomposition of a general time–harmonic wavefield into Gaussian beams, six–parametric decomposition of a general time–dependent wavefield into Gaussian packets (used, e.g., for Gaussian–packet prestack depth migrations), and the system of Gaussian packets scattered from Gabor functions forming medium perturbations.

The integral superpositions of Gaussian beams and packets are discretized into summations. The discretization step should carefully be controlled in order to preserve both accuracy and efficiency of the summation.

Keywords

Gaussian beams, Gaussian packets, coherent states, wavefield representation, summation methods, superposition integrals, Maslov method, linear canonical transform, Gabor transform, prestack depth migration.

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1. Introduction

Gaussian beams and packets may serve as building blocks of a wavefield. The summation of Gaussian beams (Červený, Popov & Pšenčík, 1982; Popov, 1982; Červený, 1985) and packets overcomes the problems of the standard ray theory with caustics. The summation of Gaussian beams and packets is considerably comprehensive and flexible. It may be formulated in many ways, including the Maslov method and its various generalizations as special cases. The form of the summation depends primarily on the specification of the wavefield, comprising also the system of Gaussian packets scattered from Gabor functions forming medium perturbations.

Under the *wave equation* we understand here a non–dissipative hyperbolic second–order partial differential equation (e.g., elastodynamic equation, system of Maxwell equations). The basic principles of the summation of Gaussian beams and packets are independent of a particular selection of the wave equation.

We concentrate here on the representation of non–directional wavefields without pronounced diffractions rather than on the decomposition of directional beams or of diffracted waves. We thus omit the higher–order Gaussian beams (Bessel–Gaussian beams, Hermite–Gaussian beams, Laguerre–Gaussian beams) and the diffracted Gaussian beams.

A Gaussian beam or packet is expressed in terms of a complex–valued vectorial amplitude and complex–valued travel time (phase function, eikonal). We shall prefer term "phase function" specifically for Gaussian packets, otherwise we shall use term "travel time". The travel time depends on the wave equation through the eikonal equation, while the amplitude depends on the wave equation through the transport equation.

2. Gaussian beams

A Gaussian beam is a high–frequency asymptotic time–harmonic solution of the wave equation, with an approximately Gaussian profile perpendicularly to the central ray. It represents a bundle of complex–valued rays concentrated in a vicinity of a real–valued central ray. A paraxial Gaussian beam is a Gaussian beam approximated by the paraxial ray approximation in the vicinity of the central ray or in the vicinity of a selected reference point. The paraxial ray approximation consists in the second–order Taylor expansion of travel time and in the constant amplitude.

Evolution of the shape of a Gaussian beam along the central ray is determined by the respective ordinary differential equations. The travel time and its first derivatives along the real–valued central ray are determined, together with the central ray, by ray tracing and are real–valued in non–dissipative media. The complex–valued second derivatives of travel time, which control the shape of the Gaussian beam, are determined by "dynamic ray tracing" (Červený, 2001). Dynamic ray tracing also determines the complex–valued amplitude of the Gaussian beam.

Infinitely broad paraxial Gaussian beams correspond to the paraxial approximation of time–harmonic ray–theory wavefields. If used in a superposition, they must differ from the paraxial approximation of the decomposed ray–theory wavefield. Ribbon Gaussian beams are paraxial Gaussian beams infinitely broad only in a "singular" direction perpendicular to the central ray. If used in a superposition, they correspond to the paraxial approximation of the decomposed ray–theory wavefield in the singular direction. If the ribbon Gaussian beams used in a superposition are infinitely broad in all directions, they coincide with the paraxial approximation of the decomposed ray–theory wavefield only in the singular direction, but must differ from it in other directions perpendicular to the central ray. Note that infinitely broad Gaussian beams correspond to the standard Maslov method. However, these infinitely broad Gaussian beams are not recommended.

3. Gaussian packets

Gaussian packets, also called (space–time) Gaussian beams (Ralston, 1983), quasi-photons (Babich & Ulin, 1981) or coherent states (Combescure, Ralston & Robert, 1999), are high–frequency asymptotic space–time solutions of the wave equation. Their envelopes at any given time are nearly Gaussian functions. A Gaussian packet is concentrated to a real–valued space–time ray, like a Gaussian beam to a spatial ray. In a stationary medium, a Gaussian packet propagates along its real–valued spatial central ray. A Gaussian packet has an approximately Gaussian profile in all spatial directions and in time. A paraxial Gaussian packet is a Gaussian packet approximated by the paraxial ray approximation in the vicinity of its central point. The time–dependent central point is the spatial position of the maximum of the envelope of the Gaussian packet.

A paraxial Gaussian packet centred at point y_κ may be expressed in the form

$$\mathbf{U}(x_\kappa) = \mathbf{A}\exp\{\mathrm{i}\omega[N_\alpha r_\alpha + \tfrac{1}{2}N_{\alpha\beta}r_\alpha r_\beta]\} \quad , \tag{1}$$

where the lower–case Greek subscripts $_{\alpha,\beta,\ldots} = {}_{1,2,3,4}$ correspond to four space–time coordinates: spatial coordinates x_i with $_i = {}_{1,2,3}$, and time x_4. Here

$$r_\alpha = x_\alpha - y_\alpha \tag{2}$$

is the distance from the space–time coordinates y_α of the central point,

$$N_i = p_i^{\mathrm{R}} \tag{3}$$

is the spatial ray–theory slowness vector at the central point, and

$$N_4 = -1 \quad . \tag{4}$$

The complex–valued second partial derivatives $N_{\alpha\beta}$ of the phase function with respect to Cartesian coordinates x_i at the central point y_α determine the shape of the Gaussian packet. The imaginary part of the 3×3 spatial submatrix N_{kl} of the 4×4 matrix $N_{\alpha\beta}$ should be positive–definite. The imaginary part of the whole 4×4 matrix $N_{\alpha\beta}$ is singular.

The central point of a Gaussian packet moves along the spatial central ray according to the ray tracing equations. The first derivatives N_α of the phase function at the central point of the Gaussian packet are determined by ray tracing and are real–valued in non–dissipative media. Evolution of the complex–valued second derivatives $N_{\alpha\beta}$ of the phase function along the spatial central ray is determined by quantities calculated by dynamic ray tracing (Klimeš, 2004). Dynamic ray tracing also determines the complex–valued amplitude of the Gaussian packet.

Infinitely long paraxial Gaussian packets correspond to the paraxial approximations of Gaussian beams. Infinitely broad paraxial Gaussian packets correspond to the paraxial approximation of space–time ray–theory wavefields.

4. Optimization of the shape of Gaussian beams or packets

Gaussian beams and packets are high–frequency approximate solutions of the wave equation. The accuracy of these approximate solutions depends on their shape, especially on their width. The "optimum" shape depends on the distance from the source (Klimeš, 1989a).

The optimum shape of Gaussian beams depends on the error of the Gaussian–beam solution of the wave equation, which is, unfortunately, unknown for general media. However, in considerably heterogeneous media, the optimum Gaussian beams are very close to the narrowest Gaussian beams.

The root–mean–square width of each individual Gaussian beam between the source and receiver can be minimized using the algorithm by Klimeš (1989a).

The same algorithm can be used to minimize the width of a Gaussian packet, measured perpendicularly to the central ray. The optimum Gaussian packet is *symmetric* with respect to its central ray. The error of the Gaussian–packet solution of the wave equation is smaller for longer Gaussian packets, but the length of gaussian packets is limited by the accuracy of their paraxial approximation. This problem with long Gaussian packets may be overcome by decomposing each long Gaussian packet into summation of shorter ones, similarly as composing a Gaussian beam of Gaussian packets in the asymptotic summation of Gaussian packets. Each long Gaussian packet is then numerically calculated by combining the paraxial approximations of a Gaussian packet from different reference points (Žáček, 2004).

To preserve the accuracy of the approximate decomposition of a wavefield into Gaussian beams or packets, optimization of the shape of individual beams or packets should be supplemented with smoothing the dependence of the shape on summation parameters (Žáček, 2001a, 2001b).

A possibility to clearly optimize the shape of Gaussian beams or packets in the superposition integral is probably the main difference of the summation of Gaussian beams and packets from the integral representations formally derived using the Maslov method, coherent–state method, or their various generalizations, notwithstanding formally equal integral representations.

5. Asymptotic summation of Gaussian beams and packets

5.1. Asymptotic decomposition into Gaussian beams

A general time–harmonic wavefield given along a surface, decomposed into individual polarizations and expressed in terms of the amplitude and travel time, can be asymptotically expressed as the two–parametric integral superposition of Gaussian beams (Klimeš, 1984a).

The two–parametric integral superposition of Gaussian beams, corresponding to ray–theory wavefield

$$\mathbf{u}^{\mathrm{R}} = \mathbf{A}^{\mathrm{R}} \exp(\mathrm{i}\omega\tau^{\mathrm{R}}) \quad , \tag{5}$$

may be expressed as (Klimeš, 1984a, eq. 77)

$$\mathbf{u}(x_i,\omega) = \frac{\omega}{2\pi} \iint \mathrm{d}\gamma_1 \mathrm{d}\gamma_2 \; \mathbf{A}^{\mathrm{R}} \, |\det(\mathbf{Q}^{\mathrm{R}})| \; \sqrt[2]{\det}[\mathrm{i}\,(\mathbf{M}^{\mathrm{R}} - \mathbf{M})] \times \exp\{\mathrm{i}\omega[\tau^{\mathrm{R}} + p_k^{\mathrm{R}}(x_k - y_k) + \tfrac{1}{2}(x_k - y_k)N_{kl}(x_l - y_l)]\} \quad , \tag{6}$$

where the function $\sqrt[n]{\det}(\mathbf{B})$ of symmetric complex–valued matrix $\mathbf{B}$ is defined as the product of the n^{th} roots of the eigenvalues of matrix $\mathbf{B}$, and the roots with the greatest

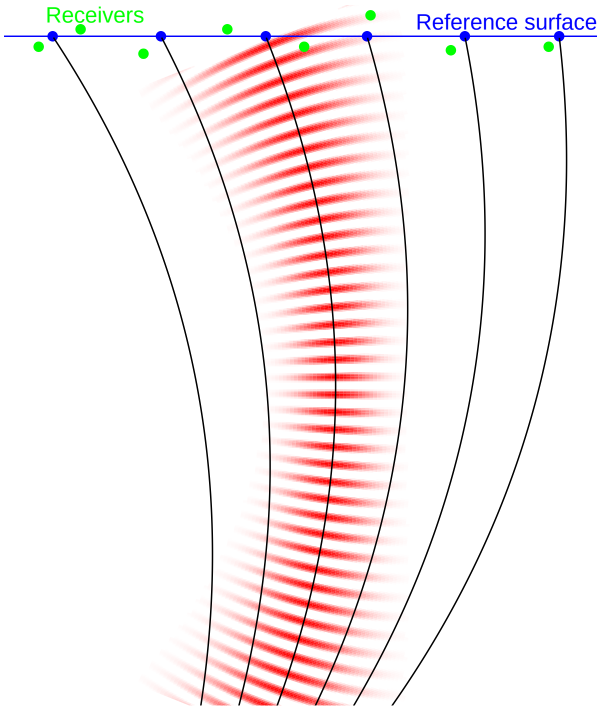

Figure 1. Asymptotic summation of Gaussian beams. Receivers are situated in the vicinity of the reference surface. The points of intersection of rays with the reference surface are selected as the reference points for the paraxial approximation of Gaussian beams.

real parts are selected. All the quantities on the right–hand side (except x_i) are taken at the reference point $y_i = y_i(\gamma_1, \gamma_2)$ selected for the paraxial approximation of a Gaussian beam concentrated to the ray with ray parameters γ_1, γ_2. The reference points are usually determined as the points of intersection of rays with a given reference surface. Receivers should be situated in the vicinity of the reference surface.

Here τ^{R}, p_k^{R} and N_{kl}^{R} are the ray–theory travel time, ray–theory slowness vector and the matrix of second derivatives of the ray–theory travel time in Cartesian coordinates x_i, taken at point y_i. Similarly $\mathbf{A}^{\mathrm{R}}$ is the complex–valued vectorial ray–theory amplitude at point y_i. Matrix $\mathbf{M}^{\mathrm{R}}$ is the 2×2 matrix of second derivatives of the ray–theory travel time with respect to ray–centred coordinates q_1, q_2 perpendicular to the central ray. Matrix $\mathbf{M}$ is the 2×2 matrix of second derivatives of the complex–valued travel time of the Gaussian beam with respect to ray–centred coordinates q_1, q_2. Matrix $\mathbf{M}$ controls the shape of Gaussian beams and should be selected with the positive–definite imaginary part. Matrix N_{kl} is the 3×3 matrix of second derivatives of the complex–valued travel time of the Gaussian beam with respect to Cartesian coordinates x_i, and is completely determined by 2×2 matrix $\mathbf{M}$ (Klimeš, 1984a, eqs. 77 and 78). The transformation Jacobian from ray parameters γ_1, γ_1 to ray–centred coordinates q_1, q_2 is $|\det(\mathbf{Q}^{\mathrm{R}})|$, where $\mathbf{Q}^{\mathrm{R}}$ is the 2×2 matrix of geometrical spreading in ray–centred coordinates (Červený, 2001).

The asymptotic integral superposition (6) is independent on the selection of the surface for decomposition of the ray–theory wavefield into Gaussian beams (Klimeš, 1984a). That is why the ray–theory wavefield may asymptotically be decomposed into Gaussian beams locally, even in its singular regions. The accuracy of the integral superposition of Gaussian beams thus depends on the optimization of the shape of beams only, not on the surface for decomposition. The variation of the optimum shape of beams with the distance from the source is quite different from the evolution of individual Gaussian beams along the same central ray. In another words, we decompose the wavefield into different Gaussian beams for different propagation distances.

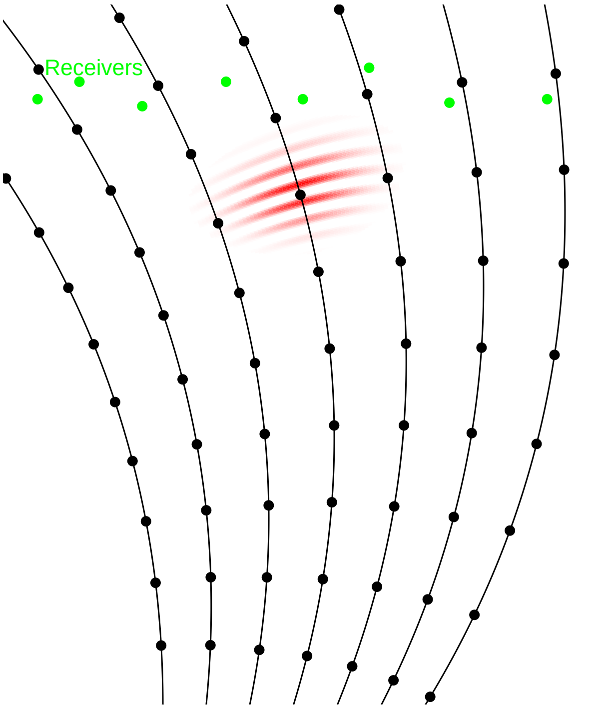

Figure 2. Asymptotic summation of Gaussian packets. The central points for the paraxial approximation of Gaussian packets are regularly distributed along rays. The density of rays and of central points depends on the shape of Gaussian packets. Receivers may be situated arbitrarily, no reference surfaces are required.

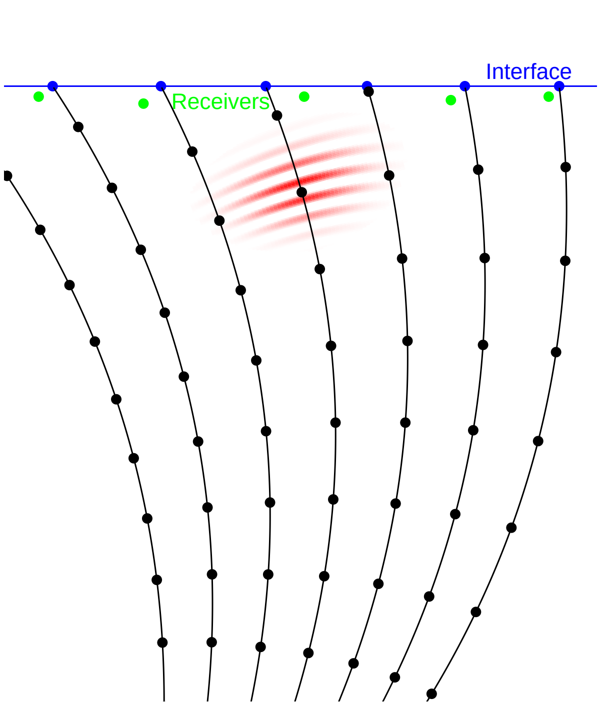

Figure 3. Summation of Gaussian packets at a structural interface. Paraxial approximation cannot be applied across the interface. The points of intersection of rays with the interface serve as the reference points for the paraxial approximation of Gaussian packets centred behind the interface. Receivers may be situated arbitrarily.

5.2. Asymptotic decomposition into Gaussian packets

A time–harmonic Gaussian beam may be expressed as a one–parametric integral superposition of space–time Gaussian packets. A general time–harmonic wavefield, specified in terms of the amplitude and travel time, can be asymptotically expressed as the three–parametric integral superposition of space–time Gaussian packets (Klimeš, 1984b, 1989b).

The three–parametric integral superposition of Gaussian packets corresponding to the ray–theory wavefield (5) may be expressed as (Klimeš, 1984b, eq. 51; 1989b, eq. 31)

$$\mathbf{u}(x_i,\omega) = \left(\frac{\omega}{2\pi}\right)^{3/2} \iiint \mathrm{d}\gamma_1 \mathrm{d}\gamma_2 \mathrm{d}\gamma_3 \; \mathbf{A}^{\mathrm{R}} \; v \, |\det(\mathbf{Q}^{\mathrm{R}})| \; \sqrt[2]{\det[\mathrm{i}\,(\mathbf{N}^{\mathrm{R}}-\mathbf{N})]}$$
$$\times \exp\{\mathrm{i}\omega[\tau^{\mathrm{R}} + p_k^{\mathrm{R}}(x_k-y_k) + \tfrac{1}{2}(x_k-y_k)N_{kl}(x_l-y_l)]\} \quad , \quad (7)$$

where all the quantities on the right–hand side (except x_k) are taken at the central point $y_i = y_i(\gamma_1,\gamma_2,\gamma_3)$ of a Gaussian packet. Points $y_i = y_i(\gamma_1,\gamma_2,\gamma_3)$ are the points of spatial rays corresponding to ray–theory wavefield $\mathbf{u}^{\mathrm{R}}$. The quadrature is performed over ray parameters γ_1, γ_2 and over travel time γ_3 along the rays.

Here τ^{R}, p_k^{R} and N_{kl}^{R} are the ray–theory travel time, ray–theory slowness vector and the matrix of second derivatives of the ray–theory travel time in Cartesian coordinates x_i, taken at point y_i. Similarly $\mathbf{A}^{\mathrm{R}}$ is the matrix of the complex–valued vectorial ray–theory amplitude at point y_i. The transformation Jacobian from ray parameters γ_l to Cartesian coordinates x_i is $v\,|\det(\mathbf{Q}^{\mathrm{R}})|$, where v is the wave–propagation velocity and $\mathbf{Q}^{\mathrm{R}}$ is the 2×2 matrix of geometrical spreading in ray–centred coordinates (Červený, 2001).

Matrix N_{kl} is the 3×3 matrix of second derivatives of the complex–valued phase function of the Gaussian packet with respect to Cartesian coordinates x_i. Matrix N_{kl} controls the shape of Gaussian packets and should be selected with the positive–definite imaginary part.

The properties of the integral superposition of space–time Gaussian packets follow from the properties of the integral superposition of Gaussian beams.

5.3. Discretization error

In numerical algorithms, the integral superposition of Gaussian beams is discretized into the summation. The error due to the discretization depends on frequency and on the shape of Gaussian beams and can be controlled by the selection of a discretization step. A conservative upper error bound for the discretization error of an arbitrary wavefield was derived by Daubechies (1991). A considerably more "optimistic" estimation of the discretization error applicable to smooth variation of the weights of Gaussian beams in the decomposition was derived by Klimeš (1986), for 2-D also by Hill (1990). This smooth variation includes wavefields generated by various point sources, or wavefields with smooth variation of the amplitude and travel time along an initial surface.

The error due to the discretization of the integral superposition of Gaussian packets can be controlled analogously (Klimeš, 1989b).

5.4. Linear canonical transforms and coherent state transforms

Spatial coordinate x_i in phase space corresponds to operator $\hat{x}_i$ of the multiplication by x_i. Momentum coordinate p_i^x in phase space corresponds to the differential operator

$$\hat{p}_i = \frac{1}{\mathrm{i}\omega}\frac{\partial}{\partial x_i} \quad . \tag{8}$$

Let us consider a linear coordinate transform in phase space,

$$\begin{pmatrix} \mathbf{x} \\ \mathbf{p}^x \end{pmatrix} = \mathcal{M} \begin{pmatrix} \mathbf{x}' \\ \mathbf{p}^{x'} \end{pmatrix} \quad , \tag{9}$$

where the complex–valued 6×6 matrix $\mathcal{M}$ may be composed of four 3×3 matrices,

$$\mathcal{M} = \begin{pmatrix} \tilde{\mathbf{Q}} & \mathbf{Q} \\ \tilde{\mathbf{P}} & \mathbf{P} \end{pmatrix} \quad . \tag{10}$$

We shall see later on that matrix $\mathcal{M}$ should be symplectic. The linear coordinate transform corresponds to the linear transform

$$\begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{p}}^x \end{pmatrix} = \mathcal{M} \begin{pmatrix} \hat{\mathbf{x}}' \\ \hat{\mathbf{p}}^{x'} \end{pmatrix} \tag{11}$$

of the operators. We now wish to find the linear transform

$$f_{\mathcal{M}}(\mathbf{x}) = \int \mathrm{d}^3\mathbf{x}' \; C_{\mathcal{M}}(\mathbf{x},\mathbf{x}')\, f(\mathbf{x}') \tag{12}$$

of functions, with integration kernel of the form

$$C_{\mathcal{M}}(\mathbf{x},\mathbf{x}') = c_{\mathcal{M}} \exp\left[\frac{\mathrm{i}\omega}{2} \begin{pmatrix} \mathbf{x} \\ \mathbf{x}' \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} \mathbf{N}_{\mathcal{M}}^{xx} & \mathbf{N}_{\mathcal{M}}^{xx'} \\ \mathbf{N}_{\mathcal{M}}^{x'x} & \mathbf{N}_{\mathcal{M}}^{x'x'} \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{x}' \end{pmatrix}\right] \quad , \tag{13}$$

which corresponds to transform (11) of operators. We thus apply transform (11) with (8) and (10) to equation (12). Assuming that the integrand vanishes at the boundary of the integration volume, we obtain equations

$$\mathbf{x}\, C_{\mathcal{M}}(\mathbf{x},\mathbf{x}') = \left[\tilde{\mathbf{Q}}\,\mathbf{x}' - \mathbf{Q}\frac{\partial}{\mathrm{i}\omega\partial\mathbf{x}'}\right] C_{\mathcal{M}}(\mathbf{x},\mathbf{x}') \tag{14}$$

and

$$\frac{\partial}{\mathrm{i}\omega\partial\mathbf{x}} C_{\mathcal{M}}(\mathbf{x},\mathbf{x}') = \left[\tilde{\mathbf{P}}\,\mathbf{x}' - \mathbf{P}\frac{\partial}{\mathrm{i}\omega\partial\mathbf{x}'}\right] C_{\mathcal{M}}(\mathbf{x},\mathbf{x}') \tag{15}$$

for the integration kernel. Differentiating the kernel in equations (14) and (15), we arrive at equations

$$\mathbf{x} = \tilde{\mathbf{Q}}\,\mathbf{x}' - \mathbf{Q}\,[\mathbf{N}^{x'x}_{\mathcal{M}}\mathbf{x} + \mathbf{N}^{x'x'}_{\mathcal{M}}\mathbf{x}'] \quad , \tag{16}$$

$$\mathbf{N}^{xx}_{\mathcal{M}}\mathbf{x} + \mathbf{N}^{xx'}_{\mathcal{M}}\mathbf{x}' = \tilde{\mathbf{P}}\,\mathbf{x}' - \mathbf{P}\,[\mathbf{N}^{x'x}_{\mathcal{M}}\mathbf{x} + \mathbf{N}^{x'x'}_{\mathcal{M}}\mathbf{x}'] \quad . \tag{17}$$

Equations (16) and (17) can be separated into the terms with $\mathbf{x}$ and $\mathbf{x}'$ to obtain matrix equations

$$\mathbf{1} = -\mathbf{Q}\mathbf{N}^{x'x}_{\mathcal{M}} \quad , \tag{18}$$

$$\mathbf{0} = \tilde{\mathbf{Q}} - \mathbf{Q}\mathbf{N}^{x'x'}_{\mathcal{M}} \quad , \tag{19}$$

$$\mathbf{N}^{xx}_{\mathcal{M}} = -\mathbf{P}\mathbf{N}^{x'x}_{\mathcal{M}} \quad , \tag{20}$$

$$\mathbf{N}^{xx'}_{\mathcal{M}} = \tilde{\mathbf{P}} - \mathbf{P}\mathbf{N}^{x'x'}_{\mathcal{M}} \quad . \tag{21}$$

Equations (18), (19) and (20) yield

$$\mathbf{N}^{x'x}_{\mathcal{M}} = (\mathbf{N}^{xx'}_{\mathcal{M}})^{\mathrm{T}} = -\mathbf{Q}^{-1} \quad , \tag{22}$$

$$\mathbf{N}^{x'x'}_{\mathcal{M}} = \mathbf{Q}^{-1}\tilde{\mathbf{Q}} \quad , \tag{23}$$

$$\mathbf{N}^{xx}_{\mathcal{M}} = \mathbf{P}\mathbf{Q}^{-1} \quad . \tag{24}$$

Equation (21) then restricts matrix $\mathcal{M}$ to a symplectic matrix. The relation of symplectic matrix (10) and integration kernel (13) resembles the relation of the paraxial ray propagator matrix and the paraxial two–point eikonal (Arnaud, 1972, eq. 11.1; Červený, Klimeš & Pšenčík, 1984, eq. 22).

The inverse transforms to (9) and (11) are

$$\begin{pmatrix} \mathbf{x}' \\ \mathbf{p}^{x'} \end{pmatrix} = \mathcal{M}^{-1} \begin{pmatrix} \mathbf{x} \\ \mathbf{p}^{x} \end{pmatrix} \tag{25}$$

and

$$\begin{pmatrix} \hat{\mathbf{x}}' \\ \hat{\mathbf{p}}^{x'} \end{pmatrix} = \mathcal{M}^{-1} \begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{p}}^{x} \end{pmatrix} \quad , \tag{26}$$

where the inverse matrix to symplectic matrix $\mathcal{M}$ is

$$\mathcal{M}^{-1} = \begin{pmatrix} \mathbf{P}^{\mathrm{T}} & -\mathbf{Q}^{\mathrm{T}} \\ -\tilde{\mathbf{P}}^{\mathrm{T}} & \tilde{\mathbf{Q}}^{\mathrm{T}} \end{pmatrix} \quad . \tag{27}$$

The inverse transform to (12) is

$$f_{\mathcal{M}^{-1}}(\mathbf{x}') = \int \mathrm{d}^3\mathbf{x}\; C_{\mathcal{M}^{-1}}(\mathbf{x}',\mathbf{x})\, f(\mathbf{x}) \quad . \tag{28}$$

Integration kernel $C_{\mathcal{M}^{-1}}(\mathbf{x}',\mathbf{x})$ is obtained from symplectic matrix $\mathcal{M}^{-1}$ by equations analogous to (13) and (22) to (24).

The product of integration kernels of (28) and (12) is

$$C_{\mathcal{M}^{-1}}(\mathbf{x}'',\mathbf{x})\;C_{\mathcal{M}}(\mathbf{x},\mathbf{x}')$$
$$= c_{\mathcal{M}^{-1}}c_{\mathcal{M}}\exp\left\{\mathrm{i}\omega\left[\tfrac{1}{2}\mathbf{x}'^{\mathrm{T}}\mathbf{Q}^{-1}\tilde{\mathbf{Q}}\mathbf{x}'-\tfrac{1}{2}\mathbf{x}''^{\mathrm{T}}\mathbf{Q}^{-1}\tilde{\mathbf{Q}}\mathbf{x}''+(\mathbf{x}''-\mathbf{x}')^{\mathrm{T}}\mathbf{Q}^{-1}\mathbf{x}\right]\right\} \quad . \tag{29}$$

Integration of (29) over $\mathbf{x}$ yields

$$\int \mathrm{d}^3\mathbf{x}\; C_{\mathcal{M}^{-1}}(\mathbf{x}'',\mathbf{x}) = c_{\mathcal{M}^{-1}}c_{\mathcal{M}}\;\sqrt[1]{\det}\!\left(\frac{2\pi}{\omega}\mathbf{Q}\right)\delta(\mathbf{x}''-\mathbf{x}') \quad , \tag{30}$$

where the matrix function $\sqrt[n]{\det}(\mathbf{A})$ is defined as the product of the $(2n)^{\mathrm{th}}$ roots of the eigenvalues of matrix $\mathbf{A}^{\mathrm{T}}\mathbf{A}$, and the roots with the greatest real parts are selected. The right–hand side of (30) should be equal to the Dirac distribution $\delta(\mathbf{x}''-\mathbf{x}')$. Since $\sqrt[2]{\det}(\mathbf{Q}) = \sqrt[2]{\det}(-\mathbf{Q}^{\mathrm{T}})$, we naturally choose

$$c_{\mathcal{M}} = \left[\sqrt[2]{\det}\!\left(\frac{2\pi}{\omega}\mathbf{Q}\right)\right]^{-1} \quad . \tag{31}$$

For more details refer to Wolf (1974, 1979) or Ozaktas, Zalevsky & Kutay (2001).

Interesting special cases of 3-D linear canonical transform (12) are 3-D Fourier transform

$$\mathcal{M} = \begin{pmatrix} \mathbf{0} & -\mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix} \quad , \tag{32}$$

and 3-D separable fractional Fourier transform (Condon, 1937; Ozaktas, Zalevsky & Kutay, 2001)

$$\mathcal{M} = \begin{pmatrix} \cos(\alpha_1) & 0 & 0 & -\sin(\alpha_1) & 0 & 0 \\ 0 & \cos(\alpha_2) & 0 & 0 & -\sin(\alpha_2) & 0 \\ 0 & 0 & \cos(\alpha_3) & 0 & 0 & -\sin(\alpha_3) \\ \sin(\alpha_1) & 0 & 0 & \cos(\alpha_1) & 0 & 0 \\ 0 & \sin(\alpha_2) & 0 & 0 & \cos(\alpha_2) & 0 \\ 0 & 0 & \sin(\alpha_3) & 0 & 0 & \cos(\alpha_3) \end{pmatrix} \quad . \tag{33}$$

For real–valued, imaginary–valued or complex–valued α_i, (33) defines real–ordered, imaginary–ordered or complex–ordered fractional Fourier transforms, respectively.

Paraxial approximation of a Gaussian packet (coherent state) in the vicinity of its central point $\mathbf{y}$ is performed in local Cartesian coordinates

$$\mathbf{r} = \mathbf{x} - \mathbf{y} \quad . \tag{34}$$

It is thus reasonable also to consider linear canonical transforms (12) and (28) with respect to local Cartesian coordinates $\mathbf{r}$. We define

$$\varphi(\mathbf{r},\mathbf{y}) = f(\mathbf{r}+\mathbf{y}) \quad . \tag{35}$$

The local inverse linear canonical transform

$$\psi(\mathbf{r}',\mathbf{y}) = \varphi_{\mathcal{M}^{-1}}(\mathbf{r}',\mathbf{y}) \tag{36}$$

of function $\varphi(\mathbf{r},\mathbf{y})$ with respect to the first argument is called the *coherent state transform* of function $f(\mathbf{x})$ (Klauder, 1987, eq. 3; Foster & Huang, 1991, eq. 39; Thomson, 2001, eq. 2.4). Function $\varphi(\mathbf{r},\mathbf{y})$ may be obtained from the coherent state transform $\psi(\mathbf{r}',\mathbf{y})$ by local linear canonical transform

$$\varphi(\mathbf{r},\mathbf{y}) = \psi_{\mathcal{M}}(\mathbf{r},\mathbf{y}) \quad . \tag{37}$$

Function $f(\mathbf{x})$ may be restored from $\varphi(\mathbf{r},\mathbf{y})$ in many ways, yielding various linear transforms of operators $\hat{\mathbf{x}}$ and $\hat{\mathbf{p}}^x$ to operators $\hat{\mathbf{r}}'$, $\hat{\mathbf{p}}^{r'}$, $\hat{\mathbf{y}}'$ and $\hat{\mathbf{p}}^{y'}$. Functional transform

$$f(\mathbf{x}) = \varphi(\mathbf{0},\mathbf{x}) \tag{38}$$

(Klauder, 1987, eq. 4; Foster & Huang, 1991, eq. 40; Thomson, 2001, eq. 2.6) yields operator transform

$$\begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{p}}^x \end{pmatrix} = \begin{pmatrix} \hat{\mathbf{y}} \\ \hat{\mathbf{p}}^y \end{pmatrix} \quad , \tag{39}$$

whereas functional transform

$$f(\mathbf{x}) = \varphi(\mathbf{y},\mathbf{x}-\mathbf{y}) \tag{40}$$

yields operator transform

$$\begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{p}}^x \end{pmatrix} = \begin{pmatrix} \hat{\mathbf{y}}+\hat{\mathbf{r}} \\ \hat{\mathbf{p}}^y \end{pmatrix} = \begin{pmatrix} \hat{\mathbf{y}}+\tilde{\mathbf{Q}}\hat{\mathbf{r}}'+\mathbf{Q}\hat{\mathbf{p}}^{r'} \\ \hat{\mathbf{p}}^y \end{pmatrix} \tag{41}$$

actually used by Foster & Huang (1991, eq. 46) and Thomson (2001, eqs. 4.1 and 4.2). Functional transform

$$f(\mathbf{x}) = \varphi(\mathbf{x}-\mathbf{y},\mathbf{y}) \quad , \tag{42}$$

naturally corresponding to equations (34) and (35), yields operator transform

$$\begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{p}}^x \end{pmatrix} = \begin{pmatrix} \hat{\mathbf{y}}+\hat{\mathbf{r}} \\ \hat{\mathbf{p}}^r \end{pmatrix} = \begin{pmatrix} \hat{\mathbf{y}} \\ \mathbf{0} \end{pmatrix} + \mathcal{M}\begin{pmatrix} \hat{\mathbf{r}}' \\ \hat{\mathbf{p}}^{r'} \end{pmatrix} \quad . \tag{43}$$

Functional transform

$$f(\mathbf{x}) = \varphi(\mathbf{x},\mathbf{0}) \tag{44}$$

reduces the coherent state transform to the linear canonical transform.

5.5. Maslov methods

The standard ray theory is derived and expressed in the "coordinate representation", i.e., with respect to the spatial coordinates. In order to obtain various special cases of the summation of Gaussian beams and packets, the high–frequency approximation may be developed with respect to 3 appreciably general "representation coordinates" $\mathbf{x}'$ chosen in 6-D phase space, see (25), and then transformed to the coordinate representation.

The original Maslov method (Maslov, 1965; Chapman & Drummond, 1982) consists in weighted combination of the standard ray–theory approximation with the Maslov methods of the first, second and third order. The Maslov method of the first order corresponds to one spatial coordinate replaced by the respective momentum coordinate. The Maslov method of the second or third order corresponds to two or three spatial coordinates replaced by the respective momentum coordinates, see (32). We obtain the generalized eikonal and transport equations by the high–frequency approximation in this representation, and solve them. The travel time in this representation is the Legendre transform of the ray–theory travel time with respect to 1, 2 or 3 coordinates. The high–frequency asymptotic approximation of the wavefield is then transformed to the coordinate representation by the 1-D, 2-D or 3-D Fourier transform, respectively. This Fourier transform forms the superposition integral. The Maslov method of the first order represents the one–parametric superposition of infinitely broad "ribbon" Gaussian beams with second–order derivatives of travel time vanishing along the summation lines.

The Maslov method of the second order represents the two–parametric superposition of infinitely broad Gaussian beams with second–order derivatives of travel time vanishing along the summation surfaces. The Maslov method of the third order represents the superposition of infinitely broad Gaussian packets with vanishing second–order spatial derivatives of the phase function.

Alonso & Forbes (1998) selected each representation coordinate x'_i as a real–valued linear combination of a spatial coordinate x_i and the corresponding momentum coordinate p^x_i, see (33). They then solved the generalized eikonal and transport equations obtained by the high–frequency approximation in this representation. The travel time in this representation is the separable real–ordered fractional Legendre transform of the ray–theory travel time (Alonso & Forbes 1995). The high–frequency asymptotic approximation of the wavefield is then transformed to the coordinate representation by the 1-D, 2-D or 3-D separable real–ordered fractional Fourier transform (Condon, 1937; Ozaktas, Zalevsky & Kutay, 2001). This real–ordered fractional Fourier transform forms the superposition integral. The resulting approximation represents the superposition of infinitely broad Gaussian beams (1-D, 2-D) or packets (3-D). Equivalent results have been achieved by application of the Maslov method in local curvilinear coordinates or with respect to the "reference travel time" (Kendall & Thomson, 1993). This approximation may artificially be supplemented with Gaussian windowing through the imaginary–ordered fractional Fourier transform, sometimes also called Gaussian–windowed Fourier transform (Alonso & Forbes, 1998; Forbes & Alonso, 1998; Kravtsov & Orlov, 1999). Analogous Gaussian windowing may be introduced using the coherent state transform (Foster & Huang, 1991).

The Maslov method yields general superpositions of Gaussian beams or packets if the representation coordinates $\mathbf{x}'$ are sufficiently general complex–valued linear combinations of phase–space coordinates $\mathbf{x}$ and $\mathbf{p}^x$ (Klimeš, 1984b, eq. 24). We then solve the generalized eikonal and transport equations obtained by the high–frequency approximation in the new representation. The travel time in the new representation may be obtained by the generalized Legendre transform of the ray–theory travel time (Klimeš, 1984b, eq. 39). The high–frequency asymptotic approximation of the wavefield is then transformed to the coordinate representation by the 3-D complex linear canonical transform (Klimeš, 1984b, eq. 27). Analogous superpositions may also be obtained by means of the coherent state transform (Thomson, 2001).

To obtain a general superposition of Gaussian packets by the Maslov method, we transform the wave equation to phase–space coordinates $\mathbf{x}'$, $\mathbf{p}^{x'}$, given by general complex–valued symplectic transform (25), by inserting expressions (11) for operators $\hat{\mathbf{x}}$, $\hat{\mathbf{p}}^x$. The high–frequency approximation then yields the generalized eikonal and transport equations. The solutions of these equations may be expressed in terms of the solutions of standard eikonal and transport equations. The obtained approximate high–frequency solution can then be transformed back to coordinates $\mathbf{x}'$ by by 3-D complex linear canonical transform (12), which is a generalization of both the real–ordered fractional Fourier transform and imaginary–ordered fractional Fourier transform (Gaussian–windowed Fourier transform). The resulting approximate high–frequency solution is identical to general 3–parametric superposition (7) of Gaussian packets, including general 2–parametric superposition (6) of Gaussian beams as a special case. Matrix $\mathbf{N}$ controlling the shape of Gaussian packets in (7) is identical to matrix (24),

$$\mathbf{N} \equiv \mathbf{N}^{xx}_{\mathcal{M}} \quad . \tag{45}$$

The approximate high–frequency solution (7) derived in this way is thus independent of the choice of $\mathbf{Q}$ in symplectic matrix (10), provided that we set $\mathbf{P} = \mathbf{NQ}$. Solution (7) is also independent of matrices $\tilde{\mathbf{Q}}$ and $\tilde{\mathbf{P}}$ forming with $\mathbf{Q}$ and $\mathbf{P}$ symplectic matrix $\mathcal{M}$. Klimeš (1984b) thus performed the above described derivation with matrix (10) selected in the special form

$$\mathcal{M} = \begin{pmatrix} \mathbf{0} & -\mathbf{1} \\ \mathbf{1} & -\mathbf{N} \end{pmatrix} \quad , \tag{46}$$

which corresponds to the form

$$\mathcal{M}^{-1} = \begin{pmatrix} -\mathbf{N} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{pmatrix} \tag{47}$$

of matrix (27).

Note that Foster & Huang (1991) and Thomson (2001) selected matrix (27) for the coherent state transform (36) in the separable form

$$\mathcal{M}^{-1} = \begin{pmatrix} \mathrm{i}\Omega & 0 & 0 & 1 & 0 & 0 \\ 0 & \mathrm{i}\Omega & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad . \tag{48}$$

Operators $\hat{\mathbf{r}}$ and $\hat{\mathbf{p}}^r$ are then transformed to operators $\hat{\mathbf{r}}'$ and $\hat{\mathbf{p}}^{r'}$ using equation (11) with matrix

$$\mathcal{M} = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & \mathrm{i}\Omega & 0 & 0 \\ 0 & 1 & 0 & 0 & \mathrm{i}\Omega & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad . \tag{49}$$

Compared with equations (45) to (47), where matrix $\mathbf{N}$ has the positive–definite imaginary part, Foster & Huang (1991) and Thomson (2001) selected the analogous 2×2 matrix with the negative–definite imaginary part.

Unfortunately, the mathematical formalism of Maslov methods obscures the evolution of Gaussian beams or Gaussian packets along rays and makes the optimization of their shapes more difficult.

6. Decomposition of a general wavefield into Gaussian packets or beams

Assume a time–dependent wavefield specified along a given surface, and call it "time section". The trace of a Gaussian packet in the time section is approximately a Gabor function. The widths of the envelopes of Gabor functions are inversely proportional to the square root of frequency, not constant as for the Gabor transforms (discrete, integral) nor inversely proportional to frequency as for the wavelet transforms. Moreover, the shape of the packets has to be to some extent optimized with respect to the wave equation. and is thus often dependent on time and on the coordinates and wavenumbers along the surface. The Gabor functions corresponding to optimized Gaussian packets have envelopes considerably dependent on frequency and on the direction of propagation, and moderately dependent on the position and time. This makes the decomposition of

Figure 4. Sensitivity of seismic waves to structural perturbations: Single Gabor function $g(\mathbf{x})$ centred at point $\mathbf{x}$.

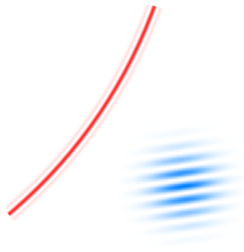

Figure 5. Sensitivity of seismic waves to structural perturbations: Short–duration broad–band wave incident at the Gabor function.

Figure 6. Scattered wave. The wavenumber vector of the scattered Gaussian packet is determined by the structural wavenumber vector $\mathbf{k}$ and by the wavenumber vector of the incident wave.

a general wavefield into Gaussian packets intricate. Žáček (2003a, 2003b) generalized the integral Gabor transform towards the approximate integral expansion of a time section into the Gabor functions of varying shape. The approximate expansion into Gaussian packets has the form of a linear integral transform. The expansion is exact if the envelopes of Gaussian packets depend on frequency and on the angle of incidence only. The system of Gaussian packets is four–parametric in 2-D and six–parametric in 3-D.

Decomposition of a general wavefield into Gaussian packets of given, optimized envelopes is crucial for Gaussian–packet true–amplitude prestack depth migrations. It may also enable to develop general hybrid methods, combining the Gaussian packet summation method with finite differences, finite elements, and other highly accurate methods which can be applied only to small parts of large models at high frequencies.

A time–dependent wavefield specified along a given surface can also be Fourier transformed into the frequency domain, where it consists of individual time–harmonic wavefields. Each time–harmonic wavefield may then be decomposed using the oversampled Gabor transform into four–parametric (in 3-D) system of Gabor functions, which are the traces of Gaussian beams along the surface (Hill, 1990, 2001; Shlivinski et al., 2004).

7. Sensitivity of waves to heterogeneities

We decompose perturbations of the coefficients of the wave equation (e.g., elastic moduli and density in the elastodynamic equation) into Gabor functions

$$g(\mathbf{x}) = \exp[\,\mathrm{i}\mathbf{k}^{\mathrm{T}}(\mathbf{x}-\mathbf{y}) - \tfrac{1}{2}(\mathbf{x}-\mathbf{y})^{\mathrm{T}}\mathbf{K}(\mathbf{x}-\mathbf{y})] \tag{50}$$

centred at various spatial positions $\mathbf{y}$ and having various structural wavenumber vectors $\mathbf{k}$. We consider a short–duration incident wavefield with a smooth broad–band frequency spectrum. The wavefield scattered by the perturbations is then composed of waves scattered by individual Gabor functions. The scattered waves are estimated using the first–order Born approximation with paraxial ray approximation. Each Gabor function usually generates only few narrow–band space–time Gaussian packets propagating in specific directions (Žáček & Klimeš, 2003). The only exceptions are broad–band forward scattering and broad–band narrow–angle scattering from the lowest structural wavenumbers, and rather rare broad–band critical scattering with mode conversion. Each scattered Gaussian packet is sensitive to just a single linear combination of the coefficients of the wave equation. This information about the Gabor function is lost if the scattered Gaussian packet does not fall into the aperture covered by the receivers and into the legible frequency band.

8. Migrations

A "prestack depth migration" is a simple back–projection of a wavefield, roughly approximating the inversion of wide–angle scattering. It often includes even additional rough approximations. The back–propagated wavefield is compared with the incident wavefield, forming an "image" (convolutional transform) of the gradient of a particular linear combination of the coefficients of the wave equation.

8.1. Gaussian packet migrations

The recorded wavefield (time section) is decomposed into the Gaussian packets as described in Section 6. Individual Gaussian packets are back–propagated and compared with the incident wavefield (Žáček, 2004). The image of each back–propagated Gaussian packet is approximately formed by one or few Gabor functions. We thus obtain not only the image of small–scale structural heterogeneities, but also the relation between the time section and the heterogeneities.

The algorithm of the true–amplitude common–source prestack depth migrations based on Gaussian packets consists of the following basic steps: optimization of the model for ray tracing and for Gaussian packets, calculation of travel times and other ray–theory quantities from the source to the dense rectangular grid of points covering the target zone, optimization of the envelopes of Gaussian packets travelling from various parts of the target zone to the receivers, decomposition of the time section into Gaussian packets, and the backprojection of individual Gaussian packets from the time section onto the migrated image of the target zone.

8.2. Gaussian beam migrations

The recorded wavefield, Fourier transformed into the frequency domain, may be decomposed into the Gaussian beams as described in Section 6. Individual Gaussian beams are back–propagated and compared with the incident wavefield (Hill, 1990, 2001). In the frequency domain, the image of each back–propagated Gaussian beam extends along its whole central ray.

To improve the numerical efficiency, this procedure of Gaussian beam migration can be transformed from the frequency domain to the time domain. Refer to Hill (1990, 2001) for more details.

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