Relation between the propagator matrix of geodesic deviation and the second-order derivatives of the characteristic function for a general Hamiltonian function

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Summary

The Hamiltonian geometry is a generalization of the Finsler geometry, which is in turn a generalization of the Riemann geometry. The Hamiltonian geometry is based on the first–order partial differential Hamilton–Jacobi equations for the characteristic function which represents the distance between two points. The Hamiltonian equations of geodesic deviation may serve to calculate geodesic deviations, amplitudes of waves, and the second–order spatial derivatives of the characteristic function or action. The propagator matrix of geodesic deviation contains all the linearly independent solutions of the linear ordinary differential equations of geodesic deviation.

In this paper, we derive the relations between the propagator matrix of the Hamiltonian equations of geodesic deviation and the second–order spatial derivatives of the characteristic function for a general Hamiltonian function. The derived relations represent the generalization of the analogous relations, previously derived for the Finsler geometry, to an arbitrary Hamiltonian function.

Keywords

Hamilton–Jacobi equation, geodesics (rays), geodesic deviation, characteristic function, wave propagation, Hamiltonian geometry, Finsler geometry.

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1. Introduction

The basics of a very general geometry of geodesics (rays) were formulated by Sir William Rowan Hamilton in 1832 (Hamilton, 1837). Hamilton's formulation is based on the first–order partial differential Hamilton–Jacobi equations for the characteristic function which represents the distance between points. The form of the Hamilton–Jacobi equation is specified in terms of the Hamiltonian function. Hamilton (1837) considered the Hamiltonian functions which are homogeneous functions of the first degree with respect to the spatial gradient of the characteristic function, but his theory is applicable to general Hamiltonian functions as well. Hamilton's formulation with a general Hamiltonian function represents a very useful generalization of the Finsler geometry, and describes the propagation of various waves (e.g., elastic, electromagnetic, Dirac) in the high–frequency approximation. A general Hamiltonian function is especially useful for describing waves which propagate with different velocities in opposite directions (e.g., electrons in an electromagnetic field, sound waves in flowing media). The Finsler geometry (Finsler, 1918) represents a special case of the Hamiltonian geometry, with the Hamiltonian functions which are homogeneous functions of the second degree with respect to the spatial gradient of the characteristic function.

The non–linear ordinary differential equations of geodesics (Hamilton's equations) may serve to calculate geodesics, the characteristic function (point–to–point distance, two–point travel time) with its first–order spatial derivatives, or the action (distance for general initial conditions, travel time for general initial conditions) with its first–order spatial derivatives. The linear ordinary differential equations of geodesic deviation derived by Červený (1972) may serve to calculate geodesic deviations, amplitudes of waves, the second–order spatial derivatives of the characteristic function, or the second–order spatial derivatives of the action. Geodesic perturbations, higher–order geodesic deviations, and the perturbation derivatives and higher–order spatial derivatives of the characteristic function or of the action can be calculated by quadratures along geodesics (Klimeš, 2002; 2010).

The characteristic function and the propagator matrix of geodesic deviation have found many important applications in wave propagation (Červený, 2001). Klimeš (2009) derived the relations between the propagator matrix of geodesic deviation and the second–order spatial derivatives of the characteristic function in the Finsler geometry, which corresponds to a Hamiltonian function which is a homogeneous function of the second degree with respect to the spatial gradient of the characteristic function.

In this paper, we generalize the relations between the propagator matrix of geodesic deviation and the second–order spatial derivatives of the characteristic function to an arbitrary Hamiltonian function. This generalization is important for describing waves which propagate with different velocities in opposite directions (e.g., electrons in an electromagnetic field, sound waves in flowing media), but may also be very useful for analytical calculations in simple media, because the analytical solutions are often derived for Hamiltonian functions which are not homogeneous (Červený, 2001, secs. 3.4 and 4.8).

2. Hamiltonian formulation of the geometry of geodesics (rays)

2.1. Hamiltonian function

We consider a smooth manifold (differentiable manifold), and coordinates x^i of its coordinate chart. At each point x^i, we have the tangent space containing contravariant vectors y^i and the cotangent space containing covariant vectors y_i such as the gradients of functions. We consider Hamiltonian function $H(x^i, y_j)$, which is a real–valued function of coordinates x^i and of covariant vector y_j from the cotangent space at point x^i, and which is differentiable within its definition domain. The Hamiltonian function may be represented by any reasonably smooth function of x^i and y_j.

Hamilton (1837) called the subset of the "unit" vectors y_j in the cotangent space at point x^i, defined by equation

$$H(x^i, y_j) = C \quad , \tag{1}$$

the *surface of components of normal slowness*. Now it is often called the *phase–slowness surface* or briefly the *slowness surface*, sometimes the *index surface*. In the Finsler geometry, it is referred to as the *figuratrix*. Constant C is determined by the meaning of the Hamiltonian function.

2.2. Action, the characteristic function and the Hamilton–Jacobi equations

The Hamilton–Jacobi equation is a partial differential equation of the first order. The Hamilton–Jacobi equation for *action* (*distance for general initial conditions*, *travel time for general initial conditions*) $S(x^m)$ reads

$$H\big(x^i, \tfrac{\partial S}{\partial x^j}(x^m)\big) = C \quad . \tag{2}$$

Hamilton (1837) also defined the *characteristic function* (*point–to–point distance*, *two–point travel time*)

$$V(x^m, \tilde{x}^n) \tag{3}$$

from point $\tilde{x}^n$ to point x^m. Note that the characteristic function need not be reciprocal,

$$V(x^m, \tilde{x}^n) \neq V(\tilde{x}^n, x^m) \quad . \tag{4}$$

The characteristic function satisfies the Hamilton–Jacobi equations

$$H\big(x^m, \tfrac{\partial V}{\partial x^n}(x^a, \tilde{x}^b)\big) = C \tag{5}$$

and

$$H\big(\tilde{x}^m, -\tfrac{\partial V}{\partial \tilde{x}^n}(x^a, \tilde{x}^b)\big) = C \tag{6}$$

(Hamilton, 1837, eq. C). Note that one of equations (5) and (6) serves as the initial conditions for the other. The Hamilton–Jacobi equations express the requirement that the gradient of the action or of the characteristic function is "unit", see the definition (1) of unit covariant vectors.

2.3. Equations of geodesics

Hamilton's equations (*equations of geodesics, equations of rays, ray tracing equations*) read

$$\frac{\mathrm{d}}{\mathrm{d}\gamma} x^i = \frac{\partial H}{\partial y_i}(x^m, y_n) \quad , \tag{7}$$

$$\frac{\mathrm{d}}{\mathrm{d}\gamma} y_i = -\frac{\partial H}{\partial x^i}(x^m, y_n) \quad . \tag{8}$$

Hamilton (1837) referred to these equations as the *general equations of rays.* Hamilton's equations (7)–(8) can simply be derived by differentiating the Hamilton–Jacobi equation (2) or (5) with respect to coordinates x^j, and putting $y_i = \frac{\partial V}{\partial x^i}(x^a, \tilde{x}^b)$. If we differentiate the Hamilton–Jacobi equation (6) with respect to coordinates $\tilde{x}^j$ and put $\tilde{y}_i = -\frac{\partial V}{\partial \tilde{x}^i}(x^a, \tilde{x}^b)$, we obtain Hamilton's equations

$$\frac{\mathrm{d}}{\mathrm{d}\gamma} \tilde{x}^i = -\frac{\partial H}{\partial \tilde{y}_i}(\tilde{x}^m, \tilde{y}_n) \quad , \tag{9}$$

$$\frac{\mathrm{d}}{\mathrm{d}\gamma} \tilde{y}_i = \frac{\partial H}{\partial \tilde{x}^i}(\tilde{x}^m, \tilde{y}_n) \tag{10}$$

for initial point $\tilde{x}^j$. The meaning of the independent parameter γ along the geodesic and the sensitivity of the geodesic to the initial conditions depend on the form of the Hamiltonian function. Covariant vector y_i in (7)–(8), which represents the first–order partial derivatives of the characteristic function with respect to spatial coordinates, analogously as covariant vector $\tilde{y}_i$ in (9)–(10), was called the *normal slowness* by Hamilton (1837). Now it is usually called the *slowness vector.*

Characteristic function $V(x^m, \tilde{x}^n)$ can be calculated by quadrature

$$V(x^m, \tilde{x}^n) = \int_0^\gamma y^r \frac{\mathrm{d}}{\mathrm{d}\gamma} x^r \, \mathrm{d}\gamma \tag{11}$$

along the geodesic obtained using Hamilton's equations (7)–(8), or by quadrature

$$V(x^m, \tilde{x}^n) = -\int_0^\gamma \tilde{y}^r \frac{\mathrm{d}}{\mathrm{d}\gamma} \tilde{x}^r \, \mathrm{d}\gamma \tag{12}$$

along the geodesic obtained using Hamilton's equations (9)–(10).

Hamilton's equations (7)–(8) and (9)–(10) also define function

$$\gamma(x^m, \tilde{x}^n) \tag{13}$$

from point $\tilde{x}^n$ to point x^m, with initial conditions $\gamma(\tilde{x}^m, \tilde{x}^n) = 0$. Note that this function need not be reciprocal,

$$\gamma(x^m, \tilde{x}^n) \neq \gamma(\tilde{x}^n, x^m) \quad . \tag{14}$$

We shall need function $\gamma(x^m, \tilde{x}^n)$ in the relations derived in this paper.

2.4. Equations of geodesic deviation

We define vectors

$$X^i{}_\alpha = \frac{\partial x^i}{\partial \gamma^\alpha} \tag{15}$$

and

$$Y_{i\alpha} = \frac{\partial y_i}{\partial \gamma^\alpha} \tag{16}$$

representing the geodesic deviation corresponding to some parameter γ^α parametrizing the initial conditions for the geodesics. Since derivatives $\frac{\mathrm{d}}{\mathrm{d}\gamma}$ and $\frac{\partial}{\partial\gamma^\alpha}$ commute, the equations for $X^i{}_\alpha$ and $Y_{i\alpha}$ are obtained by differentiating Hamilton's equations (7)–(8) with respect to γ^α. The resulting *Hamiltonian equations of geodesic deviation* (*paraxial ray equations*, *dynamic ray tracing equations*) derived by Červený (1972) read

$$\frac{\mathrm{d}}{\mathrm{d}\gamma} X^i{}_\alpha = \quad H_{,j}^{,i}\, X^j{}_\alpha + H^{,ij}\, Y_{j\alpha} \quad , \tag{17}$$

$$\frac{\mathrm{d}}{\mathrm{d}\gamma} Y_{i\alpha} = -H_{,ij}\, X^j{}_\alpha - H_{,i}^{,j}\, Y_{j\alpha} \quad , \tag{18}$$

where

$$H_{,ij} = \frac{\partial^2 H}{\partial x^i \partial x^j}(x^m, y_n) \quad , \tag{19}$$

$$H_{,j}^{,i} = \frac{\partial^2 H}{\partial y_i \partial x^j}(x^m, y_n) \quad , \tag{20}$$

$$H^{,ij} = \frac{\partial^2 H}{\partial y_i \partial y_j}(x^m, y_n) \quad . \tag{21}$$

Equations (17)–(18) may differ for different Hamiltonian functions corresponding to equivalent Hamilton–Jacobi equations.

2.5. Propagator matrix of geodesic deviation

The propagator matrix of geodesic deviation from point $\tilde{x}^b$ to point x^a is defined by equation

$$\mathbf{\Pi}(x^a, \tilde{x}^b) = \begin{pmatrix} \frac{\partial x^i}{\partial \tilde{x}^j} & \frac{\partial x^i}{\partial \tilde{y}_j} \\ \frac{\partial y_i}{\partial \tilde{x}^j} & \frac{\partial y_i}{\partial \tilde{y}_j} \end{pmatrix} \quad , \tag{22}$$

where the derivatives with respect to initial conditions $\tilde{x}^j$, $\tilde{y}_j$ for Hamilton's equations (7)–(8) are taken at fixed parameter γ along geodesics. The propagator matrix of geodesic deviation is symplectic and obeys the chain rule,

$$\mathbf{\Pi}(x^a, \tilde{x}^c) = \mathbf{\Pi}(x^a, \overline{x}^b)\, \mathbf{\Pi}(\overline{x}^c, \tilde{x}^d) \quad , \tag{23}$$

where $\tilde{x}^d$, $\overline{x}^c$ and x^a are the coordinates of three points along a geodesic.

The propagator matrix of geodesic deviation contains all linearly independent solutions of the equations of geodesic deviation, and may thus be used to calculate the geodesic deviation for any initial conditions.

The Hamiltonian equations (17)–(18) of geodesic deviation for the propagator matrix read

$$\tfrac{\mathrm{d}}{\mathrm{d}\gamma}\mathbf{\Pi}(x^a, \tilde{x}^b) = \begin{pmatrix} H_{,j}^{,i} & H^{,ij} \\ -H_{,ij} & -H_{,i}^{,j} \end{pmatrix} \mathbf{\Pi}(x^a, \tilde{x}^b) \quad , \tag{24}$$

with unit initial conditions.

3. Derivatives of the characteristic function

3.1. First–order spatial derivatives of the characteristic function

The first–order spatial derivatives

$$\frac{\partial V}{\partial x^i} = y_i \quad , \tag{25}$$

$$\frac{\partial V}{\partial \tilde{x}^i} = -\tilde{y}_i \tag{26}$$

of the characteristic function result from the solution of Hamilton's equations (7)–(8) or (9)–(10) (Hamilton, 1837).

3.2. Relation between the propagator matrix of geodesic deviation and the second–order spatial derivatives of the characteristic function

The linear ordinary differential Hamiltonian equations (24) of geodesic deviation can be used to calculate the propagator matrix (22) of geodesic deviation. The second–order spatial derivatives of the characteristic function can be obtained from the propagator matrix (22) of geodesic deviation.

The unique relations between the second–order spatial derivatives of characteristic function (3) and the propagator matrix (22) of geodesic deviation read

$$\left(\frac{\partial^2 V}{\partial x^i \partial x^j} + \frac{1}{\Gamma} \frac{\partial \gamma}{\partial x^i} \frac{\partial \gamma}{\partial x^j} \right) \frac{\partial x^j}{\partial \tilde{y}_k} = \frac{\partial y_i}{\partial \tilde{y}_k} \quad , \tag{27}$$

$$\left(\frac{\partial^2 V}{\partial \tilde{x}^i \partial x^j} + \frac{1}{\Gamma} \frac{\partial \gamma}{\partial \tilde{x}^i} \frac{\partial \gamma}{\partial x^j} \right) \frac{\partial x^j}{\partial \tilde{y}_k} = -\delta_i^k \quad , \tag{28}$$

$$\frac{\partial x^i}{\partial \tilde{y}_j} \left(\frac{\partial^2 V}{\partial \tilde{x}^j \partial \tilde{x}^k} + \frac{1}{\Gamma} \frac{\partial \gamma}{\partial \tilde{x}^j} \frac{\partial \gamma}{\partial \tilde{x}^k} \right) = \frac{\partial x^i}{\partial \tilde{x}^k} \quad , \tag{29}$$

where integral

$$\Gamma = \int_0^\gamma \left(\frac{\partial \gamma}{\partial x^r} H^{,rs} \frac{\partial \gamma}{\partial x^s} \right) \mathrm{d}\gamma \tag{30}$$

is calculated along the geodesic, and Kronecker delta δ_i^k represents the components of the identity matrix. Function $\gamma(x^m, \tilde{x}^n)$ is defined in Section 2.3. Relations (27)–(29) are not applicable if integral (30) is equal to zero, which may happen, e.g., if Hamiltonian function $H(x^i, y_j)$ is a homogeneous function of the first degree with respect to y_n. Relations (27)–(29) are proved in Section 4.

Relations (27)–(29) represent the generalization of the analogous relations (Klimeš, 2009, eqs. 27–29) in the Finsler geometry, from a homogeneous Hamiltonian function of the second degree with respect to the spatial gradient of the characteristic function to an arbitrary Hamiltonian function.

Only three submatrices of the propagator matrix (22) of geodesic deviation are used in equations (27)–(29). Note that the fourth submatrix of matrix (22) carries no additional information; it can be calculated from the three submatrices used in equations (27)–(29) thanks to the symplectic property of the propagator matrix (22) of geodesic deviation.

4. Proof of the relations between the propagator matrix of geodesic deviation and the second–order spatial derivatives of the characteristic function

We first calculate the limits of equations (27)–(29) at the initial point of a geodesic, i.e. for point x^a approaching the initial point $\tilde{x}^a$ of the geodesic, and prove that equations (27)–(29) hold at the initial point (Sections 4.3–4.6).

We then assume that equations (27)–(29) are satisfied at point x^a of the geodesic, calculate the derivatives of equations (27)–(29) along the geodesic at point x^a, and prove that these derivatives hold (Sections 4.7–4.9). In this way, we prove that equations (27)–(29) hold along the whole geodesic.

Before the actual proof of equations (27)–(29), we derive some relations, useful for further derivations, in Sections 4.1–4.2.

Hereinafter, a subscript following a comma denotes the partial derivative with respect to coordinate x^i, e.g., $H_{,i} = \frac{\partial H}{\partial x^i}$ or $V_{,i} = \frac{\partial V}{\partial x^i}$. A subscript with a tilde following a comma denotes the partial derivative with respect to initial coordinate $\tilde{x}^a$, e.g., $V_{,\tilde{a}} = \frac{\partial V}{\partial \tilde{x}^a}$. A superscript following a comma denotes the partial derivative with respect to slowness vector component y_i, e.g., $H^{,i} = \frac{\partial H}{\partial y_i}$. The Einstein summation applies also to the pair of an index without a tilde and an equal index with a tilde. If not stated otherwise, the Hamiltonian function and its derivatives are taken with arguments $\big(x^m, V_{,n}\big)$.

4.1. Differentiating the Hamilton–Jacobi equations

We differentiate the Hamilton–Jacobi equation (5) with respect to x^i and obtain equation

$$H_{,i}\big(x^m, V_{,n}\big) + H^{,k}\big(x^m, V_{,n}\big)V_{,ki} = 0 \tag{31}$$

(Hamilton, 1837, eqs. Q, I, K). We differentiate the Hamilton–Jacobi equation (5) with respect to $\tilde{x}^a$ and obtain equation

$$H^{,k}\big(x^m, V_{,n}\big)V_{,k\tilde{a}} = 0 \tag{32}$$

(Hamilton, 1837, eqs. U, I). We differentiate the Hamilton–Jacobi equation (6) with respect to $\tilde{x}^i$ and obtain equation

$$H_{,i}\big(\tilde{x}^m, -V_{,\tilde{n}}\big) - H^{,k}\big(\tilde{x}^m, -V_{,\tilde{n}}\big)V_{,\tilde{k}\tilde{i}} = 0 \tag{33}$$

(Hamilton, 1837, eqs. X, I, K).

We differentiate equation (31) with respect to x^j and obtain equation

$$H_{,ij} + H_{,i}^{,m}V_{,mj} + V_{,im}H_{,j}^{,m} + V_{,im}H^{,mn}V_{,nj} + H^{,k}V_{,kij} = 0 \quad . \tag{34}$$

We differentiate equation (31) with respect to $\tilde{x}^a$ or equation (32) with respect to x^j and obtain equation

$$V_{,\tilde{a}m}H_{,j}^{,m} + V_{,\tilde{a}m}H^{,mn}V_{,nj} + H^{,k}V_{,k\tilde{b}j} = 0 \quad . \tag{35}$$

We differentiate equation (32) with respect to $\tilde{x}^b$ and obtain equation

$$V_{,\tilde{a}m}H^{,mn}V_{,n\tilde{b}} + H^{,k}V_{,k\tilde{a}\tilde{b}} = 0 \quad . \tag{36}$$

4.2. Derivatives along geodesics

Equation (7) yields

$$\frac{\mathrm{d}}{\mathrm{d}\gamma} = H^{,k}\frac{\partial}{\partial x^k} \quad . \tag{37}$$

Equation (34) with (37) yields Riccati equation

$$\frac{\mathrm{d}}{\mathrm{d}\gamma}V_{,ij} = -H_{,ij} - H_{,i}^{,m}V_{,mj} - V_{,im}H_{,j}^{,m} - V_{,im}H^{,mn}V_{,nj} \tag{38}$$

for $V_{,ij}$. Equation (35) with (37) yields equation

$$\frac{\mathrm{d}}{\mathrm{d}\gamma}V_{,\tilde{a}j} = -V_{,\tilde{a}m}H_{,j}^{,m} - V_{,\tilde{a}m}H^{,mn}V_{,nj} \quad , \tag{39}$$

which represents, for given $V_{,ij}$, the linear ordinary differential equation for $V_{,\tilde{a}j}$. Equation (36) with (37) yields expression

$$\frac{\mathrm{d}}{\mathrm{d}\gamma}V_{,\tilde{a}\tilde{b}} = -V_{,\tilde{a}m}H^{,mn}V_{,n\tilde{b}} \tag{40}$$

for the derivative of $V_{,\tilde{a}\tilde{b}}$ along the geodesic in terms of $V_{,\tilde{a}j}$. Equations (24) of geodesic deviation read

$$\frac{\mathrm{d}}{\mathrm{d}\gamma}\frac{\partial x^i}{\partial \tilde{x}^j} = H_{,k}^{,i}\frac{\partial x^k}{\partial \tilde{x}^j} + H^{,ik}\frac{\partial y_k}{\partial \tilde{x}^j} \quad , \tag{41}$$

$$\frac{\mathrm{d}}{\mathrm{d}\gamma}\frac{\partial x^i}{\partial \tilde{y}_j} = H_{,k}^{,i}\frac{\partial x^k}{\partial \tilde{y}_j} + H^{,ik}\frac{\partial y_k}{\partial \tilde{y}_j} \quad , \tag{42}$$

$$\frac{\mathrm{d}}{\mathrm{d}\gamma}\frac{\partial y_i}{\partial \tilde{x}^j} = -H_{,ik}\frac{\partial x^k}{\partial \tilde{x}^j} - H_{,i}^{,k}\frac{\partial y_k}{\partial \tilde{x}^j} \quad , \tag{43}$$

$$\frac{\mathrm{d}}{\mathrm{d}\gamma}\frac{\partial y_i}{\partial \tilde{y}_j} = -H_{,ik}\frac{\partial x^k}{\partial \tilde{y}_j} - H_{,i}^{,k}\frac{\partial y_k}{\partial \tilde{y}_j} \quad . \tag{44}$$

4.3. Propagator matrix of geodesic deviation at the initial point of a geodesic

From the Hamiltonian equations (24) of geodesic deviation, we see that

$$\begin{pmatrix} \frac{\partial x^i}{\partial \tilde{x}^j} & \frac{\partial x^i}{\partial \tilde{y}_j} \\ \frac{\partial y_i}{\partial \tilde{x}^j} & \frac{\partial y_i}{\partial \tilde{y}_j} \end{pmatrix} = \begin{pmatrix} \delta_j^i + H_{,j}^{,i}\gamma & H^{,ij}\gamma \\ -H_{,ij}\gamma & \delta_i^j - H_{,i}^{,j}\gamma \end{pmatrix} + O(\gamma^2) \quad . \tag{45}$$

Since

$$H^{,i}\big(x^m, V_{,n}(x^p,\tilde{x}^q)\big) = H^{,i}\big(\tilde{x}^m, V_{,n}(x^p,\tilde{x}^q)\big) + O(\gamma^1) \quad , \tag{46}$$

we do not need to distinguish the Hamiltonian function and its derivatives at point x^m and initial point $\tilde{x}^m$ in Sections 4.3–4.6.

4.4. Equation (27) at the initial point of a geodesic

When point x^a approaches initial point $\tilde{x}^a$, the second derivatives $V_{,ij}(x^a,\tilde{x}^b)$ of the characteristic function with respect to x^a increase with $[\gamma(x^a,\tilde{x}^b)]^{-1}$ according to expansion

$$V_{,ij}(x^m,\tilde{x}^n) = T_{ij}\,[\gamma(x^m,\tilde{x}^n)]^{-1} + O(\gamma^0) \quad , \tag{47}$$

where matrix T_{ij} differs for different geodesics (rays).

We assume that $\gamma_{,r}H^{,rs}\gamma_{,s}$ is not equal to zero, and approximate definition (30) by

$$\Gamma = \gamma_{,r}H^{,rs}\gamma_{,s}\gamma + O(\gamma^2) \quad . \tag{48}$$

We insert expansions (45), (47) and (48) into the left–hand side of relation (27), and arrive at approximation

$$\left(\frac{\partial^2 V}{\partial x^i \partial x^j} + \frac{1}{\Gamma}\frac{\partial\gamma}{\partial x^i}\frac{\partial\gamma}{\partial x^j}\right)\frac{\partial x^j}{\partial \tilde{y}_k} = \left(T_{ij} + \frac{\gamma_{,i}\gamma_{,j}}{\gamma_{,r}H^{,rs}\gamma_{,s}}\right)H^{,jk} + O(\gamma^1) \quad . \tag{49}$$

The relation between a small coordinate difference between points $\tilde{x}^a$ and x^a and the independent parameter γ along the geodesic may be approximated by expansion

$$x^i - \tilde{x}^i = H^{,i}\big(\tilde{x}^m, V_{,n}(x^p, \tilde{x}^q)\big)\gamma(x^r, \tilde{x}^s) + O(\gamma^2) \tag{50}$$

resulting from the first Hamilton's equation (7). Differentiating approximation (50) with respect to x^j and considering expansion (47), we obtain relation

$$\delta^i_j = H^{,ir} T_{rj} + H^{,i}\gamma_{,j} + O(\gamma^1) \quad . \tag{51}$$

Inserting relation (51) into approximation (49), we obtain approximation

$$\left(\frac{\partial^2 V}{\partial x^i \partial x^j} + \frac{1}{\Gamma}\frac{\partial\gamma}{\partial x^i}\frac{\partial\gamma}{\partial x^j}\right)\frac{\partial x^j}{\partial \tilde{y}_k} = \delta^k_i + \gamma_{,i}\left(\frac{\gamma_{,j}H^{,jk}}{\gamma_{,r}H^{,rs}\gamma_{,s}} - H^{,k}\right) + O(\gamma^1) \quad . \tag{52}$$

The definition of $\gamma_{,j}$ together with Hamilton's first equation (7) yield identity

$$\gamma_{,j} H^{,j}(x^m, y_n) = 1 \quad . \tag{53}$$

Identity (53) yields

$$\left(\frac{\gamma_{,j}H^{,jk}}{\gamma_{,r}H^{,rs}\gamma_{,s}} - H^{,k}\right)\gamma_{,k} = 0 \quad . \tag{54}$$

Multiplying relation (51) by $\gamma_{,i}$ and considering relation (53), we obtain relation

$$\gamma_{,i} H^{,ir} T_{rj} = O(\gamma^1) \quad . \tag{55}$$

Inserting expansion (47) into equation (31), we obtain the approximate relation

$$T_{ir} H^{,r} = O(\gamma^1) \quad . \tag{56}$$

Approximations (55) and (56) yield

$$\left(\frac{\gamma_{,j}H^{,jk}}{\gamma_{,r}H^{,rs}\gamma_{,s}} - H^{,k}\right) T_{kl} = O(\gamma^1) \quad . \tag{57}$$

We assume that matrix T_{ij} has only one zero eigenvalue which corresponds to eigenvector $H^{,j}$, see relation (56). Since vector $H^{,j}$ is not perpendicular to $\gamma_{,j}$, see relation (53), relations (57) and (54) result in

$$\frac{\gamma_{,j}H^{,jk}}{\gamma_{,r}H^{,rs}\gamma_{,s}} - H^{,k} = O(\gamma^1) \quad . \tag{58}$$

Approximation (52) of the left–hand side of relation (27) then reads

$$\left(\frac{\partial^2 V}{\partial x^i \partial x^j} + \frac{1}{\Gamma}\frac{\partial\gamma}{\partial x^i}\frac{\partial\gamma}{\partial x^j}\right)\frac{\partial x^j}{\partial \tilde{y}_k} = \delta^k_i + O(\gamma^1) \quad . \tag{59}$$

Approximation (45) of the right–hand side of relation (27) reads

$$\frac{\partial y_i}{\partial \tilde{y}_k} = \delta^k_i + O(\gamma^1) \quad . \tag{60}$$

Relation (27) is thus satisfied for small distances between $\tilde{x}^a$ and x^a with the accuracy of $O(\gamma^1)$, and is thus satisfied for $x^a \to \tilde{x}^a$.

4.5. Equation (28) at the initial point of a geodesic

When point x^a approaches initial point $\tilde{x}^a$, the second derivatives $V_{,\tilde{i}j}(x^a, \tilde{x}^b)$ of the characteristic function with respect to x^a increase with $[\gamma(x^a, \tilde{x}^b)]^{-1}$ according to expansion

$$V_{,\tilde{i}j}(x^m, \tilde{x}^n) = T_{\tilde{i}j}\,[\gamma(x^m, \tilde{x}^n)]^{-1} + O(\gamma^0) \quad , \tag{61}$$

where matrix $T_{\tilde{i}j}$ differs for different geodesics (rays).

We assume that $\gamma_{,r}H^{,rs}\gamma_{,s}$ is not equal to zero, and insert expansions (45), (61) and (48) into the left–hand side of relation (28), and arrive at approximation

$$\left(\frac{\partial^2 V}{\partial\tilde{x}^i\partial x^j} + \frac{1}{\Gamma}\frac{\partial\gamma}{\partial\tilde{x}^i}\frac{\partial\gamma}{\partial x^j}\right)\frac{\partial x^j}{\partial\tilde{y}_k} = \left(T_{\tilde{i}j} + \frac{\gamma_{,\tilde{i}}\gamma_{,j}}{\gamma_{,r}H^{,rs}\gamma_{,s}}\right)H^{,jk} + O(\gamma^1) \quad . \tag{62}$$

Differentiating approximation (50) with respect to $\tilde{x}^j$ and considering expansion (61), we obtain relation

$$-\delta^i_j = H^{,ir}T_{r\tilde{j}} + H^{,i}\gamma_{,\tilde{j}} + O(\gamma^1) \quad . \tag{63}$$

Inserting relation (63) into approximation (62), we obtain approximation

$$\left(\frac{\partial^2 V}{\partial\tilde{x}^i\partial x^j} + \frac{1}{\Gamma}\frac{\partial\gamma}{\partial\tilde{x}^i}\frac{\partial\gamma}{\partial x^j}\right)\frac{\partial x^j}{\partial\tilde{y}_k} = -\delta^k_i + \gamma_{,\tilde{i}}\left(\frac{\gamma_{,j}H^{,jk}}{\gamma_{,r}H^{,rs}\gamma_{,s}} - H^{,k}\right) + O(\gamma^1) \quad . \tag{64}$$

Considering approximate identity (58), approximation (64) of the left–hand side of relation (28) reads

$$\left(\frac{\partial^2 V}{\partial\tilde{x}^i\partial x^j} + \frac{1}{\Gamma}\frac{\partial\gamma}{\partial\tilde{x}^i}\frac{\partial\gamma}{\partial x^j}\right)\frac{\partial x^j}{\partial\tilde{y}_k} = -\delta^k_i + O(\gamma^1) \quad . \tag{65}$$

Relation (28) is thus satisfied for small distances between $\tilde{x}^a$ and x^a with the accuracy of $O(\gamma^1)$, and is thus satisfied for $x^a \to \tilde{x}^a$.

4.6. Equation (29) at the initial point of a geodesic

When point x^a approaches initial point $\tilde{x}^a$, the second derivatives $V_{,\tilde{i}\tilde{j}}(x^a, \tilde{x}^b)$ of the characteristic function with respect to x^a increase with $[\gamma(x^a, \tilde{x}^b)]^{-1}$ according to expansion

$$V_{,\tilde{i}\tilde{j}}(x^m, \tilde{x}^n) = T_{\tilde{i}\tilde{j}}\,[\gamma(x^m, \tilde{x}^n)]^{-1} + O(\gamma^0) \quad , \tag{66}$$

where matrix $T_{\tilde{i}\tilde{j}}$ differs for different geodesics (rays).

Definition of $\gamma_{,\tilde{j}}$ together with Hamilton's first equation (9) yield identity

$$\gamma_{,\tilde{j}}H^{,j}(\tilde{x}^m, \tilde{y}_{,n}) = -1 \quad . \tag{67}$$

Since identities (53) and (67) apply to all directions of $H^{,j}(x^m, y_n) = H^{,j}(\tilde{x}^m, \tilde{y}_{,n}) + O(\gamma^1)$, we obtain approximation

$$\gamma_{,\tilde{j}} = -\gamma_{,j} + O(\gamma^1) \tag{68}$$

for short distances between $\tilde{x}^m$ and x^m, and approximate relation (48) by

$$\Gamma = \gamma_{,\tilde{r}}H^{,rs}\gamma_{,\tilde{s}}\gamma + O(\gamma^2) \quad . \tag{69}$$

Here we assume that $\gamma_{,\tilde{r}}H^{,rs}\gamma_{,\tilde{s}}$ is not equal to zero. We insert expansions (45), (66) and (69) into the left–hand side of relation (29), and arrive at approximation

$$\frac{\partial x^i}{\partial\tilde{y}_j}\left(\frac{\partial^2 V}{\partial\tilde{x}^j\partial\tilde{x}^k} + \frac{1}{\Gamma}\frac{\partial\gamma}{\partial\tilde{x}^j}\frac{\partial\gamma}{\partial\tilde{x}^k}\right) = H^{,ij}\left(T_{\tilde{j}\tilde{k}} + \frac{\gamma_{,\tilde{j}}\gamma_{,\tilde{k}}}{\gamma_{,\tilde{r}}H^{,rs}\gamma_{,\tilde{s}}}\right) + O(\gamma^1) \quad . \tag{70}$$

The relation between a small coordinate difference between points $\tilde{x}^a$ and x^a and the independent parameter γ along the geodesic may be approximated by expansion

$$\tilde{x}^i - x^i = -H^{,i}\big(\tilde{x}^m, -V_{,\tilde{n}}(x^p, \tilde{x}^q)\big)\gamma(x^r, \tilde{x}^s) + O(\gamma^2) \tag{71}$$

resulting from Hamilton's first equation (9). Differentiating approximation (71) with respect to $\tilde{x}^j$ and considering expansion (66), we obtain relation

$$\delta^i_j = H^{,ir} T_{\tilde{r}\tilde{j}} - H^{,i}\gamma_{,\tilde{j}} + O(\gamma^1) \quad . \tag{72}$$

Inserting relation (72) into approximation (70), we obtain approximation

$$\frac{\partial x^i}{\partial \tilde{y}_j}\left(\frac{\partial^2 V}{\partial \tilde{x}^j \partial \tilde{x}^k} + \frac{1}{\Gamma}\frac{\partial \gamma}{\partial \tilde{x}^j}\frac{\partial \gamma}{\partial \tilde{x}^k}\right) = \delta^i_k + \left(\frac{H^{,ij}\gamma_{,\tilde{j}}}{\gamma_{,\tilde{r}} H^{,rs}\gamma_{,\tilde{s}}} + H^{,i}\right)\gamma_{,\tilde{k}} + O(\gamma^1) \quad . \tag{73}$$

Multiplying relation (72) by $\gamma_{,\tilde{i}}$ and considering relation (67), we obtain relation

$$\gamma_{,\tilde{i}} H^{,ir} T_{\tilde{r}\tilde{j}} = O(\gamma^1) \quad . \tag{74}$$

Inserting expansion (47) into equation (33), we obtain approximate relation

$$T_{\tilde{a}\tilde{r}} H^{,r} = O(\gamma^1) \quad . \tag{75}$$

Analogously as we derived approximate identity (58) using relations (53), (55) and (56), we may derive approximation

$$\frac{H^{,ij}\gamma_{,\tilde{j}}}{\gamma_{,\tilde{r}} H^{,rs}\gamma_{,\tilde{s}}} + H^{,i} = O(\gamma^1) \tag{76}$$

using approximate relations (67), (74) and (75).

Approximation (73) of the left–hand side of relation (29) then reads

$$\frac{\partial x^i}{\partial \tilde{y}_j}\left(\frac{\partial^2 V}{\partial \tilde{x}^j \partial \tilde{x}^k} + \frac{1}{\Gamma}\frac{\partial \gamma}{\partial \tilde{x}^j}\frac{\partial \gamma}{\partial \tilde{x}^k}\right) = \delta^i_k + O(\gamma^1) \quad . \tag{77}$$

Approximation (45) of the right–hand side of relation (29) reads

$$\frac{\partial x^i}{\partial \tilde{x}^k} = \delta^i_k + O(\gamma^1) \quad . \tag{78}$$

Relation (29) is thus satisfied for small distances between $\tilde{x}^a$ and x^a with the accuracy of $O(\gamma^1)$, and is thus satisfied for $x^a \to \tilde{x}^a$.

4.7. Derivative of (27)

We denote

$$F_{ij} = \Gamma^{-1}\gamma_{,i}\gamma_{,j} \quad , \tag{79}$$

and express the derivative of equation (27) along the geodesic using equations (38), (42) and (44),

$$\begin{aligned}\big(-H_{,ij} - H^{,m}_{,i} V_{,mj} - V_{,im} H^{,m}_{,j} - V_{,im} H^{,mn} V_{,nj} + \tfrac{\mathrm{d}}{\mathrm{d}\gamma} F_{ij}\big)\tfrac{\partial x^j}{\partial \tilde{y}_k} \\ + \big(V_{,ij} + F_{ij}\big)\big(H^{,j}_{,r}\tfrac{\partial x^r}{\partial \tilde{y}_k} + H^{,jr}\tfrac{\partial y_r}{\partial \tilde{y}_k}\big) = -H_{,ir}\tfrac{\partial x^r}{\partial \tilde{y}_k} - H^{,r}_{,i}\tfrac{\partial y_r}{\partial \tilde{y}_k} \quad .\end{aligned} \tag{80}$$

We collect the terms containing $\frac{\partial x^j}{\partial \tilde{y}_k}$ and the terms containing $\frac{\partial y_j}{\partial \tilde{y}_k}$,

$$\begin{aligned}\big(-H^{,m}_{,i} V_{,mj} - V_{,im} H^{,mn} V_{,nj} + F_{ir} H^{,r}_{,j} + \tfrac{\mathrm{d}}{\mathrm{d}\gamma} F_{ij}\big)\tfrac{\partial x^j}{\partial \tilde{y}_k} \\ +\big(V_{,ir} H^{,rj} + F_{ir} H^{,rj} + H^{,j}_{,i}\big]\tfrac{\partial y_j}{\partial \tilde{y}_k} = 0 \quad .\end{aligned} \tag{81}$$

We insert (27) for $\frac{\partial y_j}{\partial \tilde{y}_k}$,

$$\big(-H^{,m}_{,i}V_{,mj}-V_{,im}H^{,mn}V_{,nj}+F_{ir}H^{,r}_{,j}+\tfrac{\mathrm{d}}{\mathrm{d}\gamma}F_{ij}\big)\tfrac{\partial x^j}{\partial \tilde{y}_k} +\big(V_{,ir}H^{,rs}+F_{ir}H^{,rs}+H^{,s}_{,i}\big)\big(V_{,sj}+F_{sj}\big)\tfrac{\partial x^j}{\partial \tilde{y}_k}=0 \quad . \tag{82}$$

We now execute the summation in (82) and arrive at

$$\big[\tfrac{\mathrm{d}}{\mathrm{d}\gamma}F_{ij}+F_{ir}\big(H^{,r}_{,j}+H^{,rs}V_{,sj}\big)+\big(H^{,r}_{,i}+H^{,rs}V_{,si}\big)F_{rj}+F_{ir}H^{,rs}F_{sj}\big]\tfrac{\partial x^j}{\partial \tilde{y}_k}=0 \quad . \tag{83}$$

We insert (79) into (83) and consider that $\frac{\mathrm{d}}{\mathrm{d}\gamma}\gamma_{,i}=\gamma_{,ir}H^{,r}$,

$$\Big\{-\Gamma^{-2}\gamma_{,i}\gamma_{,j}\tfrac{\mathrm{d}}{\mathrm{d}\gamma}\Gamma+\Gamma^{-1}\gamma_{,i}\big[\gamma_{,jr}H^{,r}+\gamma_{,r}\big(H^{,r}_{,j}+H^{,rs}V_{,sj}\big)\big] +\Gamma^{-1}\big[\gamma_{,ir}H^{,r}+\gamma_{,r}\big(H^{,r}_{,i}+H^{,rs}V_{,si}\big)\big]\gamma_{,j}+\Gamma^{-2}\gamma_{,i}\gamma_{,r}H^{,rs}\gamma_{,s}\gamma_{,j}\Big\}\tfrac{\partial x^j}{\partial \tilde{y}_k}=0 \quad . \tag{84}$$

Terms

$$\gamma_{,ir}H^{,r}+\gamma_{,r}\big(H^{,r}_{,i}+H^{,rs}V_{,si}\big)=\tfrac{\partial}{\partial x^i}\big[\gamma_{,r}H^{,r}\big(x^m,V_{,n}(x^p,\tilde{x}^q)\big)\big] \tag{85}$$

are zero because of identity (53),

$$\gamma_{,ir}H^{,r}+\gamma_{,r}\big(H^{,r}_{,i}+H^{,rs}V_{,si}\big)=0 \quad . \tag{86}$$

Equation (84) with (86) reads

$$\Gamma^{-2}\gamma_{,i}\big(\tfrac{\mathrm{d}}{\mathrm{d}\gamma}\Gamma-\gamma_{,r}H^{,rs}\gamma_{,s}\big)\gamma_{,j}\tfrac{\partial x^j}{\partial \tilde{y}_k}=0 \quad , \tag{87}$$

which is satisfied as a consequence of definition (30).

4.8. Derivative of (28)

We denote

$$F_{\tilde{i}j}=\Gamma^{-1}\gamma_{,\tilde{i}}\gamma_{,j} \quad , \tag{88}$$

and express the derivative of equation (28) along the geodesic using equations (39) and (42),

$$\big(-V_{,\tilde{i}m}H^{,m}_{,j}-V_{,\tilde{i}m}H^{,mn}V_{,nj}+\tfrac{\mathrm{d}}{\mathrm{d}\gamma}F_{\tilde{i}j}\big)\tfrac{\partial x^j}{\partial \tilde{y}_k}+\big(V_{,\tilde{i}j}+F_{\tilde{i}j}\big)\big(H^{,j}_{,r}\tfrac{\partial x^r}{\partial \tilde{y}_k}+H^{,jr}\tfrac{\partial y_r}{\partial \tilde{y}_k}\big)=0\ . \tag{89}$$

We collect the terms containing $\frac{\partial x^j}{\partial \tilde{y}_k}$ and the terms containing $\frac{\partial y_j}{\partial \tilde{y}_k}$,

$$\big(-V_{,\tilde{i}m}H^{,mn}V_{,nj}+F_{\tilde{i}r}H^{,r}_{,j}+\tfrac{\mathrm{d}}{\mathrm{d}\gamma}F_{\tilde{i}j}\big)\tfrac{\partial x^j}{\partial \tilde{y}_k}+\big(V_{,\tilde{i}r}H^{,rj}+F_{\tilde{i}r}H^{,rj}\big]\tfrac{\partial y_j}{\partial \tilde{y}_k}=0 \quad . \tag{90}$$

We insert (27) for $\frac{\partial y_j}{\partial \tilde{y}_k}$,

$$\big(-V_{,\tilde{i}m}H^{,mn}V_{,nj}+F_{\tilde{i}r}H^{,r}_{,j}+\tfrac{\mathrm{d}}{\mathrm{d}\gamma}F_{\tilde{i}j}\big)\tfrac{\partial x^j}{\partial \tilde{y}_k} +\big(V_{,\tilde{i}r}H^{,rs}+F_{\tilde{i}r}H^{,rs}\big)\big(V_{,sj}+F_{sj}\big)\tfrac{\partial x^j}{\partial \tilde{y}_k}=0 \quad . \tag{91}$$

We now execute the summation in (91) and arrive at

$$\big[\tfrac{\mathrm{d}}{\mathrm{d}\gamma}F_{\tilde{i}j}+F_{\tilde{i}r}\big(H^{,r}_{,j}+H^{,rs}V_{,sj}\big)+V_{,\tilde{i}r}H^{,rs}F_{sj}+F_{\tilde{i}r}H^{,rs}F_{sj}\big]\tfrac{\partial x^j}{\partial \tilde{y}_k}=0 \quad . \tag{92}$$

We insert (88) into (92) and consider that $\frac{\mathrm{d}}{\mathrm{d}\gamma}\gamma_{,i}=\gamma_{,ir}H^{,r}$ and $\frac{\mathrm{d}}{\mathrm{d}\gamma}\gamma_{,\tilde{i}}=\gamma_{,\tilde{i}r}H^{,r}$,

$$\Big\{-\Gamma^{-2}\gamma_{,\tilde{i}}\gamma_{,j}\tfrac{\mathrm{d}}{\mathrm{d}\gamma}\Gamma+\Gamma^{-1}\gamma_{,\tilde{i}}\big[\gamma_{,jr}H^{,r}+\gamma_{,r}\big(H^{,r}_{,j}+H^{,rs}V_{,sj}\big)\big] +\Gamma^{-1}\big[\gamma_{,\tilde{i}r}H^{,r}+\gamma_{,r}H^{,rs}V_{,s\tilde{i}}\big]\gamma_{,j}+\Gamma^{-2}\gamma_{,\tilde{i}}\gamma_{,r}H^{,rs}\gamma_{,s}\gamma_{,j}\Big\}\tfrac{\partial x^j}{\partial \tilde{y}_k}=0 \quad . \tag{93}$$

Term

$$\gamma_{,\tilde{i}r} H^{,r} + \gamma_{,r} H^{,rs} V_{,s\tilde{i}} = \frac{\partial}{\partial \tilde{x}^i} \big[\gamma_{,r} H^{,r}\big(x^m, V_{,n}(x^p, \tilde{x}^q)\big) \big] \tag{94}$$

is zero because of identity (53),

$$\gamma_{,\tilde{i}r} H^{,r} + \gamma_{,r} H^{,rs} V_{,s\tilde{i}} = 0 \quad . \tag{95}$$

Equation (93) with identities (86) and (95) reads

$$\Gamma^{-2} \gamma_{,\tilde{i}} \big(\tfrac{\mathrm{d}}{\mathrm{d}\gamma} \Gamma - \gamma_{,r} H^{,rs} \gamma_{,s} \big) \gamma_{,j} \frac{\partial x^j}{\partial \tilde{y}_k} = 0 \quad , \tag{96}$$

which is satisfied as a consequence of definition (30).

4.9. Derivative of (29)

We denote

$$F_{\tilde{i}\tilde{j}} = \Gamma^{-1} \gamma_{,\tilde{i}} \gamma_{,\tilde{j}} \quad , \tag{97}$$

and express the derivative of equation (29) along the geodesic using equations (40), (41) and (42),

$$\frac{\partial x^i}{\partial \tilde{y}_j} \big(\tfrac{\mathrm{d}}{\mathrm{d}\gamma} F_{\tilde{j}\tilde{k}} - V_{,\tilde{j}m} H^{,mn} V_{,n\tilde{k}} \big) + \big(H^{;i}_{,r} \frac{\partial x^r}{\partial \tilde{y}_j} + H^{,ir} \frac{\partial y_r}{\partial \tilde{y}_j} \big) \big(V_{,\tilde{j}\tilde{k}} + F_{\tilde{j}\tilde{k}} \big) = H^{;i}_{,r} \frac{\partial x^r}{\partial \tilde{x}^k} + H^{,ir} \frac{\partial y_r}{\partial \tilde{x}^k} \; . \tag{98}$$

Considering equation (29), the terms containing $H^{;i}_{,r}$ cancel out,

$$\frac{\partial x^i}{\partial \tilde{y}_j} \big(\tfrac{\mathrm{d}}{\mathrm{d}\gamma} F_{\tilde{j}\tilde{k}} - V_{,\tilde{j}m} H^{,mn} V_{,n\tilde{k}} \big) + H^{,ir} \frac{\partial y_r}{\partial \tilde{y}_j} \big(V_{,\tilde{j}\tilde{k}} + F_{\tilde{j}\tilde{k}} \big) = H^{,ir} \frac{\partial y_r}{\partial \tilde{x}^k} \quad . \tag{99}$$

We multiply equation (99) from the right–hand side by matrix $\frac{\partial x^l}{\partial \tilde{y}_k}$,

$$\frac{\partial x^i}{\partial \tilde{y}_j} \big(\tfrac{\mathrm{d}}{\mathrm{d}\gamma} F_{\tilde{j}\tilde{k}} - V_{,\tilde{j}m} H^{,mn} V_{,n\tilde{k}} \big) \frac{\partial x^l}{\partial \tilde{y}_k} + H^{,ir} \frac{\partial y_r}{\partial \tilde{y}_j} \big(V_{,\tilde{j}\tilde{k}} + F_{\tilde{j}\tilde{k}} \big) \frac{\partial x^l}{\partial \tilde{y}_k} = H^{,ir} \frac{\partial y_r}{\partial \tilde{x}^k} \frac{\partial x^l}{\partial \tilde{y}_k} \quad . \tag{100}$$

We insert equation (29) in order to remove $(V_{,\tilde{j}\tilde{k}} + F_{\tilde{j}\tilde{k}})$,

$$\frac{\partial x^i}{\partial \tilde{y}_j} \big(\tfrac{\mathrm{d}}{\mathrm{d}\gamma} F_{\tilde{j}\tilde{k}} - V_{,\tilde{j}m} H^{,mn} V_{,n\tilde{k}} \big) \frac{\partial x^l}{\partial \tilde{y}_k} + H^{,ir} \frac{\partial y_r}{\partial \tilde{y}_j} \frac{\partial x^l}{\partial \tilde{x}^j} = H^{,ir} \frac{\partial y_r}{\partial \tilde{x}^k} \frac{\partial x^l}{\partial \tilde{y}_k} \quad . \tag{101}$$

We apply the consequence

$$\frac{\partial y_i}{\partial \tilde{y}_r} \frac{\partial x^k}{\partial \tilde{x}^r} - \frac{\partial y_i}{\partial \tilde{x}^r} \frac{\partial x^k}{\partial \tilde{y}_r} = \delta^k_i \tag{102}$$

of the symplectic property of propagator matrix (22):

$$\frac{\partial x^i}{\partial \tilde{y}_j} \big(\tfrac{\mathrm{d}}{\mathrm{d}\gamma} F_{\tilde{j}\tilde{k}} - V_{,\tilde{j}m} H^{,mn} V_{,n\tilde{k}} \big) \frac{\partial x^l}{\partial \tilde{y}_k} + H^{,il} = 0 \quad . \tag{103}$$

We add $H^{,mn}$ multiplied from both sides by the left–hand side of equation (28), and subtract $H^{,mn}$ multiplied from both sides by the right–hand side of equation (28):

$$\frac{\partial x^i}{\partial \tilde{y}_j} \big(\tfrac{\mathrm{d}}{\mathrm{d}\gamma} F_{\tilde{j}\tilde{k}} + F_{,\tilde{j}m} H^{,mn} F_{,n\tilde{k}} + V_{,\tilde{j}m} H^{,mn} F_{,n\tilde{k}} + F_{,\tilde{j}m} H^{,mn} V_{,n\tilde{k}} \big) \frac{\partial x^l}{\partial \tilde{y}_k} = 0 \quad . \tag{104}$$

We insert (97) into (104) and consider that $\frac{\mathrm{d}}{\mathrm{d}\gamma} \gamma_{,\tilde{i}} = \gamma_{,\tilde{i}r} H^{,r}$:

$$\frac{\partial x^i}{\partial \tilde{y}_j} \big[\Gamma^{-2} \gamma_{,\tilde{j}} \big(- \tfrac{\mathrm{d}}{\mathrm{d}\gamma} \Gamma + \gamma_{,r} H^{,rs} \gamma_{,s} \big) \gamma_{,\tilde{k}} + \big(\gamma_{,\tilde{j}r} H^{,r} + V_{,\tilde{j}r} H^{,rs} \gamma_{,s} \big) \gamma_{,\tilde{k}} + \gamma_{,\tilde{j}} \big(\gamma_{,\tilde{k}r} H^{,r} + V_{,\tilde{k}r} H^{,rs} \gamma_{,s} \big) \big] \frac{\partial x^l}{\partial \tilde{y}_k} = 0 \quad . \tag{105}$$

Equation (105) with identity (95) reads

$$-\Gamma^{-2} \frac{\partial x^i}{\partial \tilde{y}_j} \gamma_{,\tilde{j}} \big(\tfrac{\mathrm{d}}{\mathrm{d}\gamma} \Gamma - \gamma_{,r} H^{,rs} \gamma_{,s} \big) \gamma_{,\tilde{k}} \frac{\partial x^l}{\partial \tilde{y}_k} = 0 \quad , \tag{106}$$

which is satisfied as a consequence of definition (30).

5. Conclusions

The propagator matrix (22) of geodesic deviation contains all the linearly independent solutions of the equations of geodesic deviation. It can be calculated using the Hamiltonian equations (17)–(18) of geodesic deviation.

The derived general relations (27)–(29) between the propagator matrix (22) of geodesic deviation and the second–order spatial derivatives of the characteristic function are applicable to the high–frequency approximations of propagation of various waves (e.g., elastic, electromagnetic, Dirac), to the Finsler geometry, to the Riemann geometry, and to their various applications such as general relativity.

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