Approximating the complex-valued Green-tensor amplitude by a real-valued Green-tensor amplitude

Luděk Klimeš

Department of Geophysics, Faculty of Mathematics and Physics, Charles University in Prague, Ke Karlovu 3, 121 16 Praha 2, Czech Republic, http://sw3d.cz/staff/klimes.htm

Summary

The paper is devoted to the approximation of the complex–valued Green–tensor amplitude by a real–valued Green–tensor amplitude with a phase shift. This approximation is required in the method of Jan Šílený for determining the real–valued seismic moment tensor from the maximum real–valued vectorial amplitude picked in the polarization diagram.

Keywords

Wave propagation, elastic waves, Green tensor, amplitude, seismic moment tensor.

1. Introduction

The tensorial amplitude of the S–wave Green tensor is often complex–valued, especially if the wavefield is recorded at the Earth surface (Červený, 2001).

The method of Šílený & Milev (2008), Šílený et al. (2009) and Horálek & Šílený (2013) for determining the real–valued seismic moment tensor from the maximum real–valued vectorial amplitude picked in the polarization diagram requires the approximation of the complex–valued Green–tensor amplitude by a real–valued Green–tensor amplitude with a phase shift. When studying this approximation, we shall assume the ray–theory approximation of the Green tensor expressed in terms of the 3×3 tensorial amplitude and travel time. The approximation is applicable to Green–tensor amplitudes in both isotropic and anisotropic heterogeneous media, including the isotropic ray theory, the anisotropic ray theory, and the prevailing–frequency approximation of the coupling ray theory for S waves (Klimeš & Bulant, 2012).

2. Approximation of the complex–valued Green–tensor amplitude

Complex–valued Green–tensor amplitude $\mathbf{G}^{\mathrm{C}}$ maps seismic force $\mathbf{f}$ onto the complex–valued displacement $\mathbf{u}^{\mathrm{C}}$,

$$\mathbf{u}^{\mathrm{C}} = \mathbf{G}^{\mathrm{C}}\,\mathbf{f} \quad . \tag{1}$$

We wish to approximate complex–valued Green–tensor amplitude $\mathbf{G}^{\mathrm{C}}$ by real–valued Green–tensor amplitude $\mathbf{G}^{\mathrm{R}}$ with phase shift factor $\exp(\mathrm{i}\varphi)$, which map seismic force $\mathbf{f}$ onto the approximate complex–valued displacement $\mathbf{u}^{\mathrm{R}}$,

$$\mathbf{u}^{\mathrm{R}} = \mathbf{G}^{\mathrm{R}}\,\mathbf{f}\,\exp(\mathrm{i}\varphi) \quad . \tag{2}$$

The square of the wavefield error of this approximation is

$$\delta^2 = (\mathbf{u}^{\mathrm{C}} - \mathbf{u}^{\mathrm{R}})^{+}(\mathbf{u}^{\mathrm{C}} - \mathbf{u}^{\mathrm{R}}) \quad , \tag{3}$$

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where $^{+}$ denotes the Hermitian adjoint. We insert relations (1) and (2) into definition (3) and obtain

$$\delta^2 = \mathbf{f}^{+} \left[\mathbf{G}^{\mathrm{C}} - \mathbf{G}^{\mathrm{R}} \exp(\mathrm{i}\varphi)\right]^{+} \left[\mathbf{G}^{\mathrm{C}} - \mathbf{G}^{\mathrm{R}} \exp(\mathrm{i}\varphi)\right] \mathbf{f} \quad . \tag{4}$$

We average the square of wavefield error δ^2 over all spatial directions of unit seismic force $\mathbf{f}$,

$$\langle\delta^2\rangle = \tfrac{1}{3} \operatorname{Tr}\left\{ \left[\mathbf{G}^{\mathrm{C}} - \mathbf{G}^{\mathrm{R}} \exp(\mathrm{i}\varphi)\right]^{+} \left[\mathbf{G}^{\mathrm{C}} - \mathbf{G}^{\mathrm{R}} \exp(\mathrm{i}\varphi)\right] \right\} \quad . \tag{5}$$

After multiplication, relation (5) reads

$$\langle\delta^2\rangle = \tfrac{1}{3} \operatorname{Tr}\left[(\mathbf{G}^{\mathrm{C}})^{+}\mathbf{G}^{\mathrm{C}} - (\mathbf{G}^{\mathrm{R}})^{\mathrm{T}}\mathbf{G}^{\mathrm{C}} \exp(-\mathrm{i}\varphi) - (\mathbf{G}^{\mathrm{C}})^{+}\mathbf{G}^{\mathrm{R}} \exp(\mathrm{i}\varphi) + (\mathbf{G}^{\mathrm{R}})^{\mathrm{T}}\mathbf{G}^{\mathrm{R}}\right] \ , \tag{6}$$

where $^{\mathrm{T}}$ denotes the transpose. We decompose the complex–valued Green–tensor amplitude into its real and imaginary parts and obtain

$$\langle\delta^2\rangle = \tfrac{1}{3} \operatorname{Tr}\left[(\mathbf{G}^{\mathrm{C}})^{+}\mathbf{G}^{\mathrm{C}} - 2\,(\mathbf{G}^{\mathrm{R}})^{\mathrm{T}}\mathrm{Re}(\mathbf{G}^{\mathrm{C}}) \cos(\varphi) - 2\,(\mathbf{G}^{\mathrm{R}})^{\mathrm{T}}\mathrm{Im}(\mathbf{G}^{\mathrm{C}}) \sin(\varphi) + (\mathbf{G}^{\mathrm{R}})^{\mathrm{T}}\mathbf{G}^{\mathrm{R}}\right] . \tag{7}$$

3. Minimizing the wavefield error

We wish to select real–valued Green–tensor amplitude $\mathbf{G}^{\mathrm{R}}$ and phase shift φ so that the average square (7) of wavefield error is minimal.

Differentiating wavefield error (7) with respect to real–valued Green–tensor amplitude $\mathbf{G}^{\mathrm{R}}$, we obtain equation

$$\tfrac{2}{3} \left[- \mathrm{Re}(\mathbf{G}^{\mathrm{C}}) \cos(\varphi) - \mathrm{Im}(\mathbf{G}^{\mathrm{C}}) \sin(\varphi) + \mathbf{G}^{\mathrm{R}}\right] = \mathbf{0} \tag{8}$$

for the minimum of the average square (7) of wavefield error.

Differentiating wavefield error (7) with respect to phase shift φ, we obtain equation

$$\tfrac{2}{3} \operatorname{Tr}\left[(\mathbf{G}^{\mathrm{R}})^{\mathrm{T}}\mathrm{Re}(\mathbf{G}^{\mathrm{C}}) \sin(\varphi) - (\mathbf{G}^{\mathrm{R}})^{\mathrm{T}}\mathrm{Im}(\mathbf{G}^{\mathrm{C}}) \cos(\varphi)\right] = 0 \quad . \tag{9}$$

for the minimum of the average square (7) of wavefield error.

Equation (8) yields

$$\mathbf{G}^{\mathrm{R}} = \mathrm{Re}(\mathbf{G}^{\mathrm{C}}) \cos(\varphi) + \mathrm{Im}(\mathbf{G}^{\mathrm{C}}) \sin(\varphi) \quad . \tag{10}$$

We insert relation (10) into equation (9) and arrive at

$$\begin{aligned} &\tfrac{2}{3} \operatorname{Tr}\left[\mathrm{Re}(\mathbf{G}^{\mathrm{C}})^{\mathrm{T}}\mathrm{Re}(\mathbf{G}^{\mathrm{C}}) \cos(\varphi) \sin(\varphi) + \mathrm{Im}(\mathbf{G}^{\mathrm{C}})^{\mathrm{T}}\mathrm{Re}(\mathbf{G}^{\mathrm{C}}) \sin(\varphi) \sin(\varphi)\right. \\ &\qquad \left. - \mathrm{Re}(\mathbf{G}^{\mathrm{C}})^{\mathrm{T}}\mathrm{Im}(\mathbf{G}^{\mathrm{C}}) \cos(\varphi) \cos(\varphi) - \mathrm{Im}(\mathbf{G}^{\mathrm{C}})^{\mathrm{T}}\mathrm{Im}(\mathbf{G}^{\mathrm{C}}) \sin(\varphi) \cos(\varphi)\right] = 0 \ , \end{aligned} \tag{11}$$

which yields

$$\operatorname{Tr}\left[\mathrm{Re}(\mathbf{G}^{\mathrm{C}})^{\mathrm{T}}\mathrm{Re}(\mathbf{G}^{\mathrm{C}}) - \mathrm{Im}(\mathbf{G}^{\mathrm{C}})^{\mathrm{T}}\mathrm{Im}(\mathbf{G}^{\mathrm{C}})\right] \sin(2\varphi) = 2 \operatorname{Tr}\left[\mathrm{Re}(\mathbf{G}^{\mathrm{C}})^{\mathrm{T}}\mathrm{Im}(\mathbf{G}^{\mathrm{C}})\right] \cos(2\varphi) . \tag{12}$$

The solution of equation (12) is

$$\tan(2\varphi) = 2 \operatorname{Tr}\left[\mathrm{Re}(\mathbf{G}^{\mathrm{C}})^{\mathrm{T}}\mathrm{Im}(\mathbf{G}^{\mathrm{C}})\right] / \operatorname{Tr}\left[\mathrm{Re}(\mathbf{G}^{\mathrm{C}})^{\mathrm{T}}\mathrm{Re}(\mathbf{G}^{\mathrm{C}}) - \mathrm{Im}(\mathbf{G}^{\mathrm{C}})^{\mathrm{T}}\mathrm{Im}(\mathbf{G}^{\mathrm{C}})\right] . \tag{13}$$

We can thus calculate phase shift φ using relation (13) and insert it into relation (10) to obtain the best real–valued Green–tensor amplitude $\mathbf{G}^{\mathrm{R}}$.

4. Relative wavefield error

For the best real–valued Green–tensor amplitude (10), the average square (7) of wavefield error reads

$$\langle\delta^2\rangle = \tfrac{1}{3}\,\mathrm{Tr}\big[(\mathbf{G}^{\mathrm{C}})^{+}\mathbf{G}^{\mathrm{C}} - (\mathbf{G}^{\mathrm{R}})^{\mathrm{T}}\mathbf{G}^{\mathrm{R}}\big] \quad . \tag{14}$$

The square of the norm of the complex–valued displacement is

$$|\mathbf{u}^{\mathrm{C}}|^2 = \mathbf{f}^{+}(\mathbf{G}^{\mathrm{C}})^{+}\mathbf{G}^{\mathrm{C}}\,\mathbf{f} \quad . \tag{15}$$

We average the square of the norm over all spatial directions of unit seismic force $\mathbf{f}$,

$$\langle|\mathbf{u}^{\mathrm{C}}|^2\rangle = \tfrac{1}{3}\,\mathrm{Tr}\big[(\mathbf{G}^{\mathrm{C}})^{+}\mathbf{G}^{\mathrm{C}}\big] \quad . \tag{16}$$

The relative root mean square error of the approximation of the complex–valued Green–tensor amplitude by the real–valued Green–tensor amplitude is

$$\rho = \sqrt{\langle\delta^2\rangle \,/\, \langle|\mathbf{u}^{\mathrm{C}}|^2\rangle} \quad , \tag{17}$$

where $\langle\delta^2\rangle$ and $\langle|\mathbf{u}^{\mathrm{C}}|^2\rangle$ are given by expressions (14) and (16) with

$$\mathrm{Tr}\big[(\mathbf{G}^{\mathrm{C}})^{+}\mathbf{G}^{\mathrm{C}}\big] = \mathrm{Tr}\big[\mathrm{Re}(\mathbf{G}^{\mathrm{C}})^{\mathrm{T}}\mathrm{Re}(\mathbf{G}^{\mathrm{C}})\big] + \mathrm{Tr}\big[\mathrm{Im}(\mathbf{G}^{\mathrm{C}})^{\mathrm{T}}\mathrm{Im}(\mathbf{G}^{\mathrm{C}})\big] \quad . \tag{18}$$

If relative error (17) is acceptable, the real–valued Green–tensor amplitude can be used in the inversion according to Šílený & Milev (2008), Šílený et al. (2009), and Horálek & Šílený (2013). If relative error (17) is not acceptable, the corresponding data from the receiver should not be used in that inversion.

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