Sensitivity of electromagnetic waves to a heterogeneous bianisotropic structure

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Summary

We study how the perturbations of a generally heterogeneous bianisotropic structure manifest themselves in the wave field, and which perturbations can be detected within a limited aperture and a limited frequency band. A short–duration broad–band incident wave field with a smooth frequency spectrum is considered. Infinitesimally small perturbations of the constitutive tensor are decomposed into Gabor functions. The wave field scattered by the perturbations is then composed of waves scattered by the individual Gabor functions. The scattered waves are estimated using the first–order Born approximation with the paraxial ray approximation.

For each incident wave, each Gabor function acts like a 3–D Bragg grating and generates at the most 3 scattered Gaussian packets propagating in specific directions.

For a particular source, each Gaussian packet scattered by a Gabor function centred at a given spatial location is sensitive to just a single linear combination of the elements of the constitutive tensor corresponding to the Gabor function. This information about the Gabor function is lost if the scattered Gaussian packet does not fall into the aperture covered by the receivers and into the legible frequency band.

1. Introduction

We shall study how the perturbations of a generally heterogeneous bianisotropic structure manifest themselves in the wave field, and which perturbations can be detected within a limited aperture and a limited frequency band. We consider a smoothly varying heterogeneous generally bianisotropic background medium, with an isotropic background medium as a special case. We consider generally bianisotropic perturbations of the medium, with isotropic perturbations as a special case.

We assume a short–duration broad–band incident wave field with a smooth frequency spectrum. The scattered waves are estimated using the first–order Born approximation with the paraxial ray approximation.

We decompose infinitesimally small perturbations of the constitutive tensor into Gabor functions. The wave field scattered by the perturbations is then composed of waves scattered by individual Gabor functions. The wave scattered by one Gabor function is composed of 0 to 3 Gaussian packets, see Figures 1–3.

We present the resulting equations for the scattered Gaussian packets without derivation. The derivation for the electromagnetic waves is analogous to the derivation for the elastic waves by Klimeš (2007), which is based on the ray theory and paraxial ray approximation described in detail by Červený (2001).

Invited paper for the *URSI 2010 International Symposium on Electromagnetic Theory*, Berlin, Germany, August 16–19, 2010, but with corrected equations (14) and (33).
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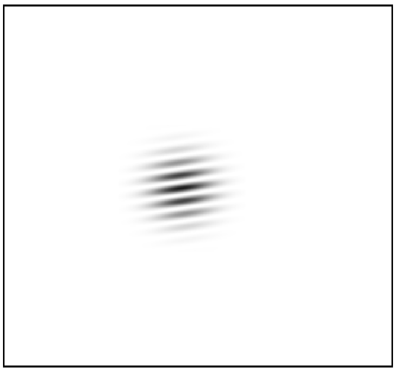

Figure 1. A single Gabor function (8) centred at point x^m_Ω.

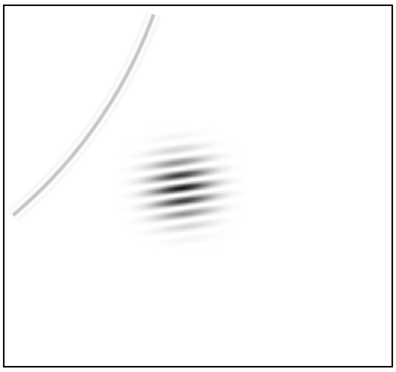

Figure 2. Broad–band wave (10) incident at the Gabor function.

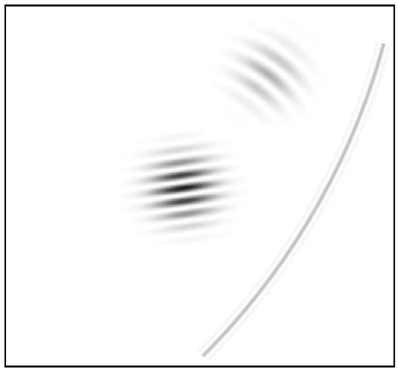

Figure 3. Scattered wave $a^\Omega_\alpha(x^\mu)$, see (9).

2. Constitutive equations and Maxwell equations

We consider a 4–D space–time manifold. We denote the three spatial coordinates by x^i, $i = 1, 2, 3$, time by x^4, and the space–time coordinates by x^α, $\alpha = 1, 2, 3, 4$. The coordinates may generally be curvilinear. The metric structure of the manifold will be determined by the Maxwell equations. We use the notation

$$f_{,\alpha} = \frac{\partial f}{\partial x^\alpha} \tag{1}$$

for the partial derivatives with respect to the space–time coordinates. The Einstein summation over repetitive lower–case Roman indices (spatial coordinates) and lower–case Greek indices (space–time coordinates) is used throughout the paper. No implicit summation is applied to upper–case Greek indices corresponding to Gabor functions.

We shall describe the electromagnetic field in terms of the covariant 4–vector potential A_α. The Maxwell equations with the generally bianisotropic constitutive equations in the Boys–Post representation read (Post, 2003, eq. 26)

$$[\chi^{\alpha\beta\gamma\delta}(x^\mu)\, A_{\gamma,\delta}(x^\nu)]_{,\beta} + J^\alpha(x^\mu) = 0 \quad , \tag{2}$$

where constitutive tensor $\chi^{\alpha\beta\gamma\delta}$ is a contravariant tensor density of weight -1, and current density J^α is a contravariant 4–vector density of weight -1.

The constitutive tensor is skew with respect to its first and second indices, and with respect to its third and fourth indices,

$$\chi^{\alpha\beta\gamma\delta} = -\chi^{\beta\alpha\gamma\delta} = -\chi^{\alpha\beta\delta\gamma} \quad . \tag{3}$$

In order to simplify the application of the ray–theory approximation, we are assuming here that the constitutive tensor is real–valued, and is symmetric with respect to its first and second pairs of indices,

$$\chi^{\alpha\beta\gamma\delta} = \chi^{\gamma\delta\alpha\beta} \quad . \tag{4}$$

For the sake of simplicity, we are also assuming here that the structure is time–independent,

$$\chi^{\alpha\beta\gamma\delta} = \chi^{\alpha\beta\gamma\delta}(x^m) \quad . \tag{5}$$

3. Gabor representation of medium perturbations

We consider infinitesimally small perturbations $\delta\chi^{\alpha\beta\gamma\delta}$ of the constitutive tensor $\chi^{\alpha\beta\gamma\delta}$ in the Maxwell equations (2).

The first–order perturbation (variation) δ of the Maxwell equations (2) without source term $J^\alpha(x^\mu)$ yields equations

$$[\chi^{\alpha\beta\gamma\delta}(x^m)\,\delta A_{\gamma,\delta}(x^\nu)]_{,\beta} = -[\delta\chi^{\alpha\beta\gamma\delta}(x^m)\,A_{\gamma,\delta}(x^\nu)]_{,\beta} \tag{6}$$

for wave–field perturbation $\delta A_\alpha(x^\mu)$ due to medium perturbation $\delta\chi^{\alpha\beta\gamma\delta}(x^m)$. We assume that the background medium $\chi^{\alpha\beta\gamma\delta}(x^m)$ is sufficiently smooth for the application of the ray–theory approximation.

We decompose the perturbations of the constitutive tensor into Gabor functions $g^\Omega(x^m)$,

$$\delta\chi^{\alpha\beta\gamma\delta}(x^m) = \sum_\Omega \chi_\Omega^{\alpha\beta\gamma\delta}\; g^\Omega(x^m) \quad , \tag{7}$$

where Gabor functions

$$g^\Omega(x^m) = \exp\bigl[\mathrm{i}\,k_i^\Omega\,(x^i - x_\Omega^i) - \tfrac{1}{2}(x^i - x_\Omega^i)\,K_{ij}^\Omega\,(x^j - x_\Omega^j)\bigr] \tag{8}$$

are centred at various spatial positions x_Ω^i and have various structural wavenumber vectors k_i^Ω.

The wave field scattered by the perturbations is then composed of waves $A_\alpha^\Omega(x^\mu)$ scattered by the individual Gabor functions,

$$\delta A_\alpha(x^\mu) = \sum_\Omega a_\alpha^\Omega(x^\mu) \quad . \tag{9}$$

Wave $a_\alpha^\Omega(x^\mu)$ scattered by one Gabor function is composed of 0 to 3 Gaussian packets $a_\alpha^{\Omega\,\mathbf{GP}}(x^\mu)$ described in Section 5, see Figures 1–3.

4. Applied approximations

4.1. First–order Born approximation

The first–order sensitivity of the wave field to the infinitesimally small structural perturbations is described exactly by the first–order Born approximation of solution $\delta A_\alpha(x^\nu)$ of equation (6).

4.2. Ray–theory approximation

We assume that the wave field is expressed in terms of amplitude $A(x^m)$, polarization vector $E_\alpha(x^m)$, travel time $T(x^m)$ and waveform $f(t)$,

$$A_\alpha(x^\mu) \approx A(x^m)\,E_\alpha(x^n)\,f[(x^4 - T(x^k)] \quad . \tag{10}$$

The polarization vector and the travel time should satisfy the Christoffel equation

$$\Gamma^{\alpha\gamma}\,E_\gamma = 0 \quad , \tag{11}$$

where

$$\Gamma^{\alpha\gamma}(x^m, p_n, p_4) = \chi^{\alpha\beta\gamma\delta}(x^m)\,p_\beta\,p_\delta \tag{12}$$

is the Christoffel matrix. Here

$$p_i = T_{,i} \quad , \qquad p_4 = -1 \tag{13}$$

is the slowness vector.

The Christoffel equation (11) is always satisfied by longitudinal polarization $E_\alpha = p_\alpha$ generating no electromagnetic field. The Christoffel equation (11) has another solution E_α if

$$\det\bigl[\Gamma^{ik}(x^m, p_n, p_4)\bigr] = 0 \quad . \tag{14}$$

For given p_n, the sixth–order polynomial equation (14) for p_4 has two negative solutions corresponding to two different polarizations E_α of electromagnetic waves, two positive solutions, and two imaginary solutions. We select one of the negative solutions and define the dependence

$$p_4 = -H(x^m, p_n) \tag{15}$$

of p_4 on p_i. Corresponding polarization vector E_α is a vector from the null space of the Christoffel matrix (12), different from 4–vector p_α. Note that we may or may not require $E_4 = 0$.

We refer to homogeneous function $H(x^m, p_n)$ of the first degree with respect to p_n as the Hamiltonian function. Travel time T then should satisfy the Hamilton–Jacobi equation

$$H(x^m, T_{,n}) = 1 \quad . \tag{16}$$

The corresponding rays satisfy Hamilton equations

$$\frac{\mathrm{d}x^i}{\mathrm{d}\gamma} = H^{,i}(x^m, p_n) \quad , \tag{17}$$

$$\frac{\mathrm{d}p_i}{\mathrm{d}\gamma} = -H_{,i}(x^m, p_n) \quad , \tag{18}$$

where

$$H_{,i} = \frac{\partial H}{\partial x^i} \quad , \qquad H^{,i} = \frac{\partial H}{\partial p_i} \quad . \tag{19}$$

For our homogeneous Hamiltonian function with respect to p_n, independent parameter γ along rays coincides with travel time T, and $H^{,i}$ represents the ray–velocity vector.

Differentiating the Christoffel equation

$$\Gamma^{\alpha\gamma}\bigl[x^m, p_n, -H(x^m, p_n)\bigr]\, E_\gamma = 0 \tag{20}$$

and multiplying the result by polarization vector E_α, we obtain expressions

$$H_{,i} = \tfrac{1}{2}\, \chi^{\alpha\beta\gamma\delta}_{,i} E_\alpha\, p_\beta\, E_\gamma\, p_\delta / \varrho \ , \quad H^{,i} = \chi^{\alpha i\gamma\delta} E_\alpha\, E_\gamma\, p_\delta / \varrho \tag{21}$$

with

$$\varrho = \chi^{\alpha 4\gamma\delta} E_\alpha E_\gamma\, p_\delta \tag{22}$$

for the phase–space derivatives (19) of the Hamiltonian function.

Amplitude A should satisfy transport equation

$$\bigl(A^2\, \varrho\, H^{,i}\bigr)_{,i} = 0 \quad . \tag{23}$$

We assume that the incident wave is expressed in form (10) with real–valued short–duration broad–band waveform $f(t)$ having a smooth frequency spectrum. The incident wave need not necessarily be calculated by the ray–theory approximation.

On the other hand, we assume that the Green tensor corresponding to the Maxwell equations (2) is calculated from the vicinity of point x^m_Ω to the vicinity of the receivers by the ray–theory approximation.

4.3. Paraxial ray approximation

Point x^m is situated in the vicinity of point x^m_Ω. We shall apply the paraxial ray approximation to incident wave $A_\alpha(x^\mu)$, and the two–point paraxial ray approximation to the Green tensor. The paraxial ray approximation consists in a constant amplitude and in the second–order Taylor expansion of the travel time at point x^m_Ω and at the receiver.

A "constant amplitude" means that we shall apply approximations

$$A(x^m) \approx A(x^m_\Omega) \quad , \tag{24}$$

$$E_\alpha(x^m) \approx E_\alpha(x^m_\Omega) \tag{25}$$

in the vicinity of point x^m_Ω for the incident wave, and the analogous approximations for the Green tensor.

For the travel time $T(x^m)$ of the incident wave at point x^m, situated in the vicinity of point x^m_Ω, we denote the travel time of the incident wave at point x^m_Ω by

$$T^\Omega = T(x^m_\Omega) \quad , \tag{26}$$

the slowness vector of the incident wave at point x^m_Ω by

$$T^\Omega_{,i} = T_{,i}(x^m_\Omega) \quad , \tag{27}$$

and the second–order spatial derivatives of the travel time of the incident wave at point x^m_Ω by

$$T^\Omega_{,ij} = T_{,ij}(x^m_\Omega) \quad . \tag{28}$$

The second–order Taylor expansion of the travel time of the incident wave at point x^m_Ω is

$$T(x^m) \approx T^\Omega + T^\Omega_{,i}\Delta x_i + \tfrac{1}{2}T^\Omega_{,ij}\Delta x^i \Delta x^j \quad , \tag{29}$$

where

$$\Delta x^i = x^i - x^i_\Omega \quad . \tag{30}$$

We analogously consider the second–order Taylor expansion of the travel time of the Green tensor with respect to both the 3 coordinates x^m and the 3 coordinates of the receiver.

4.4. High–frequency approximation of spatial derivatives

We use the high–frequency approximation of the spatial derivatives of the incident wave,

$$A_{\alpha,\gamma}(x^\mu) \approx -A_{\alpha,4}(x^\mu)\, T^\Omega_{,\gamma} \quad , \tag{31}$$

where $T^\Omega_{,\alpha}$ defined by (27) is the slowness vector of the incident wave.

We also use the analogous high–frequency approximations of the spatial derivatives of the Green tensor with respect to both the 3 coordinates x^m and the 3 coordinates of the receiver.

5. Scattered Gaussian packets

Considering the above approximations, wave $a^{\Omega}_{\alpha}(x^{\mu})$ scattered by one Gabor function (8) is composed of 0 to 3 Gaussian packets $a^{\Omega\,\mathbf{GP}}_{\alpha}(x^{\mu})$, see Figures 1–3.

The 4–vector potential of each Gaussian packet scattered by the Gabor function (8) can be expressed at time $x^4 = T^{\Omega}$ in terms of the spatial paraxial ray approximation

$$a^{\Omega\,\mathbf{GP}}_{\alpha}(x^m, T^{\Omega}) \approx \mathrm{Re}\big\{a\, e_{\alpha} \exp\big[\mathrm{i}\omega\big(\theta_{,k}\Delta x^k + \tfrac{1}{2}\,\Delta x^k\theta_{,kl}\Delta x^l\big)\big]\big\} \tag{32}$$

in the vicinity of point x^i_{Ω}. Vectorial distances Δx^i from point x^i_{Ω} are defined by equation (30).

We now describe how the parameters of this Gaussian packet can be determined.

Frequency ω is the positive solution of the sixth–order polynomial equation

$$\det\big[\Gamma^{ik}(x^m_{\Omega}, T^{\Omega}_{,n} + \omega^{-1}k^{\Omega}_n, -1)\big] \quad . \tag{33}$$

For a given incident wave, there are 0 to 3 finite positive solutions ω of equation (33) corresponding to 0 to 3 scattered Gaussian packets (32).

The central point of the Gaussian packet propagates from point x^i_{Ω} at time

$$x^4_{\Omega} = T^{\Omega} \tag{34}$$

in the direction $H^{,i}(x^m_{\Omega}, \theta_{,n})$ corresponding to slowness vector

$$\theta_{,i} = T^{\Omega}_{,i} + \omega^{-1}k^{\Omega}_i \quad . \tag{35}$$

Polarization vector e_{α} of the Gaussian packet is a vector from the null space of the Christoffel matrix

$$\Gamma^{\alpha\gamma}(x^m_{\Omega}, \theta_{,i}, -1) \quad , \tag{36}$$

different from 4–vector $\theta_{,\alpha}$ with $\theta_{,4} = -1$.

The spatial shape of the Gaussian packet is determined by matrix

$$\begin{aligned}\theta_{,ij} = {} & [\delta^k_i + (1 - H^{,m}T^{\Omega}_{,m})^{-1}T^{\Omega}_{,i}H^{,k}]\,(T^{\Omega}_{,kl} + \mathrm{i}\omega^{-1}K^{\Omega}_{kl})\,[\delta^l_j + (1 - H^{,n}T^{\Omega}_{,n})^{-1}H^{,l}T^{\Omega}_{,j}] \\ & + (1 - H^{,n}T^{\Omega}_{,n})^{-1}(T^{\Omega}_{,i}H_{,j} + H_{,i}T^{\Omega}_{,j}) + (1 - H^{,n}T^{\Omega}_{,n})^{-2}\,T^{\Omega}_{,i}\,H^{,k}H_{,k}\,T^{\Omega}_{,j} \quad ,\end{aligned} \tag{37}$$

where the phase–space derivatives

$$H^{,k} = H^{,k}(x^m_{\Omega}, \theta_{,n}) \quad , \qquad H_{,k} = H_{,k}(x^m_{\Omega}, \theta_{,n}) \tag{38}$$

of the Hamiltonian function correspond to the slowness vector $\theta_{,i}$ of the Gaussian packet.

The initial amplitude of the Gaussian packet at point x^i_{Ω} is

$$a = -2\,\mathrm{i}\,\omega\, A(x^m_{\Omega})\,\widehat{f}(\omega)\, R^{\Omega} \quad , \tag{39}$$

where $A(x^m_{\Omega})$ is the amplitude of the incident wave,

$$\widehat{f}(\omega) = \int_{-\infty}^{+\infty} \mathrm{d}t \exp(\mathrm{i}\omega t)\, f(t) \tag{40}$$

is the Fourier transform of the real–valued waveform $f(t)$ of the incident wave, and

$$R^{\Omega} = \frac{\chi^{\alpha\beta\gamma\delta}_{\Omega}\, E_{\alpha}(x^m_{\Omega})\; T^{\Omega}_{,\beta}\; e_{\gamma}\, \theta_{,\delta}}{-2\,\chi^{\alpha\beta\gamma\delta}(x^n_{\Omega})\; e_{\alpha}\, T^{\Omega}_{,\beta}\; e_{\gamma}\, \theta_{,\delta}} \quad . \tag{41}$$

Here we have extended the spatial slowness vectors $T^{\Omega}_{,i}$ and $\theta_{,k}$ to 4–vectors with

$$T^{\Omega}_{,4} = -1 \quad , \qquad \theta_{,4} = -1 \quad . \tag{42}$$

The polarization vector of the incident wave is denoted by $E_{\alpha}(x^m_{\Omega})$.

Coefficient R^{Ω} represents the weak–contrast reflection–transmission coefficient at the interface at which the constitutive tensor changes by $\chi^{\alpha\beta\gamma\delta}_{\Omega}$. For analogous weak–contrast reflection–transmission coefficients for the elastic waves and for their derivation refer to Klimeš (2003).

6. Conclusions

A real–valued short–duration broad–band wave (10) with a smooth frequency spectrum incident at Gabor function (8) generates 0 to 3 scattered Gaussian packets (32). The scattered Gaussian packets have a specific frequency and propagate in a specific direction. The equations for the propagation of Gaussian packets in isotropic media are given by Červený, Klimeš & Pšenčík (2007, sec. 5.2), and in anisotropic media by Klimeš (2007, sec. 5.5).

Each Gaussian packet scattered by a Gabor function centred at a given spatial location is sensitive to just a single linear combination (41) of the elements of the constitutive tensor corresponding to the Gabor function. This information about the Gabor function is lost if the scattered Gaussian packet does not fall into the aperture covered by the receivers and into the legible frequency band.

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