Paraxial Super-Gaussian beams

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Summary

In the conventional ray theory with real–valued travel time, the initial amplitude profile is represented by the initial conditions for amplitude. Since the accuracy of the ray theory suffers from the amplitude changes along wavefronts, this approach is considerably inaccurate for beams, because it does not provide the spreading of the beams caused by diffraction.

The representation of the initial Gaussian amplitude profile in terms of the imaginary part of the initial complex–valued travel time with the constant initial conditions for amplitude yields satisfactorily accurate paraxial Gaussian beams.

In this paper, we demonstrate that the representation of the initial Super–Gaussian amplitude profile in terms of the imaginary part of the initial complex–valued travel time with the constant initial conditions for amplitude yields the Super–Gaussian beams whose lowest–order paraxial approximation is identical to the conventional ray theory solution with real–valued travel time, without the diffracted wavefield which could result from the representation theorem.

Keywords

Wave propagation, ray theory, complex–valued travel time, paraxial approximation, Gaussian beams, Super–Gaussian beams.

1. Introduction

In the conventional ray theory with real–valued travel time, the initial amplitude profile is represented by the initial conditions for amplitude. Since the accuracy of the ray theory suffers from the amplitude changes along the wavefronts, this approach is considerably inaccurate for beams, because it does not provide the spreading of the beams caused by diffraction. For example, if the initial travel time is constant along the initial plane and the initial conditions for amplitude are Gaussian, the conventional ray theory with real–valued travel time yields the incorrect beam with the constant envelope without any spreading (Kravtsov & Berczynski, 2006, eq. 15).

The representation of the initial amplitude profile in terms of the imaginary part of the initial complex–valued travel time with the constant initial conditions for amplitude yields considerably more accurate beams in many cases. Since we calculate wavefields in real–valued space, we need to avoid tracing complex–valued rays. We thus usually approximate the complex–valued travel time of beams by the paraxial expansion along the real–valued reference ray. For example, if the Gaussian initial amplitude profile $\exp(-ax^2)$ is represented by the quadratic imaginary part of the initial complex–valued travel time and the initial conditions for amplitude are constant, we obtain the Gaussian beam which is often very accurate even within the second–order paraxial approximation of the complex–valued travel time, and has no problems with caustics.

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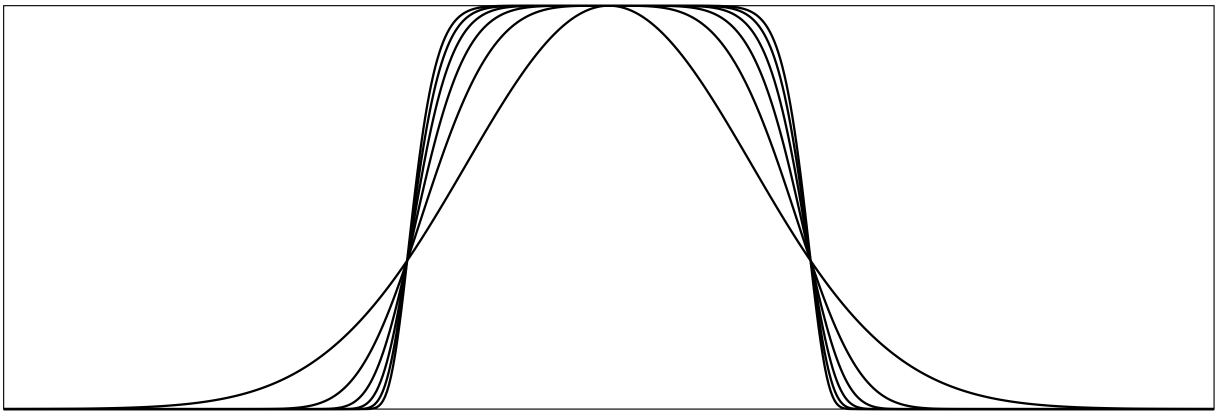

Figure 1. Gaussian function $\exp(-x^2)$ and supergauss functions $\exp(-x^4)$, $\exp(-x^6)$, $\exp(-x^8)$, $\exp(-x^{10})$ and $\exp(-x^{12})$.

Unfortunately, it is obvious that the initial amplitude cannot always be represented in terms of the imaginary part of the initial complex–valued travel time, especially if the initial amplitude changes its sign. In this paper we demonstrate that this approach fails even for well localized beams corresponding to the initial amplitude profiles of forms $\exp(-ax^4)$, $\exp(-ax^6)$, $\exp(-ax^8)$, ..., see Figure 1, which are called "supergauss functions" (Oldham, Myland & Spanier, 2009, eq. 27:12:1).

In the high–frequency approximation, the amplitude cross–section of a beam is predominantly determined by the lowest–order non–vanishing derivative of the imaginary part of its travel time at the reference ray. Since we wish to study the beams concentrated at the reference ray, this lowest–order imaginary derivative should be of an even order. If this lowest–order imaginary derivative is of the second order, we obtain the famous Gaussian beams whose amplitude cross–section is approximately a Gaussian function of form $\exp(-ax^2)$.

In this paper, we are curious to know the properties of the beams with the lowest–order imaginary derivative of the fourth, sixth, eighth, ... order. The amplitude cross–section of these beams are approximately functions of forms $\exp(-ax^4)$, $\exp(-ax^6)$, $\exp(-ax^8)$, ..., which are called "supergauss functions" (Oldham, Myland & Spanier, 2009, eq. 27:12:1). That is why we refer to these beams as the *Super–Gaussian beams*. Since we restrict the Taylor expansion of the complex–valued travel time of a Super–Gaussian beam to the order of the lowest–order non–vanishing derivative of the imaginary part of the travel time, we refer to our approximation of the Super–Gaussian beams as the *paraxial Super–Gaussian beams*.

Following Babich, Buldyrev & Molotkov (1985), Klimeš (2002) derived explicit equations for calculating the third–order and higher–order spatial derivatives of travel time. In this paper, we apply these equations for the travel–time derivatives to the paraxial Super–Gaussian beams.

2. Paraxial Super–Gaussian beams

For the calculation of travel time, we consider Hamiltonian function $H(x^i, y_j)$, which is a real–valued function of coordinates x_i and of covariant vector y_j from the cotangent space at point x^i, and is sufficiently smooth within its definition domain.

We denote the spatial derivatives of travel time $\tau(x_k)$ with respect to coordinates x_i by $\tau_{,ij...n}$. We introduce the covariant derivatives (Klimeš, 2002, eq. 16)

$$T_{ab...f} = \tau_{,ij...n}\, Q_{ia} Q_{jb} \cdots Q_{nf} \tag{1}$$

of travel time with respect to ray coordinates γ_a. Ray coordinates γ_a are composed of the ray parameters, which parametrize the rays and are constant along each ray, and of the independent parameter γ along rays which is determined by the form of the Hamiltonian function. The transformation matrix

$$Q_{ia} = \frac{\partial x_i}{\partial \gamma_a} \tag{2}$$

from ray coordinates γ_a to coordinates x_i is often referred to as the matrix of geometrical spreading.

The third–order and higher–order spatial derivatives of travel time with respect to coordinates x_i may be expressed in terms of covariant derivatives (1) as (Klimeš, 2002, eq. 20)

$$\tau_{,ij...n} = T_{ab...f}\, Q^{-1}_{ai}\, Q^{-1}_{bj} \cdots Q^{-1}_{fn} \quad , \tag{3}$$

where we have used Q^{-1}_{ai} to denote the components of the matrix inverse to matrix Q_{ia}. Matrix Q^{-1}_{ai} exists and is finite off caustics.

The covariant derivatives of travel time with respect to the ray parameters may be expressed in the form of the integral along the ray (Klimeš, 2002, eq. 19),

$$T_{ab...f}(\gamma) = T_{ab...f}(\gamma^0) + \int_{\gamma^0}^{\gamma} \mathrm{d}\gamma\; K_{ij...n}\, Q_{ia} Q_{jb} \cdots Q_{nf} \quad , \tag{4}$$

with initial conditions $T_{ab...f\alpha...\nu}(\gamma^0)$ defined by equation (1).

Integration kernels $K_{ij...n}$, corresponding to the third–order and higher–order derivatives of travel time, are composed of the lower–order derivatives of travel time (with respect to the calculated derivatives), and of the phase–space derivatives of the Hamiltonian function (Klimeš, 2002, eq. 21). We assume that the phase–space derivatives of the Hamiltonian function are real–valued along the real–valued reference ray.

We consider a Super–Gaussian beam which has real–valued derivatives of travel time up to order $N-1$ and non–vanishing N^{th}–order derivatives of the imaginary part of the travel time ($N = 4, 6, 8, ...$). Then the real–valued derivatives of travel time up to order $2N-3$ are independent of the N^{th}–order derivatives of the imaginary part of the travel time. The lowest–order derivative of the real part of the travel time influenced by the imaginary part of the travel time is the derivative of order $2N-2$.

Since the integration kernels $K_{ij...n}$ corresponding to the N^{th}–order derivatives of travel time are composed of the derivatives of travel time up to order $N-1$, the integration kernels are real–valued, and equation (4) yields relation

$$\mathrm{Im}[T_{ab...f}(\gamma)] = \mathrm{Im}[T_{ab...f}(\gamma^0)] \tag{5}$$

for the derivatives of travel time of order N.

From definition (1) with relation (5), we see that the lowest–order (N^{th}–order) paraxial approximation of the imaginary part of the travel time is constant along all paraxial rays. The corresponding paraxial approximation of the real part of the travel time is the same as it were for the real–valued travel time.

The Super–Gaussian beams are thus equivalent to the zero–order ray–theory wavefield with the real–valued travel time and with the initial Super–Gaussian amplitude profile, without the diffracted wavefield which could result from the representation theorem.

The diffraction of a beam is thus satisfactorily included in the case of paraxial Gaussian beams, but not in the case of paraxial Super–Gaussian beams.

3. Conclusions

The representation of the initial Gaussian amplitude profile in terms of the imaginary part of the initial complex–valued travel time with the constant initial conditions for amplitude yields satisfactorily accurate paraxial Gaussian beams. Unfortunately, this does not apply to the initial Super–Gaussian amplitude profiles of forms $\exp(-ax^N)$ with $N = 4, 6, 8, ...$

We have demonstrated that the representation of the initial Super–Gaussian amplitude profile $\exp(-ax^N)$ in terms of the imaginary part of the initial complex–valued travel time with the constant initial conditions for amplitude yields the Super–Gaussian beam whose lowest–order (i.e. N^{th}–order) paraxial approximation is identical to the zero–order ray theory solution with the real–valued travel time and the Super–Gaussian initial conditions for amplitude, without the diffracted wavefield which could result from the representation theorem.

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