Uniaxial bianisotropic electromagnetic medium with a split intersection slowness-surface singularity

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Summary

In this paper, we find an uniaxial bianisotropic electromagnetic medium with a split intersection slowness–surface singularity.

Keywords

Electromagnetic waves, uniaxial bianisotropic medium, constitutive tensor, slowness–surface singularity.

1. Introduction

Bulant & Klimeš (2014) and Klimeš & Bulant (2014b) demonstrated using an elastic example that the anisotropic–ray–theory rays are not applicable in the vicinity of a split intersection slowness–surface singularity, and that we cannot use the anisotropic ray theory there.

We need the *coupling ray theory* proposed, e.g., by Kravtsov (1968), Naida (1977, 1979) or Fuki, Kravtsov & Naida (1998) for electromagnetic waves, and by Coates & Chapman (1990) for elastic S waves. The frequency–dependent coupling ray theory is the generalization of both the zero–order isotropic and anisotropic ray theories and provides continuous transition between them. The coupling ray theory is applicable to coupled waves at all degrees of anisotropy, from isotropic to considerably anisotropic or bianisotropic media. The numerical algorithm for calculating the frequency–dependent coupling–ray–theory tensor Green function has been designed by Bulant & Klimeš (2002).

Klimeš & Bulant (2014a) and Bulant & Klimeš (2017) demonstrated the accuracy of the wave field calculated by the coupling ray theory in several approximately uniaxial anisotropic elastic media with split intersection slowness–surface singularities.

The question, whether an electromagnetic medium can display a split intersection slowness–surface singularity, thus naturally emerged.

In this paper, we demonstrate that an uniaxial bianisotropic electromagnetic medium may display split intersection slowness–surface singularities.

We assume Cartesian coordinates with the unit metric tensor. The lower–case Greek indices take values 1, 2, 3 and 4. The lower–case Roman indices take values 1, 2 and 3. The Einstein summation over repetitive indices is used throughout the paper.

Seismic Waves in Complex 3–D Structures, **27** (2017), 119–125 (ISSN 2336–3827, online at http://sw3d.cz)

2. Constitutive tensor

In the frequency domain, Maxwell equations (Post, 1962, eq. 6.28; 2003, eq. 26) for 4–vector potential $A_\alpha = A_\alpha(x^m, \omega)$ with linear constitutive relations in the Boys–Post representation without spatial dispersion but with time dispersion (Weiglhofer, 2000, eqs. 1.12–1.13; 2003, eqs. 57–58) read

$$(\chi^{\alpha jk\delta} A_{\delta,k})_{,j} - \mathrm{i}\omega(\chi^{\alpha j4\delta} A_\delta)_{,j} - \mathrm{i}\omega\chi^{\alpha 4k\delta} A_{\delta,k} - \omega^2 \chi^{\alpha 44\delta} A_\delta - J^\alpha = 0 \quad , \tag{1}$$

where $J^i = J^i(x^m, \omega)$ represents the electric current density and $J^4 = J^4(x^m, \omega)$ represents the electric charge density. The $4\times4\times4\times4$ frequency–domain constitutive tensor $\chi^{\alpha\beta\gamma\delta} = \chi^{\alpha\beta\gamma\delta}(x^m, \omega)$ (Post, 1962, eq. 6.12; 2003, eq. 27; Hehl & Obukhov, 2003, eq. D.1.9) is skew with respect to the first pair of indices

$$\chi^{\alpha\beta\gamma\delta} = -\chi^{\beta\alpha\gamma\delta} \quad , \tag{2}$$

and with respect to the second pair of indices

$$\chi^{\alpha\beta\gamma\delta} = -\chi^{\alpha\beta\delta\gamma} \quad , \tag{3}$$

and thus has 36 independent components. Analogously to Voigt notation in elasticity, the constitutive tensor can be expressed as the 6×6 constitutive matrix which lines correspond to the first pair of indices and columns to the second pair of indices.

In this paper, we consider an uniaxial bianisotropic electromagnetic medium (Klimeš, 2017, eq. 48) with symmetry axis x^3, which exhibits natural optical activity (Post, 2003, table 3) in the x^1x^2 plane, but quite opposite optical activity along the x^3 axis. The corresponding $4\times4\times4\times4$ constitutive tensor reads

$$\chi^{\alpha\beta\gamma\delta} = \begin{array}{c} \\ 41 \\ 42 \\ 43 \\ 23 \\ 31 \\ 12 \end{array} \begin{array}{c} \begin{array}{cccccc} 41 & 42 & 43 & 23 & 31 & 12 \end{array} \\ \begin{pmatrix} -\varepsilon & 0 & 0 & \mathrm{i}\gamma & 0 & 0 \\ 0 & -\varepsilon & 0 & 0 & \mathrm{i}\gamma & 0 \\ 0 & 0 & -\varepsilon & 0 & 0 & -\mathrm{i}(3-\delta)\gamma) \\ -\mathrm{i}\gamma & 0 & 0 & \mu^{-1} & 0 & 0 \\ 0 & -\mathrm{i}\gamma & 0 & 0 & \mu^{-1} & 0 \\ 0 & 0 & \mathrm{i}(3+\delta)\gamma & 0 & 0 & \mu^{-1} \end{pmatrix} \end{array} \quad , \tag{4}$$

where ε is the permittivity, μ^{-1} is the inverse permeability, γ is the chirality parameter in the x^1x^2 plane, and δ is a dimensionless parameter. Since we are not interested in a boring isotropic medium, we assume $\gamma \neq 0$ in this paper.

3. Kelvin–Christoffel matrix and the characteristic equation

We define 3×3 *Kelvin–Christoffel matrix* (Klimeš, 2016, eq. 41)

$$\Gamma^{il}(p_n,p_4)=\chi^{i\beta\gamma l}p_\beta p_\gamma \quad . \tag{5}$$

We define functions $p_4=p_4(p_n)$ as the solutions of characteristic equation (Klimeš, 2016, eq. 51)

$$\det\left[\Gamma^{ad}(p_n,p_4)\right]=0 \tag{6}$$

for given slowness vector p_n. Functions $p_4=p_4(p_n)$ are homogeneous functions of the first degree with respect to slowness vector p_n.

We express the dependence of the Kelvin–Christoffel matrix on p_4 as (Klimeš, 2016, eq. 56)

$$\Gamma^{ad}(p_n,p_4)=(p_4)^2\Gamma_0^{ad}+p_4\,\Gamma_1^{ad}(p_n)+\Gamma_2^{ad}(p_n) \quad , \tag{7}$$

where (Klimeš, 2016, eqs. 53–55)

$$\Gamma_0^{ad}=\chi^{a44d} \quad , \tag{8}$$

$$\Gamma_1^{ad}(p_n)=\left(\chi^{a4cd}+\chi^{ac4d}\right)p_c \tag{9}$$

and

$$\Gamma_2^{ad}(p_n)=\chi^{abcd}\,p_b\,p_c \quad . \tag{10}$$

Since the slowness surface defined by characteristic equation (6) with constitutive tensor (4) is rotationally invariant with respect to the p_3 axis, we shall study it in the p_1p_3 plane. Hereinafter, we thus put

$$p_2=0 \quad . \tag{11}$$

With constitutive tensor (4), matrices (8)–(10) read

$$\Gamma_0^{ad}=\varepsilon\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} \quad , \tag{12}$$

$$\Gamma_1^{ad}(p_n)=\mathrm{i}\gamma\begin{pmatrix}0&2p_3&0\\-2p_3&0&-(2+\delta)p_1\\0&(2-\delta)p_1&0\end{pmatrix} \tag{13}$$

and

$$\Gamma_2^{ad}(p_n)=\mu^{-1}\begin{pmatrix}-p_3^2&0&p_1p_3\\0&-p_1^2-p_3^2&0\\p_1p_3&0&-p_1^2\end{pmatrix} \quad . \tag{14}$$

We define the matrices (Klimeš, 2016, eq. 57)

$$\overline{\Gamma}_{\bullet\,ai}=\tfrac{1}{2}\,\varepsilon_{abc}\,\varepsilon_{ijk}\,\Gamma_\bullet^{bj}\,\Gamma_\bullet^{ck} \tag{15}$$

of cofactors of given 3×3 matrices $\Gamma_\bullet^{ij}$, where $\bullet=0,1,2$.

Inserting matrices (12)–(14) into definition (15), we obtain the matrices of cofactors

$$\overline{\Gamma}_{0ad}=\varepsilon^2\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} \quad , \tag{16}$$

$$\overline{\Gamma}_{1ad}(p_n)=-\gamma^2\begin{pmatrix}(4-\delta^2)p_1^2&0&-2(2-\delta)p_1p_3\\0&0&0\\-2(2+\delta)p_1p_3&0&4p_3^2\end{pmatrix} \tag{17}$$

and

$$\overline{\Gamma}_{2ad}(p_n) = (\mu^{-1})^2(p_1^2+p_3^2)\begin{pmatrix} p_1^2 & 0 & p_1p_3 \\ 0 & 0 & 0 \\ p_1p_3 & 0 & p_3^2 \end{pmatrix} \quad . \tag{18}$$

The determinant (Klimeš, 2016, eq. 52)

$$\det\big[\Gamma^{ad}(p_n,p_4)\big] = \tfrac{1}{6}\varepsilon_{abc}\varepsilon_{ijk}\Gamma^{ai}(p_n,p_4)\Gamma^{bj}(p_n,p_4)\Gamma^{ck}(p_n,p_4) \tag{19}$$

of 3×3 Kelvin–Christoffel matrix (7) reads (Klimeš, 2016, eq. 58)

$$\begin{aligned}\det\big[\Gamma^{ad}(p_n,p_4)\big] &= (p_4)^6\Gamma_0 + (p_4)^5\Gamma_1(p_n) + (p_4)^4\Gamma_2(p_n) \\ &\quad + (p_4)^3\Gamma_3(p_n) + (p_4)^2\Gamma_4(p_n) + p_4\,\Gamma_5(p_n) + \Gamma_6(p_n) \quad ,\end{aligned} \tag{20}$$

with coefficients (Klimeš, 2016, eqs. 59–65)

$$\Gamma_0 = \det\big(\Gamma_0^{ad}\big) \quad , \tag{21}$$

$$\Gamma_1 = \overline{\Gamma}_{0\,rs}\Gamma_1^{rs} \quad , \tag{22}$$

$$\Gamma_2 = \Gamma_0^{rs}\overline{\Gamma}_{1\,rs} + \overline{\Gamma}_{0\,rs}\Gamma_2^{rs} \quad , \tag{23}$$

$$\Gamma_3 = \varepsilon_{abc}\varepsilon_{ijk}\Gamma_0^{ai}\Gamma_1^{bj}\Gamma_2^{ck} + \det\big(\Gamma_1^{ad}\big) \quad , \tag{24}$$

$$\Gamma_4 = \Gamma_0^{rs}\overline{\Gamma}_{2\,rs} + \overline{\Gamma}_{1\,rs}\Gamma_2^{rs} \quad , \tag{25}$$

$$\Gamma_5 = \Gamma_1^{rs}\overline{\Gamma}_{2\,rs} \tag{26}$$

and

$$\Gamma_6 = \det\big(\Gamma_2^{ad}\big) \quad . \tag{27}$$

Since (Klimeš, 2016, eq. 69)

$$\Gamma_5 = 0 \tag{28}$$

and (Klimeš, 2016, eq. 70)

$$\Gamma_6 = 0 \quad , \tag{29}$$

the characteristic equation (6) reduces to fourth–order polynomial equation (Klimeš, 2016, eq. 71)

$$(p_4)^4\Gamma_0 + (p_4)^3\Gamma_1(p_n) + (p_4)^2\Gamma_2(p_n) + p_4\,\Gamma_3(p_n) + \Gamma_4(p_n) = 0 \tag{30}$$

for p_4.

Coefficient (21) with matrix (12) reads

$$\Gamma_0 = \varepsilon^3 \quad . \tag{31}$$

Coefficient (22) with matrices (13) and (16) reads

$$\Gamma_1 = 0 \quad . \tag{32}$$

Coefficient (23) with matrices (12), (14), (16) and (17) reads

$$\Gamma_2 = -\varepsilon\gamma^2[(4-\delta^2)p_1^2 + 4p_3^2] - 2\varepsilon^2\mu^{-1}(p_1^2+p_3^2) \quad . \tag{33}$$

We sort the terms with p_1^2 and p_3^2,

$$\Gamma_2 = -\varepsilon\big[(4-\delta^2)\gamma^2p_1^2 + 2\varepsilon\mu^{-1}\big]p_1^2 - \varepsilon\big[4\gamma^2 + 2\varepsilon\mu^{-1}\big]p_3^2 \quad . \tag{34}$$

Coefficient (24) with matrices (12)–(14) reads

$$\Gamma_3 = 0 \quad . \tag{35}$$

Coefficient (25) with matrices (12), (14), (17) and (18) reads

$$\Gamma_4 = \varepsilon(\mu^{-1})^2(p_1^2 + p_3^2)^2 + \mu^{-1}\gamma^2(16-\delta^2)p_1^2p_3^2 \quad . \tag{36}$$

Since $\Gamma_1 = 0$ and $\Gamma_3 = 0$, polynomial equation (30) for p_4 reads

$$(p_4)^4\Gamma_0 + (p_4)^2\Gamma_2(p_n) + \Gamma_4(p_n) = 0 \quad . \tag{37}$$

It has four solutions. Two solutions with negative real parts read

$$p_4 = -\sqrt{-\frac{1}{2}\frac{\Gamma_2}{\Gamma_0} \pm \frac{1}{2\Gamma_0}\sqrt{(\Gamma_2)^2 - 4\Gamma_0\Gamma_4}} \quad . \tag{38}$$

Since these two solutions must equal -1 (Klimeš, 2016, eqs. 74),

$$p_4(p_n) = -1 \quad , \tag{39}$$

we obtain two equations

$$\Gamma_0 = -\tfrac{1}{2}\Gamma_2(p_n) \pm \tfrac{1}{2}\sqrt{[\Gamma_2(p_n)]^2 - 4\Gamma_0\,\Gamma_4(p_n)} \tag{40}$$

for p_1 and p_3 describing two sheets of the slowness surface.

Coefficient Γ_2 is a linear function of p_1^2 and p_3^2. Discriminant $(\Gamma_2)^2 - 4\Gamma_0\Gamma_4$ is a quadratic function of p_1^2 and p_3^2. We express it as

$$\Gamma_2^2 - 4\Gamma_0\Gamma_4 = Ap_1^4 - 2Bp_1^2p_3^2 + Cp_3^4 \quad , \tag{41}$$

and calculate coefficients A, B and C using relations (31), (34) and (36):

$$A = \varepsilon^2\big[(4-\delta^2)\gamma^2 + 2\varepsilon\mu^{-1}\big]^2 - 4\varepsilon^4(\mu^{-1})^2 \quad , \tag{42}$$

$$A = \varepsilon^2\gamma^2(4-\delta^2)\big[(4-\delta^2)\gamma^2 + 4\varepsilon\mu^{-1}\big] \quad , \tag{43}$$

$$B = -\varepsilon^2[(4-\delta^2)\gamma^2 + 2\varepsilon\mu^{-1}][4\gamma^2 + 2\varepsilon\mu^{-1}] + 2\varepsilon^3\mu^{-1}[2\varepsilon\mu^{-1} + (16-\delta^2)\gamma^2] \quad , \tag{44}$$

$$B = 4\varepsilon^2\gamma^2\big[-(4-\delta^2)\gamma^2 + 4\varepsilon\mu^{-1}\big] \quad , \tag{45}$$

$$C = \varepsilon^2[4\gamma^2 + 2\varepsilon\mu^{-1}]^2 - 4\varepsilon^4(\mu^{-1})^2 \quad , \tag{46}$$

$$C = 16\varepsilon^2\gamma^2(\gamma^2 + \varepsilon\mu^{-1}) \quad . \tag{47}$$

We look for the intersection singularity where $\sqrt{(\Gamma_2)^2 - 4\Gamma_0\Gamma_4}$ is a linear function of p_1^2 and p_3^2. This may happen only if

$$B^2 - AC = 0 \quad . \tag{48}$$

We thus calculate expression

$$B^2 - AC = 16\varepsilon^4\gamma^4\Big\{\big[(4-\delta^2)\gamma^2 - 4\varepsilon\mu^{-1}\big]^2 - (4-\delta^2)\big[(4-\delta^2)\gamma^2 + 4\varepsilon\mu^{-1}\big]\big[\gamma^2 + \varepsilon\mu^{-1}\big]\Big\}\,, \tag{49}$$

which may be simplified to

$$B^2 - AC = 16\varepsilon^5\mu^{-1}\gamma^4\Big\{-(4-\delta^2)(16-\delta^2)\gamma^2 + 4\delta^2\varepsilon\mu^{-1}\Big\} \quad . \tag{50}$$

Considering condition (48) with expression (50), we obtain quadratic equation

$$\delta^4\gamma^2 - \delta^2\,4(\varepsilon\mu^{-1} - 5\gamma^2) + 64\gamma^2 = 0 \tag{51}$$

for δ^2 with a small coefficient at δ^4, which means that one root δ^2 is small and other root δ^2 is large. The smaller one of two roots of this quadratic equation reads

$$\delta^2 = \frac{32\gamma^2}{\varepsilon\mu^{-1} - 5\gamma^2 + \sqrt{(\varepsilon\mu^{-1} - 5\gamma^2)^2 - 16\gamma^4}} \quad , \tag{52}$$

i.e.,

$$\delta = \pm 4 \sqrt{\frac{2\gamma^2}{\varepsilon\mu^{-1} - 5\gamma^2 + \sqrt{(\varepsilon\mu^{-1} - 5\gamma^2)^2 - 16\gamma^4}}} \quad , \tag{53}$$

which may be, for small γ^2 in comparison with $\varepsilon\mu^{-1}$, approximated by

$$\delta \approx \pm 4 \sqrt{\frac{\gamma^2}{\varepsilon\mu^{-1} - 5\gamma^2}} \quad . \tag{54}$$

For δ given by (53), $\sqrt{(\Gamma_2(p_n))^2 - 4\Gamma_0(p_n)\Gamma_4(p_n)}$ is a linear function of p_1^2 and p_3^2, and two equations (40) specify two rotationally invariant ellipsoids intersecting at two line intersection singularities. For δ between values (53), discriminant (41) is always positive and the slower slowness sheet separates from the faster slowness sheet, forming smooth but very sharp edges on both sheets. The behaviour of the slowness surface becomes complex outside the interval given by values (53).

4. Conclusions

For small dimensionless parameter δ given by relation (53), the slowness surface of a medium with constitutive tensor (4) is composed of two rotationally invariant ellipsoids intersecting at two line intersection singularities. For other values of parameter δ, e.g., for $\delta = 0$, the intersection singularity is split and the slower slowness sheet separates from the faster slowness sheet, forming smooth but very sharp edges on both sheets.

When the slowness vector of a ray smoothly passes through a split intersection singularity, the ray–velocity vector rapidly changes its direction and creates a sharp bend on the ray, see Klimeš & Bulant (2014b, figs. 1–4). This sharp bend is connected with a rapid rotation of the eigenvectors of the Kelvin–Christoffel matrix. The sharply bent rays thus cannot describe the correct wave propagation and indicate a failure of the anisotropic ray theory. The actual electromagnetic waves do not propagate along the sharply bent rays, but tunnel smoothly through a split intersection singularity.

In the vicinity of a split intersection singularity where the anisotropic ray theory fails, electromagnetic waves can be calculated by means of the coupling ray theory (Kravtsov, 1968; Naida, 1977; 1979; Fuki, Kravtsov & Naida, 1998; Bulant & Klimeš; 2002; Klimeš & Bulant, 2016). However, the coupling ray theory cannot be applied to the sharply bent anisotropic–ray–theory reference rays in this case. It has to be applied to the anisotropic common reference rays (Klimeš, 2016), or to some other more accurate reference rays.

Acknowledgements

The research has been supported by the Grant Agency of the Czech Republic under contract 16-01312S, and by the members of the consortium "Seismic Waves in Complex 3+D Structures" (see "http://sw3d.cz").

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