Second-order traveltimes and first-order spreading of reflected/transmitted P waves in inhomogeneous, weakly anisotropic media

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1 Introduction

First-order ray tracing (FORT) and dynamic ray tracing (FODRT) for P waves (Pšenčík and Farra, 2005, 2007) provide approximate, but simple and quite accurate way of computing traveltimes and geometrical spreading of seismic waves propagating in inhomogeneous, weakly anisotropic media and even in media of moderate anisotropy. FORT is a technique based on perturbation theory, in which deviations of anisotropy from isotropy are considered to be small quantities, used further in the perturbation procedure. The basic idea of FORT is to replace the exact eigenvalue of the Christoffel matrix, which controls P-wave ray tracing and dynamic ray tracing, by its first-order counterpart.

We extend the applicability of FORT and FODRT to layered media. Basic step in this extension is introduction of a rule (Snell's law) for the determination of first-order slowness vectors of reflected or transmitted P waves into FORT. Transformation of FODRT quantities across an interfaces is controlled by formally the same equations as in the exact case (Farra and Le Bégat, 1995), with exact quantities replaced by their first-order counterparts. Therefore, in the following, we concentrate on description of traveltime computations including transformation of slowness vectors at an interface, and for spreading computations we refer to Pšenčík and Farra (2007) and Farra and Le Bégat (1995).

The lower-case indices $i, j, ...$ take the values of 1,2,3, the upper-case indices $I, J, ...$ take the values of 1,2. The Einstein summation convention over repeated indices is used.

2 Traveltime computations in smooth media

The first-order P-wave ray in an inhomogeneous, weakly anisotropic medium can be obtained by solving FORT equations:

$$\frac{dx_i}{d\tau} = \frac{1}{2}\frac{\partial G(x_m, p_m)}{\partial p_i}, \qquad \frac{dp_i}{d\tau} = -\frac{1}{2}\frac{\partial G(x_m, p_m)}{\partial x_i}. \tag{1}$$

Here x_i and p_i are the Cartesian coordinates of the first-order P-wave ray and the components of the corresponding first-order slowness vectors, respectively. Parameter τ is the first-order traveltime. Symbol G denotes the first-order P-wave (greatest) eigenvalue of the generalized Christoffel matrix with elements Γ_{ik}:

$$\Gamma_{ik}(x_m, p_m) = a_{ijkl}(x_m)p_j p_l \ . \tag{2}$$

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The fourth-order tensor a_{ijkl} is the tensor of density-normalized elastic moduli, $a_{ijkl} = c_{ijkl}/\rho$, c_{ijkl} being elements of the fourth-order tensor of elastic moduli and ρ the density.

The initial conditions for the ray-tracing equations (1) for $\tau = \tau_0$ read:

$$x_i(\tau_0) = x_i^0, \qquad p_i(\tau_0) = p_i^0. \tag{3}$$

Here, x_i^0 are the coordinates of source point $\mathbf{x}^0$, and $p_i^0 = n_i^0/c_0$ are the components of the first-order slowness vector $\mathbf{p}^0$ at the source. Symbol c_0 denotes the first-order approximation of P-wave phase velocity in the direction $\mathbf{n}^0$ at source point $\mathbf{x}^0$.

FORT equations (1) provide directly first-order traveltimes. Traveltimes of second-order accuracy can be simply computed by quadratures along first-order rays (Pšenčík and Farra, 2005). The second-order traveltime formula has the form

$$\tau_P(\tau, \tau_0) = \tau + \Delta\tau_P(\tau, \tau_0) \ . \tag{4}$$

Here τ denotes the first-order traveltime obtained by integrating the ray tracing system (1). Symbol $\Delta\tau_P$ denotes the second-order traveltime correction:

$$\Delta\tau_P = -\frac{1}{2}\int_{\tau_0}^{\tau} \frac{B_{13}^2(\tau) + B_{23}^2(\tau)}{1 - \frac{1}{2}[B_{11}(\tau) + B_{22}(\tau)]} d\tau. \tag{5}$$

The elements of the symmetric matrix $\mathbf{B}(\tau)$ appearing in (5) are given by the formula

$$B_{jl}(\tau) = \Gamma_{ik}(x_m, p_m) e_i^{[j]} e_k^{[l]}, \tag{6}$$

where $x_m = x_m(\tau)$, $p_m = p_m(\tau)$. Symbols $e_i^{[j]} = e_i^{[j]}(\tau)$ in (6) denote the components of vectors $\mathbf{e}^{[j]}$ forming an orthonormal triplet. Vectors $\mathbf{e}^{[1]}$ and $\mathbf{e}^{[2]}$ are perpendicular to the vector $\mathbf{e}^{[3]}$ chosen so that $\mathbf{e}^{[3]} = c\mathbf{p}$, where c is the phase velocity and $\mathbf{p}$ the slowness vector. Vector $\mathbf{e}^{[3]}$ can be determined from the second set of FORT equations (1). At any point of the first-order P-wave ray, vectors $\mathbf{e}^{[K]}$ can be chosen arbitrarily in the plane perpendicular to $\mathbf{p}$. Elements of the matrix $\mathbf{B}$ are, in fact, projections of elements of the generalized Christoffel matrix into the vectors $\mathbf{e}^{[j]}$.

3 Transformation of a slowness vector at an interface

Let us consider an interface Σ and a P wave incident at it. The slowness vector of a generated reflected or transmitted P wave at the point of incidence of a P-wave ray at Σ can be written in the form (Farra and Pšenčík, 2010)

$$\mathbf{p}^G = \mathbf{b} + \xi^G\mathbf{N} = \mathbf{p} - (\mathbf{p}\cdot\mathbf{N})\mathbf{N} + \xi^G\mathbf{N} \ . \tag{7}$$

Here, $\mathbf{p}$ and $\mathbf{p}^G$ are first-order slowness vectors of the incident and generated (G) waves, $\mathbf{N}$ is the unit normal to interface Σ at the point of the incidence at Σ. The symbol ξ^G represents the projection of $\mathbf{p}^G$ to $\mathbf{N}$, and $\mathbf{b}$ represents the vectorial component of $\mathbf{p}^G$, tangential to Σ. Projections ξ^G can be found from the first-order eikonal equations satisfied at the point of incidence by the waves generated on corresponding sides of the interface:

$$G(\mathbf{b} + \xi^G\mathbf{N}) = 1 \ . \tag{8}$$

To solve equation (8), we can use an iterative procedure proposed by Dehghan et al. (2007), described in detail by Farra and Pšenčík (2010). In it, we seek the first-order slowness vector $\mathbf{p}^G$ of a selected generated wave at the point of incidence iteratively. The j-th iteration has the form

$$\mathbf{p}^{G\{j\}} = \mathbf{b} + \xi^{G\{j\}}\mathbf{N} \ , \tag{9}$$

where

$$\xi^{G\{j\}} = \xi^{G\{j-1\}} - \frac{G(\mathbf{p}^{G\{j-1\}}) - 1}{N_k \partial G/\partial p_k(\mathbf{p}^{G\{j-1\}})} \ . \tag{10}$$

The initial value $\xi^{G(0)}$ of the quantity $\xi^{G(j)}$ is determined for a reference isotropic medium. The explicit expressions for G and $\partial G/\partial p_k$ can be found in papers dealing with FORT, see more details in Farra and Pšenčík (2010).

4 Tests of accuracy of FORT approach in layered media

It was shown by Pšenčík and Farra (2005) that even for a model with P-wave anisotropy of about 20%, the relative error of the P-wave FORT traveltimes compared with the standard ray theory traveltimes was less than 1%. In case of spreading (Pšenčík and Farra, 2007), the errors were larger, exceptionally they reached up to 20%. This picture is similar even in layered media. For a model of a two-layer HTI medium of anisotropy about 8%, we found traveltime relative errors to be less than 0.02%, spreading errors to be less than 0.5%. In the following, we present results for a two layer orthorhombic model with stronger anisotropy of about 20%. The model is based on the model of Schoenberg and Helbig (1997). The relative errors are obtained by comparing FORT results with results obtained by ANRAY package based on standard ray theory.

We consider the VSP configuration with the source and the borehole situated in a vertical plane (x, z). The borehole is parallel to the z axis, and contains 33 receivers separated by 0.6 km, with the shallowest at the depth of 0.6 km. The vertical single-force source is located on the surface at $z = 0$ km, at a distance of 1 km from the borehole.

The model consists of two orthorhombic, vertically inhomogeneous layers separated by an interface at the depth of 1 km. In each layer, the density-normalized elastic moduli are obtained by the linear interpolation of their values at the top and bottom of the layer. The moduli, in (km/s)^2, at the top ($z = 0$ km) of the first layer are: $A_{11} = 9$, $A_{12} = 3.6$, $A_{13} = 2.25$, $A_{22} = 9.84$, $A_{23} = 2.4$, $A_{33} = 5.94$, $A_{44} = 2$, $A_{55} = 1.6$ and $A_{66} = 2.18$; at the bottom ($z = 1$ km): $A_{11} = 12.6$, $A_{12} = 5.04$, $A_{13} = 3.15$, $A_{22} = 14.11$, $A_{23} = 3.36$, $A_{33} = 8.32$, $A_{44} = 2.8$, $A_{55} = 2.24$ and $A_{66} = 3.05$. The moduli at the top ($z = 1$ km) and the bottom ($z = 3$ km) of the second layer are: $A_{11} = 19.8$, $A_{12} = 7.92$, $A_{13} = 4.95$, $A_{22} = 21.65$, $A_{23} = 5.28$, $A_{33} = 13.07$, $A_{44} = 4.4$, $A_{55} = 3.52$ and $A_{66} = 4.8$. At the top of the second layer, we apply rotation by 45^0 around z-axis, followed by rotation by 45^0 around the new y-axis. No rotation is applied at its bottom. The second layer is thus vertically inhomogeneous.

In the left bottom corner of Fig.1, we can see the ray diagram of direct wave, including the wave transmitted at the interface at 1 km. The plot above the ray diagram shows the corresponding traveltime curve. Plots on the right show relative errors of traveltime (top) and spreading (bottom) of the reflected wave (between depths 0 and 1 km) and of the transmitted wave (between depths of 1 and 2 km). We can see that traveltime errors do not exceed 0.02%, spreading errors are less than 8%. The results shown in Fig.1 correspond to subcritical incidence. Preliminary results for overcritical incidence (not presented here) show traveltime errors not exceeding 0.04%, but spreading errors slightly larger than 10%.

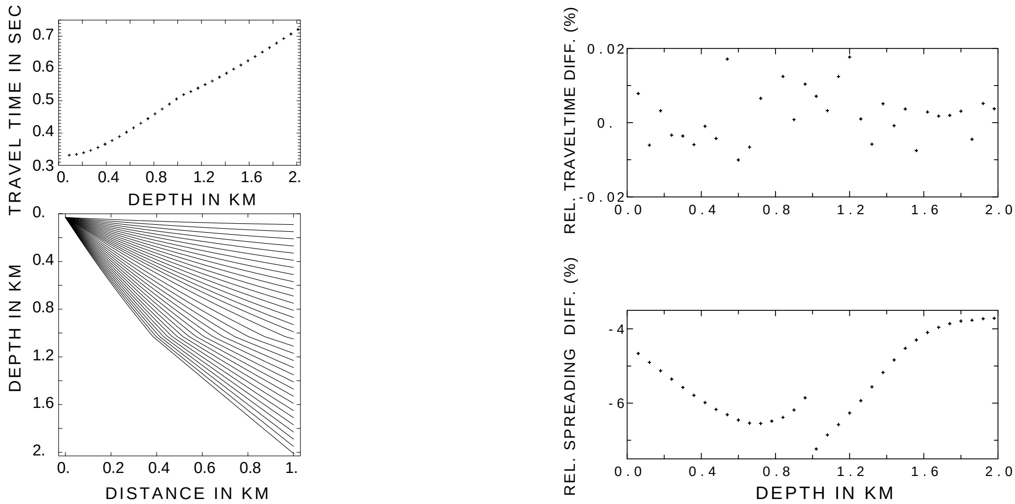

Figure 1

5 Conclusions

Presented tests indicate that accuracy of P-wave FORT computations of traveltime remains very high even in layered media of moderate anisotropy. Accuracy of spreading computations seems to be lower. Obviously, the applicability of the FORT approach to layered media requires further testing.

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