Perturbation expansions of complex-valued travel time along real-valued reference rays

Martin Klimeš
*Département de mathématiques et de statistique, Faculté des arts et des sciences, Université de Montréal,
C.P. 6128, succursale Centre-ville, Montréal, QC H3C 3J7, Canada*

Luděk Klimeš
Department of Geophysics, Faculty of Mathematics and Physics, Charles University, Ke Karlovu 3, 121 16 Praha 2, Czech Republic, http://sw3d.cz/staff/klimes.htm

Summary

The eikonal equation in an attenuating medium has the form of a complex–valued Hamilton–Jacobi equation and must be solved in terms of the complex–valued travel time (complex–valued action function). The solution of the complex–valued Hamilton–Jacobi equation for complex–valued travel time by Hamilton's equations of rays would require complex–valued rays (complex–valued geodesics). Since the material properties are known in real space only, we cannot calculate complex–valued rays. A very suitable approximate method for calculating the complex–valued travel time right in real space is represented by the perturbation from the reference travel time calculated along real–valued reference rays to the complex–valued travel time defined by the complex–valued Hamilton–Jacobi equation.

For this perturbation from the reference travel time to the complex–valued travel time, we need a complex–valued perturbation Hamiltonian function, i.e., a family of complex–valued Hamiltonian functions smoothly parametrized by one or more perturbation parameters. The perturbation Hamiltonian function must smoothly connect the reference Hamiltonian function with the Hamiltonian function corresponding to a given complex–valued Hamilton–Jacobi equation, and Hamilton's equations corresponding to the reference Hamiltonian function must yield real–valued reference rays. In order to be able to perform the perturbation from the reference travel time to the complex–valued travel time, we need the perturbation Hamiltonian function to be a holomorphic function of the complex slowness vector. This paper is devoted to the construction of the reference Hamiltonian function for a given complex–valued Hamilton–Jacobi equation, and to the construction of the corresponding complex–valued perturbation Hamiltonian function.

The perturbation Hamiltonian function may be constructed in different ways, yielding differently accurate perturbation expansions of travel time. Unlike in previous papers, we construct the reference Hamiltonian function directly using the Hamiltonian function corresponding to a given complex–valued Hamilton–Jacobi equation. The direct construction of the reference Hamiltonian function from the given complex–valued Hamilton–Jacobi equation is very general and accurate, especially for homogeneous Hamiltonian functions of degree $N = -1$ with respect to the slowness vector.

Keywords

Ray theory, complex–valued travel time (complex–valued action function), complex–valued Hamilton–Jacobi equation, complex–valued eikonal equation, perturbation methods, attenuation, anisotropy, heterogeneous media, wave propagation.

In: Seismic Waves in Complex 3–D Structures, Report 20 (Department of Geophysics, Charles University, Prague, 2010), pp. 193–205

1. Introduction

The eikonal equation in an attenuating medium has the form of a complex–valued Hamilton–Jacobi equation and must be solved in terms of the complex–valued travel time (complex–valued action function).

The solution of the complex–valued Hamilton–Jacobi equation for complex–valued travel time by Hamilton's (1837) equations of rays would require complex–valued rays (complex–valued geodesics). Since the material properties are known in real space only, we cannot calculate complex–valued rays. We thus need to calculate the complex–valued travel time right in real space. A very suitable approximate method for this purpose is represented by the perturbation from the reference travel time calculated along real–valued reference rays to the complex–valued travel time defined by the complex–valued Hamilton–Jacobi equation.

For this perturbation from the reference travel time to the complex–valued travel time, we need a complex–valued perturbation Hamiltonian function, i.e., a family of complex–valued Hamiltonian functions smoothly parametrized by one or more perturbation parameters. The perturbation Hamiltonian function must smoothly connect the reference Hamiltonian function with the Hamiltonian function corresponding to a given complex–valued Hamilton–Jacobi equation, and Hamilton's equations corresponding to the reference Hamiltonian function must yield real–valued reference rays. In order to be able to perform the perturbation from the reference travel time to the complex–valued travel time, we need the perturbation Hamiltonian function to be a holomorphic function of the complex slowness vector. This paper is devoted to the construction of the reference Hamiltonian function for a given complex–valued Hamilton–Jacobi equation, and to the construction of the corresponding complex–valued perturbation Hamiltonian function.

The perturbation Hamiltonian function may be constructed in different ways. Different forms of the perturbation Hamiltonian function yield different expressions for the perturbation expansion of travel time of a given order, with considerably different accuracies. In this paper, we construct the reference Hamiltonian function directly using the Hamiltonian function corresponding to a given complex–valued Hamilton–Jacobi equation, whereas Červený, Klimeš & Pšenčík (2008) and Červený & Pšenčík (2009) considered the complex–valued Hamilton–Jacobi equation for travel time in an anisotropic attenuating medium, selected a reference anisotropic non–attenuating medium, and then put the reference Hamiltonian function equal to the Hamiltonian function corresponding to the reference anisotropic non–attenuating medium. The direct construction of the reference Hamiltonian function from a given complex–valued Hamilton–Jacobi equation is more general and should usually be more accurate, which was numerically demonstrated by Vavryčuk (2009) in a special case of an isotropic attenuating medium.

In this paper, we propose to prefer homogeneous Hamiltonian functions of degree $N=-1$ with respect to the slowness vector. Homogeneous Hamiltonian functions of degree $N=-1$ usually yield the most accurate linear perturbations of travel time, which was theoretically explained by Klimeš (2002, sec. 4.4), numerically demonstrated by Bulant & Klimeš (2008) in examples of perturbations from isotropic reference rays and common anisotropic reference rays in an anisotropic elastic medium, and also numerically demonstrated by Vavryčuk (2009) in examples of perturbations from real–valued reference rays in two isotropic attenuating media.

When a perturbation Hamiltonian function is constructed, we can calculate the perturbation derivatives (derivatives with respect to perturbation parameters) of travel time according to equations by Klimeš (2002), and construct the perturbation expansion (Taylor expansion with respect to perturbation parameters) of travel time. For the calculation of the n-th order perturbation derivatives of travel time, we need the phase–space, perturbation and mixed derivatives of the perturbation Hamiltonian function up to the n-th order at the real–valued reference rays. The perturbation derivatives of travel time of all orders are calculated by simple numerical quadratures along unperturbed reference rays.

In Section 2, which represents the key part of the paper, we propose a general method for the construction of the reference Hamiltonian function for an arbitrary given complex–valued Hamilton–Jacobi equation, and for the construction of the corresponding complex–valued perturbation Hamiltonian function. The subsequent sections are devoted to the application of this general method. In Section 3, we review general Hamilton's equations for reference rays and the corresponding general Hamiltonian equations of geodesic deviation (dynamic ray tracing system), and apply them to the reference Hamiltonian function derived in Section 2. In Section 4, we review the general equations for the first–order and second–order perturbation derivatives of travel time, and apply them to the perturbation Hamiltonian function derived in Section 2. In Section 5, we apply the final general equations of Sections 3 and 4 to the complex–valued eikonal equation (Hamilton–Jacobi equation) for viscoelastic waves propagating in an anisotropic attenuating medium. The final equations for the approximate calculation of the complex–valued travel time of viscoelastic waves, propagating in an anisotropic attenuating medium, along real–valued reference rays can be found in Section 5.2.

Under phase space, we understand a spatial manifold parametrized by coordinates x^i with cotangent spaces parametrized by slowness–vector components p_i. We use the componental notation for vectors and matrices. For example, p_i stands for the covariant vector with components p_i. The Einstein summation over repetitive lower–case Roman indices is used throughout the paper. The summation does not apply to subscripts α corresponding to the derivatives with respect to the perturbation parameter.

2. Construction of the perturbation Hamiltonian function corresponding to a given complex–valued Hamiltonian function

The equations of Klimeš (2002) for perturbation expansions of travel time are applicable to the complex–valued travel time and complex slowness vector, if the complex–valued perturbation Hamiltonian function is a holomorphic function of the slowness vector. For the perturbation expansions of complex–valued travel time along real–valued rays (geodesics), we thus need a holomorphic perturbation Hamiltonian function which yields real–valued reference rays.

2.1. Complex–valued Hamiltonian function

We consider the complex–valued Hamiltonian function

$$H(x^m, p_n) \tag{1}$$

of real spatial coordinates x^m and of complex slowness vector p_n. We assume that $H(x^m, p_n)$ is a holomorphic function of p_n in the domain of our interest. The corresponding Hamilton–Jacobi equation for complex–valued travel time $\tau = \tau(x^k)$ reads

$$H\big(x^m, \tau_{,n}(x^k)\big) = C \quad , \tag{2}$$

where

$$\tau_{,i}(x^k) = \frac{\partial}{\partial x^i}\tau(x^k) \quad . \tag{3}$$

The value of constant C is determined by the form and meaning of the Hamiltonian function.

2.2. Reference Hamiltonian function

In order to be able to perform the perturbation from the reference travel time to the complex–valued travel time, we need the reference Hamiltonian function $\widetilde{H}(x^m, p_n)$ to be a holomorphic function of the complex slowness vector.

We assume that the reference travel time is real–valued. In order to obtain real–valued reference rays, the reference Hamiltonian function $\widetilde{H}(x^m, p_n)$ should take real values for real slowness vectors p_n. Since the reference Hamiltonian function $\widetilde{H}(x^m, p_n)$ should be as close to the given Hamiltonian function $H(x^m, p_n)$ as possible, we want $\widetilde{H}(x^m, p_n)$ to be equal to the real part $\mathrm{Re}[H(x^m, p_n)]$ for real p_n. Requiring that the reference Hamiltonian function $\widetilde{H}(x^m, p_n)$ should be a holomorphic function of the slowness vector then determines $\widetilde{H}(x^m, p_n)$ uniquely.

The reference Hamiltonian function may be constructed in the following way. We choose a real slowness vector p_n^0, and take the infinite Taylor expansion of the complex–valued Hamiltonian function $H(x^m, p_n)$ with respect to p_i at phase–space point (x^m, p_n^0),

$$H(x^m, p_n) = \sum_{\Omega=0}^{+\infty} \frac{1}{\Omega!}\, H^{,k_1 k_2 \ldots k_\Omega}(x^m, p_n^0)\; (p_{k_1} - p_{k_1}^0)\,(p_{k_2} - p_{k_2}^0)\ldots(p_{k_\Omega} - p_{k_\Omega}^0) \quad , \tag{4}$$

where

$$H^{,k_1 k_2 \ldots k_\Omega}(x^m, p_n) = \frac{\partial}{\partial p_{k_1}}\frac{\partial}{\partial p_{k_2}}\ldots\frac{\partial}{\partial p_{k_\Omega}} H(x^m, p_n) \quad . \tag{5}$$

The reference Hamiltonian function $\widetilde{H}(x^m, p_n)$ is obtained just by replacing the coefficients of the Taylor series by their real parts:

$$\widetilde{H}(x^m, p_n) = \sum_{\Omega=0}^{+\infty} \frac{1}{\Omega!}\, \mathrm{Re}[H^{,k_1 k_2 \ldots k_\Omega}(x^m, p_n^0)]\; (p_{k_1} - p_{k_1}^0)\,(p_{k_2} - p_{k_2}^0)\ldots(p_{k_\Omega} - p_{k_\Omega}^0) \; . \tag{6}$$

For real p_i, equation (6) represents the real part of the Taylor expansion of $H(x^m, p_n)$ with respect to p_i at phase–space point (x^m, p_n^0), and its value is independent of the choice of real p_n^0 within the domain of convergence.

For each p_n^0, equation (6) defines a holomorphic function of complex p_n in the domain of convergence. Since the values of functions (6) for two different p_n^0 are equal at real p_n from the intersection of the corresponding domains of convergence, they are also equal at complex p_n from the intersection of the corresponding domains of convergence. Taking the expansion at two different real points p_n^0 close to one another will thus yield the same holomorphic function $\widetilde{H}(x^m, p_n)$. Equation (6) with proper choices of real vectors p_n^0 thus defines a holomorphic function of complex p_n in the vicinity of all real p_n. This holomorphic function coincides with $\mathrm{Re}[H(x^m, p_n)]$ at real p_n.

In order to secure the convergence of (6) in the vicinity of real p_n, we therefore equivalently express $\widetilde{H}(x^m, p_n)$ as a series at $p_n^0 = \mathrm{Re}(p_n)$ for every complex vector p_n sufficiently close to real one. This leads to the following more convenient form:

$$\widetilde{H}(x^m, p_n) = \sum_{\Omega=0}^{+\infty} \frac{\mathrm{i}^\Omega}{\Omega!}\, \mathrm{Re}[H^{,k_1 k_2 \ldots k_\Omega}(x^m, \mathrm{Re}\, p_n)]\; \mathrm{Im}(p_{k_1})\,\mathrm{Im}(p_{k_2})\ldots\mathrm{Im}(p_{k_\Omega}) \quad . \tag{7}$$

We shall assume that the modified Taylor series (7) is convergent in the domain of our interest. On this condition, the reference Hamiltonian function $\widetilde{H}(x^m, p_n)$ will be defined and will be a holomorphic function of complex p_n in the domain of our interest.

2.3. Perturbation Hamiltonian function

For a convenient perturbation from Hamiltonian function $\widetilde{H}(x^m, p_n)$ to Hamiltonian function $H(x^m, p_n)$, we define the one–parametric perturbation Hamiltonian function

$$H(x^m, p_n, \alpha) = \widetilde{H}(x^m, p_n) + [H(x^m, p_n) - \widetilde{H}(x^m, p_n)]\, \alpha \quad , \tag{8}$$

linear with respect to perturbation parameter α.

For each value of α, Hamilton–Jacobi equation (2) with the corresponding Hamiltonian function $H(x^m, p_n, \alpha)$ defines the complex–valued travel time $\tau = \tau(x^k, \alpha)$.

We now calculate derivatives

$$H^{,k_1 k_2 \dots k_\Omega}_{,j_1 j_2 \dots j_\Phi \alpha \dots \alpha}(x^m, p_n, \alpha) = \frac{\partial}{\partial x^{j_1}} \frac{\partial}{\partial x^{j_2}} \dots \frac{\partial}{\partial x^{j_\Phi}} \; \frac{\partial}{\partial p_{k_1}} \frac{\partial}{\partial p_{k_2}} \dots \frac{\partial}{\partial p_{k_\Omega}} \; \frac{\partial}{\partial \alpha} \dots \frac{\partial}{\partial \alpha} H(x^m, p_n, \alpha) \tag{9}$$

of Hamiltonian function (8) at real p_m and $\alpha = 0$.

For real p_m, the perturbation Hamiltonian function and its derivatives with respect to complex p_m read

$$H(x^m, p_n, 0) = \mathrm{Re}[H(x^m, p_n)] \tag{10}$$

and

$$H^{,k_1 k_2 \dots k_\Omega}(x^m, p_n, 0) = \mathrm{Re}[H^{,k_1 k_2 \dots k_\Omega}(x^m, p_n)] \quad . \tag{11}$$

For real p_m, the first–order perturbation derivative of the perturbation Hamiltonian function and of its derivatives with respect to complex p_m read

$$H_{,\alpha}(x^m, p_n, 0) = \mathrm{i}\, \mathrm{Im}[H(x^m, p_n)] \tag{12}$$

and

$$H^{,k_1 k_2 \dots k_\Omega}_{,\alpha}(x^m, p_n, 0) = \mathrm{i}\, \mathrm{Im}[H^{,k_1 k_2 \dots k_\Omega}(x^m, p_n)] \quad . \tag{13}$$

The second–order and higher–order perturbation derivatives of the perturbation Hamiltonian function vanish,

$$H_{,\alpha\alpha}(x^m, p_n, 0) = H_{,\alpha\alpha\alpha}(x^m, p_n, 0) = \dots = 0 \tag{14}$$

and

$$H^{,k_1 k_2 \dots k_\Omega}_{,\alpha\alpha}(x^m, p_n, 0) = H^{,k_1 k_2 \dots k_\Omega}_{,\alpha\alpha\alpha}(x^m, p_n, 0) = \dots = 0 \quad . \tag{15}$$

The spatial derivatives and other mixed derivatives of perturbation Hamiltonian function (8) can be obtained by differentiating the above equations (10)–(15) with respect to x^i.

These derivatives of the perturbation Hamiltonian function can be used for calculating the perturbation derivatives of travel time $\tau(x^k, \alpha)$ (Klimeš, 2002, eqs. 19–21, 23).

We see that all phase–space derivatives of the complex–valued perturbation Hamiltonian function at $\alpha = 0$ are real–valued, which secures the reference rays and reference travel time $\tau(x^k, 0)$ to be real–valued.

3. Reference rays and reference travel time

3.1. Reference rays

Reference rays are determined by Hamilton's equations

$$\frac{\mathrm{d}x^i}{\mathrm{d}\gamma} = H^{,i}(x^m, p_n, 0) \quad , \tag{16}$$

$$\frac{\mathrm{d}p_i}{\mathrm{d}\gamma} = -H_{,i}(x^m, p_n, 0) \quad . \tag{17}$$

where the meaning of parameter γ along rays is determined by the form of the Hamiltonian function.

The initial conditions for p_n in equations (16) and (17) are determined by the initial travel time specified along the initial surface and by condition

$$H(x^m, p_n, 0) = C \quad , \tag{18}$$

where constant C is defined by Hamilton–Jacobi equation (2).

We insert relations (10) and (11) into the above equations. Hamilton's equations (16) and (17) for reference rays then read

$$\frac{\mathrm{d}x^i}{\mathrm{d}\gamma} = \mathrm{Re}[H^{,i}(x^m, p_n)] \quad , \tag{19}$$

$$\frac{\mathrm{d}p_i}{\mathrm{d}\gamma} = -\mathrm{Re}[H_{,i}(x^m, p_n)] \quad . \tag{20}$$

The initial conditions for p_n in equations (19) and (20) are determined by the initial travel time specified along the initial surface and by condition

$$\mathrm{Re}[H(x^m, p_n, 0)] = C \tag{21}$$

following from (18). Constant C is defined by Hamilton–Jacobi equation (2).

3.2. Reference travel time

Reference travel time τ^0 is determined by equation

$$\frac{\mathrm{d}\tau^0}{\mathrm{d}\gamma} = p_i\, H^{,i}(x^m, p_n, 0) \quad . \tag{22}$$

We insert relation (11), and equation (22) for reference travel time τ^0 reads

$$\frac{\mathrm{d}\tau^0}{\mathrm{d}\gamma} = p_i\, \mathrm{Re}[H^{,i}(x^m, p_n)] \quad . \tag{23}$$

3.3. Ray parameters and Hamiltonian equations of geodesic deviation

For calculating the second–order perturbation derivatives of travel time, we need the second–order spatial derivatives of travel time. These second–order spatial derivatives are most conveniently calculated using the Hamiltonian equations of geodesic deviation (dynamic ray tracing system) by Červený (1972).

In D–dimensional space, the initial conditions for reference rays, which start from the initial surface with the initial travel time for Hamilton–Jacobi equation (2), may be parametrized by $D-1$ ray parameters γ_1, γ_2, ..., γ_{D-1}. These $D-1$ ray parameters together with parameter $\gamma_D=\gamma$ along the rays form D ray coordinates.

We define the matrices of the partial derivatives of coordinates x^i and of slowness–vector components $p_i=\tau_{,i}$ with respect to the ray coordinates,

$$Q^i{}_a = \frac{\partial x^i}{\partial \gamma_a} \quad , \tag{24}$$

$$P_{ia} = \frac{\partial \tau_{,i}}{\partial \gamma_a} \quad , \tag{25}$$

where $\frac{\partial}{\partial\gamma_D}$ is identical to $\frac{\mathrm{d}}{\partial\gamma}$ used in other equations. Paraxial matrices $Q^i{}_a$ and P_{ia} describe, by definition, the properties of the orthonomic system of rays corresponding to the travel time under consideration. Let us emphasize that the definition of paraxial matrices $Q^i{}_a$ and P_{ia} depends on the kind of parameter $\gamma=\gamma_D$ along rays, which is in turn determined by the form of the Hamiltonian.

The system

$$\frac{\mathrm{d}}{\mathrm{d}\gamma} Q^i{}_a = \quad H^{,i}_{,j}(x^q,p_r,0)\; Q^j{}_a + H^{,ij}(x^q,p_r,0)\; P_{ja} \quad , \tag{26}$$

$$\frac{\mathrm{d}}{\mathrm{d}\gamma} P_{ia} = -H_{,ij}(x^q,p_r,0)\; Q^j{}_a - H^{,j}_{,i}(x^q,p_r,0)\; P_{ja} \tag{27}$$

of Hamiltonian equations of geodesic deviation (dynamic ray tracing system) of the reference rays can be obtained by differentiating Hamilton's equations (16)–(17) with respect to γ_a (Červený, 1972).

Equation

$$P_{ia} = \tau_{,ij}\; Q^j{}_a \tag{28}$$

is a direct consequence of definitions (24)–(25).

We insert relations (10) and (11) into the above equations. Hamiltonian equations (26) and (27) of the geodesic deviation of the reference rays then read

$$\frac{\mathrm{d}}{\mathrm{d}\gamma} Q^i{}_a = \quad \mathrm{Re}[H^{,i}_{,j}(x^q,p_r)]\; Q^j{}_a + \mathrm{Re}[H^{,ij}(x^q,p_r)]\; P_{ja} \quad , \tag{29}$$

$$\frac{\mathrm{d}}{\mathrm{d}\gamma} P_{ia} = -\mathrm{Re}[H_{,ij}(x^q,p_r)]\; Q^j{}_a - \mathrm{Re}[H^{,j}_{,i}(x^q,p_r)]\; P_{ja} \quad . \tag{30}$$

4. Perturbation expansion of complex–valued travel time

The perturbation expansion of complex–valued travel time $\tau=\tau(x^k,\alpha)$ is its Taylor expansion

$$\tau(x^m,\alpha) \approx \tau(x^m,0) + \tau_{,\alpha}(x^m,0)\,\alpha + \tfrac{1}{2}\tau_{,\alpha\alpha}(x^m,0)\,\alpha^2 + \tfrac{1}{6}\tau_{,\alpha\alpha\alpha}(x^m,0)\,\alpha^3 + ... \tag{31}$$

with respect to perturbation parameter α. The Greek subscripts following a comma denote partial derivatives with respect to perturbation parameter α, here referred to as perturbation derivatives.

Since perturbation Hamiltonian (8) yields the given Hamiltonian (1) for $\alpha=1$, perturbation expansion (31) yields the solution of Hamilton–Jacobi equation (2) for $\alpha=1$.

The zero–order term

$$\tau(x^m,0) = \tau^0 \tag{32}$$

in perturbation expansion (31) is the real–valued reference travel time determined by equation (23).

Equations (10)–(15) and their derivatives with respect to real x^i can be inserted into the equations of Klimeš (2002, eqs. 19–21, 23) for calculating the perturbation derivatives of travel time. As a consequence of this insertion, the even–order terms in the perturbation expansion (31) of travel time are real–valued, and the odd–order terms in the perturbation expansion (31) of travel time are purely imaginary.

We shall now explicitly express the equations for calculating the linear and quadratic terms in perturbation expansion (31) of travel time.

4.1 First–order perturbation derivative of travel time

The first–order perturbation derivative $\tau_{,\alpha}$ in the perturbation expansion (31) of travel time is determined by equation

$$\frac{\mathrm{d}\tau_{,\alpha}}{\mathrm{d}\gamma} = -H_{,\alpha}(x^m, p_n, 0) \tag{33}$$

(Klimeš, 2002, eq. 25).

We insert relation (12), and equation (33) for the first–order perturbation derivative $\tau_{,\alpha}$ in the perturbation expansion (31) of travel time reads

$$\frac{\mathrm{d}\tau_{,\alpha}}{\mathrm{d}\gamma} = -\mathrm{i}\,\mathrm{Im}[H(x^m, p_n)] \quad . \tag{34}$$

The first–order term in the perturbation expansion (31) of travel time is purely imaginary.

4.2 Second–order perturbation derivative of travel time

To calculate the second–order perturbation derivative $\tau_{,\alpha\alpha}$ of travel time, we need to calculate the second–order mixed derivatives $\tau_{,i\alpha}$ first.

The first–order perturbation derivative $\tau_{,i\alpha}$ of the spatial travel–time gradient is determined by equation

$$\tau_{,i\alpha}(x^m, 0) = T_{a\alpha}\, Q^{-1}_{ai} \tag{35}$$

(Klimeš, 2002, eq. 20), where Q^{-1}_{ai} are the elements of the matrix inverse to matrix $Q^i{}_a$. The covariant derivatives $T_{a\alpha}$ of $\tau_{,\alpha}$ with respect to ray coordinates γ_a can be calculated using equation

$$\frac{\mathrm{d}T_{,a\alpha}}{\mathrm{d}\gamma} = -H_{,j\alpha}(x^m, p_n, 0)\, Q^j{}_a - H^{;j}_{,\alpha}(x^m, p_n, 0)\, P_{ja} \tag{36}$$

(Klimeš, 2002, eqs. 19, 27). Since

$$T_{,D\alpha} = -H_{,\alpha}(x^m, p_n, 0) \tag{37}$$

(Klimeš, 2002, eq. 28), the quadrature of equation (36) is unnecessary for $a = D$.

The second–order perturbation derivative $\tau_{,\alpha\alpha}$ in the perturbation expansion (31) of travel time is determined by equation

$$\begin{aligned}\frac{\mathrm{d}\tau_{,\alpha\alpha}}{\mathrm{d}\gamma} = -H_{,\alpha\alpha}(x^m, p_n, 0) - 2H^{;j}_{,\alpha}(x^m, p_n, 0)\,\tau_{,j\alpha}(x^m, 0)&\\ - H^{;jk}(x^m, p_n, 0)\,\tau_{,j\alpha}(x^m, 0)\,\tau_{,k\alpha}(x^m, 0)&\end{aligned} \tag{38}$$

(Klimeš, 2002, eqs. 19, 20, 30).

We insert relations (11)–(14) into equations (36)–(38). Equations (36) and (37) then read

$$\frac{\mathrm{d}T_{,a\alpha}}{\mathrm{d}\gamma} = -\mathrm{i}\,\mathrm{Im}[H_{,j}(x^m,p_n)]\; Q^j_{\ a} - \mathrm{i}\,\mathrm{Im}[H^{,j}(x^m,p_n)]\; P_{ja} \tag{39}$$

and

$$T_{,D\alpha} = -\mathrm{i}\,\mathrm{Im}[H(x^m,p_n)] \quad . \tag{40}$$

The first–order perturbation derivative $\tau_{,i\alpha}$ of the spatial travel–time gradient is then determined by equation (35). We see that $T_{,i\alpha}$ and $\tau_{,i\alpha}$ are purely imaginary. Equation (38) then reads

$$\begin{aligned}\frac{\mathrm{d}\tau_{,\alpha\alpha}}{\mathrm{d}\gamma} = &- 2\,\mathrm{i}\,\mathrm{Im}[H^{,j}(x^m,p_n,0)]\,\tau_{,j\alpha}(x^m,0) \\ &- \mathrm{Re}[H^{,jk}(x^m,p_n)]\,\tau_{,j\alpha}(x^m,0)\,\tau_{,k\alpha}(x^m,0) \quad .\end{aligned} \tag{41}$$

The second–order term in the perturbation expansion (31) of travel time is real–valued.

5. Application to anisotropic attenuating media

We now consider a time–harmonic wave, which propagates in an anisotropic attenuating medium described by the complex–valued density–reduced viscoelastic moduli $a_{ijkl}(x^m)$ specified in Cartesian coordinates x^m.

The density–reduced viscoelastic moduli $a_{ijkl}(x^m)$ obey symmetry relations

$$a_{ijkl}(x^m) = a_{jikl}(x^m) = a_{ijlk}(x^m) = a_{klij}(x^m) \quad . \tag{42}$$

The eikonal equation (Hamilton–Jacobi equation) for the corresponding complex–valued travel time reads

$$G(x^m,p_n) = 1 \quad , \tag{43}$$

where the selected eigenvalue $G(x^m,p_n)$ of the symmetric complex–valued Christoffel matrix

$$\Gamma_{ik}(x^m,p_n) = a_{ijkl}(x^m)\; p_j\, p_l \tag{44}$$

is a homogeneous function of degree 2 with respect to slowness vector p_n. The eigenvalue can also be expressed as

$$G(x^m,p_n) = a_{ijkl}(x^m)\; g_i\, p_j\, g_k\, p_l \quad , \tag{45}$$

where $g_j = g_j(x^m,p_n)$ is the unit complex–valued eigenvector,

$$g_j\, g_j = 1 \quad , \tag{46}$$

corresponding to the eigenvalue.

The first–order phase–space derivatives of the eigenvalue read

$$G^{,i}(x^m,p_n) = 2\; a_{aikl}(x^m)\; g_a\, g_k\, p_l \quad , \tag{47}$$

$$G_{,i}(x^m,p_n) = a_{ajkl,i}(x^m)\; g_a\, p_j\, g_k\, p_l \quad . \tag{48}$$

For the calculation of the second–order phase–space derivatives $G^{,ij}$, $G^{,i}_{,j}$ and $G_{,ij}$ of the eigenvalue refer, e.g., to Červený (2001, eq. 4.14.7) or to Klimeš (2006, eq. 23).

5.1. Homogeneous Hamiltonian function for anisotropic attenuating media

We choose the complex–valued Hamiltonian function $H(x^m, p_n)$ homogeneous of degree N with respect to slowness vector p_n,

$$H(x^m, p_n) = \frac{1}{N} \, [G(x^m, p_n)]^{\frac{N}{2}} \quad , \tag{49}$$

where the selected eigenvalue $G(x^m, p_n)$ of the Christoffel matrix is given by (45), and is a homogeneous function of degree 2 with respect to slowness vector p_n. For this choice (49), the constant in Hamilton–Jacobi equation (2) has the value

$$C = \frac{1}{N} \quad . \tag{50}$$

For a homogeneous Hamiltonian function, parameter γ along the rays coincides with the reference travel time determined by equation (23),

$$\gamma = \tau^0 \quad , \tag{51}$$

and Hamilton's equations (19)–(20) for reference rays read

$$\frac{\mathrm{d}x^i}{\mathrm{d}\tau^0} = \frac{1}{2} \, \mathrm{Re}\big\{ [G(x^m, p_n)]^{\frac{N-2}{2}} \, G^{,i}(x^m, p_n) \big\} \quad , \tag{52}$$

$$\frac{\mathrm{d}p_i}{\mathrm{d}\tau^0} = -\frac{1}{2} \, \mathrm{Re}\big\{ [G(x^m, p_n)]^{\frac{N-2}{2}} \, G_{,i}(x^m, p_n) \big\} \quad . \tag{53}$$

The initial conditions for equations (52) and (53) should be chosen in such a way that

$$\mathrm{Re}[H(x^m, p_n)] = \frac{1}{N} \quad . \tag{54}$$

Paraxial matrices $Q^i{}_a$ and P_{ia} of the reference rays can be calculated using the linear Hamiltonian equations (29) and (30) of geodesic deviation, with the second–order phase–space derivatives $H^{,ij}$, $H^{,i}_{,j}$ and $H_{,ij}$ of the complex–valued Hamiltonian function (49) expressed according to Klimeš (2006, eq. 27).

Equation (34) for the first–order perturbation derivative $\tau_{,\alpha}$ in the perturbation expansion (31) of travel time reads

$$\frac{\mathrm{d}\tau_{,\alpha}}{\mathrm{d}\tau^0} = -\mathrm{i} \, \frac{1}{N} \, \mathrm{Im}\big\{ [G(x^m, p_n)]^{\frac{N}{2}} \big\} \quad . \tag{55}$$

The first–order perturbation derivative $\tau_{,i\alpha}$ of the spatial travel–time gradient is determined by equation (35), with

$$\begin{aligned} \frac{\mathrm{d}T_{,a\alpha}}{\mathrm{d}\gamma} = & - \frac{\mathrm{i}}{2} \, \mathrm{Im}\big\{ [G(x^m, p_n)]^{\frac{N-2}{2}} \, G_{,j}(x^m, p_n) \big\} \, Q^j{}_a \\ & - \frac{\mathrm{i}}{2} \, \mathrm{Im}\big\{ [G(x^m, p_n)]^{\frac{N-2}{2}} \, G^{,j}(x^m, p_n) \big\} \, P_{ja} \end{aligned} \tag{56}$$

and

$$T_{,D\alpha} = -\mathrm{i} \, \frac{1}{N} \, \mathrm{Im}\big\{ [G(x^m, p_n)]^{\frac{N}{2}} \big\} \quad , \tag{57}$$

see (39) and (40). Note that $T_{,i\alpha}$ and $\tau_{,i\alpha}$ are purely imaginary. Equation (41) then reads

$$\begin{aligned} \frac{\mathrm{d}\tau_{,\alpha\alpha}}{\mathrm{d}\gamma} = & - \mathrm{i} \, \mathrm{Im}\big\{ [G(x^m, p_n)]^{\frac{N-2}{2}} \, G^{,j}(x^m, p_n) \big\} \, \tau_{,j\alpha}(x^m, 0) \\ & - \mathrm{Re}[H^{,jk}(x^m, p_n)] \, \tau_{,j\alpha}(x^m, 0) \, \tau_{,k\alpha}(x^m, 0) \quad . \end{aligned} \tag{58}$$

For the calculation of the second–order phase–space derivatives $H^{,ij}$ of complex–valued Hamiltonian function (49) refer, e.g., to Klimeš (2006, eq. 27).

5.2. Homogeneous Hamiltonian function of degree -1 for anisotropic attenuating media

The most accurate linear perturbations of travel time are usually obtained for the homogeneous Hamiltonian function of degree $N = -1$ with respect to the slowness vector (Klimeš, 2002, sec. 4.4; Bulant & Klimeš, 2008; Vavryčuk, 2009).

For complex–valued Hamiltonian function $H(x^m, p_n)$ homogeneous of degree $N = -1$ with respect to slowness vector p_n, equation (49) reads

$$H(x^m, p_n) = -[G(x^m, p_n)]^{-\frac{1}{2}} \quad , \tag{59}$$

where the selected complex–valued eigenvalue $G(x^m, p_n)$ of the Christoffel matrix is given by (45). For this choice, the complex–valued travel time satisfies Hamilton–Jacobi equation (2) with constant

$$C = -1 \quad . \tag{60}$$

Hamilton's equations (52)–(53) for the real–valued reference rays and reference travel time τ^0 read

$$\frac{dx^i}{d\tau^0} = \frac{1}{2}\,\mathrm{Re}\big\{[G(x^m, p_n)]^{-\frac{3}{2}}\, G^{,i}(x^m, p_n)\big\} \quad , \tag{61}$$

$$\frac{dp_i}{d\tau^0} = -\frac{1}{2}\,\mathrm{Re}\big\{[G(x^m, p_n)]^{-\frac{3}{2}}\, G_{,i}(x^m, p_n)\big\} \quad , \tag{62}$$

where $G^{,i}(x^m, p_n)$ and $G_{,i}(x^m, p_n)$ are given by (47) and (48). The initial conditions for equations (61) and (62) should satisfy condition

$$\mathrm{Re}\big\{[G(x^m, p_n)]^{-\frac{1}{2}}\big\} = 1 \quad . \tag{63}$$

Paraxial matrices $Q^i{}_a$ and P_{ia} of the reference rays can be calculated using linear Hamiltonian equations (29) and (30) of geodesic deviation, with the second–order phase–space derivatives $H^{,ij}$, $H^{,i}_{,j}$ and $H_{,ij}$ of the complex–valued Hamiltonian function (59) expressed according to Klimeš (2006, eq. 27).

Equation (55) for the first–order perturbation derivative $\tau_{,\alpha}$ in the perturbation expansion (31) of travel time reads

$$\frac{d\tau_{,\alpha}}{d\tau^0} = \mathrm{i}\,\mathrm{Im}\big\{[G(x^m, p_n)]^{-\frac{1}{2}}\big\} \quad . \tag{64}$$

The first–order perturbation derivative $\tau_{,i\alpha}$ of the spatial travel–time gradient is determined by equation (35), with

$$\begin{aligned}\frac{dT_{,a\alpha}}{d\gamma} = &-\frac{\mathrm{i}}{2}\,\mathrm{Im}\big\{[G(x^m, p_n)]^{-\frac{3}{2}}\, G_{,j}(x^m, p_n)\big\}\, Q^j{}_a \\ &-\frac{\mathrm{i}}{2}\,\mathrm{Im}\big\{[G(x^m, p_n)]^{-\frac{3}{2}}\, G^{,j}(x^m, p_n)\big\}\, P_{ja}\end{aligned} \tag{65}$$

and

$$T_{,D\alpha} = \mathrm{i}\,\mathrm{Im}\big\{[G(x^m, p_n)]^{-\frac{1}{2}}\big\} \quad , \tag{66}$$

see (56) and (57). Here D is the number of spatial dimensions, usually $D=3$. Note that $T_{,i\alpha}$ and $\tau_{,i\alpha}$ are purely imaginary. Equation (58) for the second–order perturbation derivative of travel time, with $N=-1$, reads

$$\begin{aligned}\frac{d\tau_{,\alpha\alpha}}{d\gamma} = &-\mathrm{i}\,\mathrm{Im}\big\{[G(x^m, p_n)]^{-\frac{3}{2}}\, G^{,j}(x^m, p_n)\big\}\, \tau_{,j\alpha}(x^m, 0) \\ &-\mathrm{Re}[H^{,jk}(x^m, p_n)]\, \tau_{,j\alpha}(x^m, 0)\, \tau_{,k\alpha}(x^m, 0) \quad .\end{aligned} \tag{67}$$

For the calculation of the second–order phase–space derivatives $H^{,ij}$ of the complex–valued Hamiltonian function (59) refer, e.g., to Klimeš (2006, eq. 27).

5.3. Homogeneous Hamiltonian function of degree -1 for isotropic attenuating media

In an isotropic medium with complex–valued propagation velocity $v(x^m)$, the corresponding complex–valued eigenvalue of the Christoffel matrix reads

$$G(x^m, p_n) = [v(x^m)]^2\, p_k p_k \quad . \tag{68}$$

Hamilton's equations (61)–(62) for real–valued reference rays and reference travel time τ^0 then read

$$\frac{\mathrm{d}x^i}{\mathrm{d}\tau^0} = \mathrm{Re}\big[[v(x^m)]^{-1}\big]\, (p_k p_k)^{-\frac{3}{2}}\, p_i \quad , \tag{69}$$

$$\frac{\mathrm{d}p_i}{\mathrm{d}\tau^0} = -\mathrm{Re}\big[[v(x^m)]^{-2} v_{,i}(x^n)\big]\, (p_k p_k)^{-\frac{1}{2}} \quad . \tag{70}$$

The initial conditions for equations (69) and (70) should satisfy condition (63) which now reads

$$\mathrm{Re}\big[[v(x^m)]^{-1}\big]\, (p_k p_k)^{-\frac{1}{2}} = 1 \quad . \tag{71}$$

This condition is then satisfied along the whole reference rays, and Hamilton's equations (69)–(70) read

$$\frac{\mathrm{d}x^i}{\mathrm{d}\tau^0} = \big\{\mathrm{Re}\big[[v(x^m)]^{-1}\big]\big\}^{-2}\, p_i \quad , \tag{72}$$

$$\frac{\mathrm{d}p_i}{\mathrm{d}\tau^0} = -\big\{\mathrm{Re}\big[[v(x^m)]^{-1}\big]\big\}^{-1}\, \mathrm{Re}\big[[v(x^m)]^{-2} v_{,i}(x^n)\big] \quad . \tag{73}$$

Equation (64) for the first–order perturbation derivative $\tau_{,\alpha}$ in the perturbation expansion (31) of travel time reads

$$\frac{\mathrm{d}\tau_{,\alpha}}{\mathrm{d}\tau^0} = \mathrm{i}\,\mathrm{Im}\big[[v(x^m)]^{-1}\big]\, (p_k p_k)^{-\frac{1}{2}} \quad . \tag{74}$$

Since condition (71) is satisfied along the reference rays,

$$\frac{\mathrm{d}\tau_{,\alpha}}{\mathrm{d}\tau^0} = \mathrm{i}\,\mathrm{Im}\big[[v(x^m)]^{-1}\big]\, \big\{\mathrm{Re}\big[[v(x^m)]^{-1}\big]\big\}^{-1} = -\mathrm{i}\,\mathrm{Im}[v(x^m)]\, \big\{\mathrm{Re}[v(x^m)]\big\}^{-1} \quad . \tag{75}$$

The first–order perturbation equations (72), (73) and (75) have already been proposed by Vavryčuk (2009, eq. 15). Vavryčuk (2009) numerically demonstrated that these equations yield considerably more accurate first–order perturbation expansions of complex–valued travel time than the commonly used equations corresponding to the homogeneous Hamiltonian function of the second degree with respect to the slowness vector.

Acknowledgements

The research has been supported by the Grant Agency of the Czech Republic under contracts 205/07/0032 and P210/10/0736, by the Ministry of Education of the Czech Republic within research project MSM0021620860, and by the members of the consortium "Seismic Waves in Complex 3–D Structures" (see "http://sw3d.cz").

References

Bulant, P. & Klimeš, L. (2008): Numerical comparison of the isotropic–common–ray and anisotropic–common–ray approximations of the coupling ray theory. *Geophys. J. int.*, **175**, 357–374.

Červený, V. (1972): Seismic rays and ray intensities in inhomogeneous anisotropic media. *Geophys. J. R. astr. Soc.*, **29**, 1–13.

Červený, V. (2001): *Seismic Ray Theory.* Cambridge Univ. Press, Cambridge.

Červený, V., Klimeš, L. & Pšenčík, I. (2008): Attenuation vector in heterogeneous, weakly dissipative, anisotropic media. *Geophys. J. int.*, **175**, 346–355.

Červený, V. & Pšenčík, I. (2009): Perturbation Hamiltonians in heterogeneous anisotropic weakly dissipative media. *Geophys. J. int.*, **178**, 939–949.

Hamilton, W.R. (1837): Third supplement to an essay on the theory of systems of rays. *Trans. Roy. Irish Acad.*, **17**, 1–144, read January 23, 1832, and October 22, 1832.

Klimeš, L. (2002): Second–order and higher–order perturbations of travel time in isotropic and anisotropic media. *Stud. geophys. geod.*, **46**, 213–248.

Klimeš, L. (2006): Common–ray tracing and dynamic ray tracing for S waves in a smooth elastic anisotropic medium. *Stud. geophys. geod.*, **50**, 449–461.

Vavryčuk, V. (2009): Real ray tracing in isotropic viscoelastic media: a numerical modelling. In: *Seismic Waves in Complex 3–D Structures, Report 19*, pp. 219–236, Dep. Geophys., Charles Univ., Prague, online at “http://sw3d.cz”.