Polarization, phase velocity and NMO velocity of qP waves in arbitrary weakly anisotropic media

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ABSTRACT

Approximate formulae for the qP wave phase velocity, polarization vector and normal moveout velocity in an arbitrary weakly anisotropic medium obtained with the first order perturbation theory are presented. All the mentioned quantities are expressed in terms of weak anisotropy (WA) parameters, which represent a natural generalization of parameters introduced by Thomsen (1986). The presented formulae and the WA parameters have properties of Thomsen's (1986) formulae and parameters: (i) the approximate equations are considerably simpler than exact equations for qP waves; (ii) the WA parameters are non-dimensional quantities; (iii) in isotropic media, the WA parameters are zero and the corresponding equations reduce to equations for isotropic media. In contrast to Thomsen's (1986) parameters, the WA parameters are linearly related to the density normalized elastic parameters. For the transversely isotropic media with vertical axis of symmetry, the presented equations and the WA parameters reduce to equations and linearized parameters of Thomsen (1986). Accuracy of presented formulae is tested on two examples of anisotropic media with relatively strong anisotropy: on a transversely isotropic medium with the horizontal axis of symmetry and on a medium with triclinic anisotropy. Although anisotropy is rather strong, the presented approximate formulae yield satisfactory results.

INTRODUCTION

On the basis of observation that most of the anisotropic media are weakly anisotropic, Thomsen (1986) proposed to substitute complicated nontransparent formulae, which describe wave propagation in media with an arbitrarily strong anisotropy by simple equations for weak anisotropy, which are easier to understand and deal with. Thomsen concentrated on the transversely isotropic (TI) media with vertical axis of symmetry (VTI media). Backus (1965) found such simple equations for phase velocities in arbitrary weakly anisotropic media, Recently, study of the phase velocities in arbitrary weakly anisotropic media attracted again attention of several authors. For example Sayers (1994) used an expansion into spherical harmonics to obtain the approximate relations for the phase velocity of the qP wave. Mensch and Rasolofosaon (1997) linearized exact equations for the phase velocities of the qP and qS waves to get approximate equations. Tsvankin (1997) obtained the approximate equation for the phase velocity of qP waves in orthorhombic media by linearizing the corresponding exact equations. As shown by Backus (1965), the approximate formulae for the phase velocity in weakly anisotropic media follow simply from the first-order perturbation theory for anisotropic media, see also Červený and Jech (1982), Jech and Pšenčík (1989), Gajewski and Pšenčík (1996), Pšenčík and Gajewski (1996). In addition, the first-order perturbation theory yields also expressions for the polarization vectors.

In this paper, we are using the perturbation formulae of Jech and Pšenčík (1989), specified for the isotropic background, to derive approximate expressions for the qP wave phase velocity, the polarization vector and the normal moveout velocity in arbitrary weakly anisotropic media. The formulae for the approximate computation of the qP wave polarization vector and the NMO velocity in arbitrary weakly anisotropic media represent an original contribution of this paper. Although some of the approximate formulae for the qP wave phase velocities have already been presented elsewhere, here they are presented in order to show their relation to the formulae for the polarization vector and the NMO velocity. It is shown that the polarization vector is controlled by the same set of parameters, called *weak anisotropy (WA) parameters*, as the corresponding phase velocity. The NMO velocity is controlled only by some of the WA parameters. For media with higher-order symmetries, the formula for the NMO velocity presented in this paper reduces to formulae of Sayers (1995), Tsvankin (1997).

The WA parameters are linear functions of elastic parameters. This distinguishes them from the parameters introduced by Thomsen (1986). In weakly anisotropic media studied in this paper, however, both types of parameters are equivalent since the non-linear terms in Thomsen's (1986) parameters are negligible second-order quantities.

All the formulae in this paper are derived for the phase normal in a weakly anisotropic medium coinciding with the phase normal in the background isotropic medium. This means that the corresponding ray direction in the anisotropic medium differs from the ray direction in the isotropic medium. In some applications it is desirable to have the ray directions in both media the same. In such cases, the phase normal used in this study must be substituted by the phase normal which yields the ray direction in the anisotropic medium coinciding with the ray direction in the isotropic medium. An approximate linear relation between the phase normal and the ray direction can be found in Eq.(21) of Pšenčík (1996).

In order to illustrate the accuracy of some of the presented approximate formulae we show maps of relative errors of the approximately determined phase velocity and of maps deviations of the approximately and exactly determined polarization vectors in a TI medium with the horizontal axis of symmetry (HTI medium) and in a triclinic (TRI) medium. Although the considered anisotropy can hardly be called weak, the approximate formulae yield satisfactory results in both cases.

In the following, all Roman lowercase indices range over the values 1, 2 and 3, uppercase indices over the values 1 and 2. The Greek indices run from 1 to 6. Einstein summation convention is used for the repeated indices. Voigt notation $A_{\alpha\beta}$ for density normalized elastic parameters, with α, β running from 1 to 6, is used in parallel with the tensor notation a_{ijkl}.

FORMULAE FOR QP WAVE PHASE VELOCITY AND POLARIZATION VECTOR IN WEAKLY ANISOTROPIC MEDIA

Let us consider a tensor of the density normalized elastic parameters a_{ijkl}, which describes a weakly anisotropic medium. Let us express the tensor a_{ijkl} in the following way

$$a_{ijkl} = a^0_{ijkl} + \Delta a_{ijkl}, \tag{1}$$

where a^0_{ijkl} is a tensor of density normalized elastic parameters in a background unperturbed medium and Δa_{ijkl} is its small perturbation, $|\Delta a_{ijkl}| \ll ||a^0_{ijkl}||$. The norm $||.||$ can be understood, for example, as $||a^0_{ijkl}|| = max|a^0_{ijkl}|$. How to understand the inequality is shown on a few examples in the section with numerical examples. Since we are considering a weak anisotropy, the background unperturbed medium is isotropic and thus a^0_{ijkl} reads

$$a^0_{ijkl} = (\alpha^2 - 2\beta^2)\delta_{ij}\delta_{kl} + \beta^2(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}). \tag{2}$$

In (2), α and β denote P and S wave velocities in the unperturbed isotropic medium. The above perturbation of elastic properties of the medium produces perturbations in quantities describing wave propagation through the medium. For a fixed phase normal direction, i.e. for the wavefield whose phase normal in the perturbed weakly anisotropic medium is the same as in the unperturbed isotropic medium, the equations given by Jech & Pšenčík (1989) reduce to the following formulae. A perturbation Δc of the qP wave phase velocity c in a general, weakly anisotropic medium is given by

$$\Delta c = \frac{1}{2\alpha} B_{33}. \tag{3}$$

The perturbation Δg_i of the P wave polarization vector e_{3i} specified in the unperturbed isotropic medium is given by

$$\Delta g_i = \frac{B_{13}e_{1i} + B_{23}e_{2i}}{\alpha^2 - \beta^2}. \tag{4}$$

Here

$$B_{mn} = \Delta\Gamma_{jk} e_{mj} e_{nk} \quad . \tag{5}$$

The symbol $\Delta\Gamma_{jk}$ represents a perturbation of the Christoffel matrix

$$\Gamma_{jk} = a_{ijkl} n_i n_l. \tag{6}$$

The symbols e_{1i}, e_{2i} and e_{3i} denote three mutually perpendicular unit vectors in the unperturbed isotropic medium. The P wave polarization vector e_{3i} is parallel to the unit phase normal n_i, i.e., $e_{3i} = n_i$ since in an isotropic medium phase normal and polarization vector of a P wave are identical. The vectors e_{1i} and e_{2i} can be arbitrarily chosen in the plane perpendicular to n_i. The most practical choice of vectors e_{1i}, e_{2i} expressed in terms of components of the vector e_{3i} seems to be the following one:

$$\vec{e}_1 = D^{-1}(n_1 n_3, n_2 n_3, n_3^2 - 1), \quad \vec{e}_2 = D^{-1}(-n_2, n_1, 0), \quad \vec{e}_3 = \vec{n} = (n_1, n_2, n_3), \tag{7}$$

where

$$D = (n_1^2 + n_2^2)^{1/2}, \quad n_1^2 + n_2^2 + n_3^2 = 1. \tag{8}$$

Let us specify the phase normal as

$$\vec{n} = (\cos\varphi \sin\theta, \sin\varphi \sin\theta, \cos\theta). \tag{9}$$

Here φ denotes an azimuthal angle, θ a polar angle, $(0 \leq \varphi \leq 2\pi,\ 0 \leq \theta \leq \pi)$. In this case we have $D = \sin\theta$ and vectors $\vec{e}_1$ and $\vec{e}_2$ from Eq.(7) can be rewritten in the form

$$\vec{e}_1 = (\cos\varphi \cos\theta, \sin\varphi \cos\theta, -\sin\theta), \quad \vec{e}_2 = (-\sin\varphi, \cos\varphi, 0), \tag{10}$$

For the sake of simplicity, we use in most of the text the specification (7) in terms of the components of the phase normal. Only in final formulae, we use the expressions in terms of angles θ and φ, see Eqs.(9) and (10).

Taking into account (2), Eq.(5) defining the matrix B_{mn} can be rewritten in the following way

$$B_{mn} = (a_{ijkl} - a^0_{ijkl}) n_i n_l e_{mj} e_{nk} = a_{ijkl} n_i n_l e_{mj} e_{nk} - c_0^2 \delta_{mn}, \tag{11}$$

Here c_0 denotes a phase velocity in the unperturbed medium. Since here we consider unperturbed medium isotropic, for $m = n = 3$, we have $c_0 = \alpha$ and for $m = n = 1$ or 2, we have $c_0 = \beta$. Note that in Eq.(11), the elements of the matrix B_{mn} are expressed in terms of elastic parameters of a weakly anisotropic medium instead of their perturbations.

For B_{33}, Eq.(11) yields

$$B_{33} = a_{ijkl} n_i n_l n_j n_k - \alpha^2. \tag{12}$$

Combining Eqs.(3) and (12), we can write the approximate formula for the qP wave phase velocity in a weakly anisotropic medium

$$c(n_m) = \alpha[1 + \frac{(a_{ijkl} - \alpha^2 \delta_{ij} \delta_{kl}) n_i n_j n_k n_l}{2\alpha^2}]. \tag{13}$$

This is a basic equation for the approximate evaluation of the phase velocity in an arbitrary weakly anisotropic medium. This simple and transparent equation is an equivalent of the combination of equations (19), (21) and Table 1 of Mensch and Rasolofosaon (1997). As it is shown later, the coefficients $\frac{a_{ijkl}-\alpha^2\delta_{ij}\delta_{kl}}{2\alpha^2}$ are the WA parameters, which represent a generalization of Thomsen's (1986) parameters.

If Eq.(13) is squared and terms of orders higher than first neglected, the approximate formula for the square of the qP wave phase velocity c^2 in a weakly anisotropic medium has the form presented by Backus (1965), see his equations (16) and (17),

$$c^2(n_m) = \alpha^2[1 + \frac{(a_{ijkl} - \alpha^2\delta_{ij}\delta_{kl})n_i n_j n_k n_l}{\alpha^2}] = a_{ijkl}n_i n_l n_j n_k. \tag{14}$$

As we can see, the approximation of c^2 in weakly anisotropic media is independent of the choice of the constant α, i.e. of the specification of the background medium. As shown in the Appendix B, c^2 is always less or, at most, equal to the square of the exact phase velocity. Equality occurs only in longitudinal directions, i.e. directions, where pure modes propagate, see e.g. Helbig (1993).

The dependence of Eq.(13) on α may be useful in applications, in which the propagation is prevailingly unidirectional like, e.g., in reflection seismics. By a proper choice of α, Eq.(13) can be "tuned" to give very precise results of the phase velocity for a selected direction of propagation. The P wave velocity α can be chosen in various ways, see also discussion of this topic in Mensch and Rasolofosaon (1997). In reflection seismics, where prevailing direction of propagation is vertical and the vertical velocity can be easily determined, it is reasonable to take $\alpha^2 = A_{33}$, where $A_{33} = a_{3333}$. This was the choice of α made by Thomsen (1986). For cross-hole experiments, it is more convenient to choose α as a horizontal qP wave phase velocity, e.g. as $\sqrt{A_{11}}$ or $\sqrt{A_{22}}$ or their average. In refraction or wide angle reflection studies, where many propagation directions are encountered, and thus an approximation giving good results in all directions is sought, α can be chosen as

$$\alpha^2 = \frac{1}{15}(a_{iikk} + 2a_{ikik}). \tag{15a}$$

This choice follows from the requirement that the global difference between considered weakly anisotropic medium and background isotropic medium is minimum, see Fedorov (1968).

By inspecting Eq.(13), we can find that, for

$$\alpha^2 = a_{ijkl}n_i n_l n_j n_k, \tag{15b}$$

Eq.(13) yields the minimum value of the approximate phase velocity, which is equal to the value following from Eq.(14). Thus Eq.(14) yields, for any direction of the phase normal, a minimum possible value of the qP wave phase velocity.

In longitudinal directions (see Helbig, 1993), the values of the approximate phase velocity obtained from Eq.(13) generally differ from exact ones, c_{ex}. In fact, Eq.(13) yields $c = (\alpha^2 + c_{ex}^2)/2\alpha$ in these cases. Thus, both approximate and exact velocities coincide only if α is chosen as $\alpha = c_{ex}$.

The perturbation of the polarization vector is controlled by the off-diagonal elements of the matrix B_{mn}, see Eq.(4). Using Eq.(4), we can write the approximate formula for the qP wave polarization vector g_i in an arbitrary weakly anisotropic medium in the form

$$g_i(n_m) = n_i + \frac{B_{13}e_{1i} + B_{23}e_{2i}}{\alpha^2 - \beta^2}. \tag{16}$$

Let us note that it is not clear from Eq.(16) how to choose the parameters α and β of the unperturbed medium so that Eq.(16) yields a best approximation of the exact polarization vector. This best choice can be found by inspecting the higher-order terms neglected by Jech and Pšenčík (1989) during their derivation of Eq.(4). The best choice of the parameters α and β is such in which the term $\alpha^2 - \beta^2$ is equal to the difference between the squares of the approximate qP wave and $qS1$ (or $qS2$) wave phase velocities for the given phase normal direction. In forward modeling, in which the approximate phase velocities are directly available, this can lead to highly accurate determination of the polarization vector, see the numerical examples below. In inverse problems, estimates of the phase velocities derived from the travel-time studies can be used for this purpose.

The above formula for the polarization vector is *local.* In contrast to formulae of, for example, Farra and Le Bégat (1995), it does not require integration along an unperturbed ray path.

DEPENDENCE OF PHASE VELOCITY, POLARIZATION VECTOR AND NORMAL MOVEOUT ON WA PARAMETERS

In the following, we derive relations expressing the phase velocity, polarization vector and normal moveout velocity in terms of elastic parameters of weakly anisotropic media of various symmetries. Since the relations are linear, they can be easily used to investigate sensitivity of studied quantities to individual elastic parameters.

Phase velocity

Within the first-order perturbation theory, the phase velocity perturbation equals the ray velocity perturbation (Backus, 1965). The phase and ray velocities in weakly anisotropic media are thus equal and it is sufficient to investigate only one of them.

General anisotropy.—Using Voigt's notation, we can rewrite Eq.(13) in a slightly modified notation of Mensch and Rasolofosaon (1997) into the form

$$\begin{aligned} c(n_m) = \alpha[1 &+ \epsilon_z n_3^4 + 2n_3^3(\epsilon_{34}n_2 + \epsilon_{35}n_1) + n_3^2(\delta_x n_1^2 + \delta_y n_2^2 + 2\chi_z n_1 n_2) \\ &+2n_3(\chi_x n_1^2 n_2 + \chi_y n_1 n_2^2 + \epsilon_{15}n_1^3 + \epsilon_{24}n_2^3) + \epsilon_x n_1^4 + \delta_z n_1^2 n_2^2 \\ &+\epsilon_y n_2^4 + 2\epsilon_{16}n_1^3 n_2 + 2\epsilon_{26}n_1 n_2^3]. \end{aligned} \tag{17a}$$

Here

$$\epsilon_x = \frac{A_{11} - \alpha^2}{2\alpha^2}, \quad \epsilon_y = \frac{A_{22} - \alpha^2}{2\alpha^2}, \quad \epsilon_z = \frac{A_{33} - \alpha^2}{2\alpha^2},$$

$$\delta_x = \frac{A_{13} + 2A_{55} - \alpha^2}{\alpha^2}, \quad \delta_y = \frac{A_{23} + 2A_{44} - \alpha^2}{\alpha^2}, \quad \delta_z = \frac{A_{12} + 2A_{66} - \alpha^2}{\alpha^2},$$

$$\chi_x = \frac{A_{14} + 2A_{56}}{\alpha^2}, \quad \chi_y = \frac{A_{25} + 2A_{46}}{\alpha^2}, \quad \chi_z = \frac{A_{36} + 2A_{45}}{\alpha^2},$$

$$\epsilon_{15} = \frac{A_{15}}{\alpha^2}, \quad \epsilon_{16} = \frac{A_{16}}{\alpha^2}, \quad \epsilon_{24} = \frac{A_{24}}{\alpha^2}, \quad \epsilon_{26} = \frac{A_{26}}{\alpha^2}, \quad \epsilon_{34} = \frac{A_{34}}{\alpha^2}, \quad \epsilon_{35} = \frac{A_{35}}{\alpha^2} \qquad (17b)$$

are the weak anisotropy parameters or briefly the WA parameters.

Equation (17a) is a generalization of Thomsen's (1986) equation for qP wave phase velocity for VTI media to arbitrary weakly anisotropic media. It has all the properties of Thomsen's (1986) equation: (i) it considerably simplifies the exact equation for the qP wave phase velocity; (ii) the coefficients of various combinations of components of the phase normal n_i, i.e. the WA parameters, are non-dimensional quantities; (iii) in the case of isotropy, all of them reduce to zero. In contrast to Thomsen's parameters, the relation of all WA parameters to elastic parameters $A_{\alpha\beta}$ is linear. In the case of a complete set of WA parameters, for whose determination also qS wave phase velocities are necessary, the last property offers an easy transformation from one set of parameters to the other.

From Eq.(17a), we can see, first of all, that the qP wave phase velocity is specified by 15 WA parameters. This is in agreement with conclusions of e.g. Klíma (1973), Every and Sachse (1992), Sayers (1994), Mensch and Rasolofosaon (1997). It means that the determination of all 21 elastic parameters from only qP wave phase velocities or qP wave travel time data in weakly anisotropic media is impossible. Only the above mentioned 15 WA parameters could be determined if a good angular coverage is available. In order to determine the remaining elastic parameters, qS wave data have to be used. The influence of the limit of 15 recoverable WA parameters of qP waves can be observed even in the case of strong anisotropy. An experiment performed by Jech (1991) implies that even for a strong anisotropy, the elastic parameters appearing in combinations in Eq.(17a) do not separate well. Let us note that the number of WA parameters can be reduced by one if α is chosen so that one of the parameters in (17b) vanishes.

Further, we can see from Eq.(17a) that in a single plane, e.g. in the (x, z) plane, i.e. for n_2=0, we are able to recover even less number of WA parameters, specifically the following five: ϵ_x, ϵ_z, δ_x, ϵ_{15} and ϵ_{35}. This is in agreement with conclusions of Chapman and Pratt (1992). In the plane (y, z), i.e. for $n_1 = 0$, further four WA parameters ϵ_y, δ_y, ϵ_{24} and ϵ_{34}, in addition to the already determined ϵ_z, can be found. From measurements in two more vertical planes, different from the previous two planes, the parameters χ_x, χ_y, χ_z and two combinations of the parameters δ_z, ϵ_{16} and ϵ_{26} can be obtained. To resolve the latter combination, measurements either in the plane (x, y) or in a fifth vertical plane are necessary. The same conclusion was made by Simões-Filho (pers. com.).

In seismic reflection methods, in which the prevailingly vertical wave propagation is considered, it is convenient to choose $\alpha^2 = A_{33}$ as discussed above. Since the terms in Eq.(17a) are ordered according to decreasing power of n_3, Eq.(17a) clearly shows, which parameters control this propagation. In this case the component n_3 of the phase normal is nearly unit while the other two components are small. This means that the nearly vertical propagation will be mostly controlled by the WA parameters ϵ_{34} and ϵ_{35} and by the parameters δ_x, δ_y and χ_z. Note that the parameters ϵ_{34} and ϵ_{35} become zero if the vertical axis represents a longitudinal direction, see Fedorov (1968), Norris (1989), Sayers (1994).

In the following, we are considering several special cases of weakly anisotropic media with a higher symmetry, which are of interest in seismological applications. Most of the following formulae have already been presented by other authors, see Thomsen (1986), Sayers (1994), Gajewski and Pšenčík (1996), Pšenčík and Gajewski (1996), Mensch and Rasolofosaon (1997), Tsvankin (1997). We present them here for completness.

Orthorhombic medium.—Let us consider an orthorhombic medium and let us consider the phase normal specified in a "crystal" coordinate system with its axes coinciding with the axes of symmetry. We denote the components of the phase normal in this system by n'_i in order to distinguish them from the components n_i in the general coordinate system. The components n'_i can be expressed in terms of the components n_i in the general coordinate system as

$$n'_i = R_{ij} n_j \quad , \tag{18}$$

where R_{ij} is the transformation matrix from the general to the crystal coordinate system. The matrix R_{ij} depends generally on three Euler angles. They represent additional parameters specifying the phase velocity in a general coordinate system, see Norris (1989). In the crystal coordinate system in an orthorhombic medium, Eq.(17a) yields

$$c(n_m) = \alpha[1 + \epsilon_z n_3'^4 + (\delta_x n_1'^2 + \delta_y n_2'^2) n_3'^2 + \epsilon_x n_1'^4 + \epsilon_y n_2'^4 + \delta_z n_1'^2 n_2'^2]. \tag{19}$$

If we put $\alpha^2 = A_{33}$, Eq.(19) is identical with Eq.(48) of Tsvankin (1997). The phase velocity is controlled by 6 WA parameters, ϵ_x, ϵ_y, ϵ_z, δ_x, δ_y and δ_z and by three Euler angles. Since in the crystal coordinate system the orthorhombic medium is described by 9 independent elastic parameters, three remaining elastic parameters or their combinations must be determined from the observations of qS waves.

In the crystal coordinate system, the following conclusions can be made. In each of the planes (x, z) and (y, z), only three WA parameters ϵ_x, ϵ_z, δ_x and ϵ_y, ϵ_z, δ_y respectively, can be determined from Eq.(19). The remaining WA parameter, δ_z can be determined from measurements either in the plane (x, y) or in a vertical plane, different from (x, z) and (y, z) planes. Thus, in this case, measurements in only three vertical planes are sufficient.

Specification of Eq.(19) for any symmetry plane leads to equations with exactly same form as equations in symmetry planes of TI media, see the next section. This only confirms earlier observations of Cheadle et al. (1991).

Prevailingly vertical propagation, for which we specify again $\alpha^2 = A_{33}$, is first of all controlled by two WA parameters δ_x and δ_y.

TI medium.—Let us consider a TI medium with elastic parameters in the crystal coordinate system satisfying the conditions:

$$A_{22} = A_{11}, \quad A_{55} = A_{44}, \quad A_{23} = A_{13}, \quad A_{12} = A_{11} - 2A_{66}, \tag{20}$$

i.e.

$$\epsilon_x = \epsilon_y = \frac{1}{2}\delta_z, \quad \delta_x = \delta_y. \tag{21}$$

This means that the axis of symmetry is along the x'_3 axis of the crystal coordinate system. In this case, Eq.(19) yields

$$c(n'_m) = \alpha[1 + \epsilon_z n'^4_3 + \delta_x n'^2_3(1 - n'^2_3) + \epsilon_x(1 - n'^2_3)^2]. \tag{22}$$

Since the axis of symmetry is along the x'_3 axis, Eq.(22) shows a well-known fact that in a TI medium, the phase velocity depends only on the projection of the phase normal n'_i into the axis of symmetry. By specifying the phase normal n'_i with respect to the general coordinate system, we can study a variety of TI media with various orientations of the symmetry axis. In the following, we concentrate on two special cases: VTI and HTI media. Formulae for the phase velocities in both media can be obtained from Eq.(22) with a proper specification of n'_i. For the case of HTI media, however, the elastic parameter A_{33}, would be related to a horizontal, and not a vertical, direction. Since we wish to have A_{33} consistently related to the vertical direction, we consider in the case of HTI media the elastic parameters in the crystal coordinate system satisfying the following conditions:

$$A_{33} = A_{22}, \quad A_{66} = A_{55}, \quad A_{13} = A_{12}, \quad A_{23} = A_{33} - 2A_{44}, \tag{23}$$

i.e.

$$\epsilon_y = \epsilon_z = \frac{1}{2}\delta_y, \quad \delta_x = \delta_z. \tag{24}$$

This means that the axis of symmetry is along the x'_1 axis of the crystal coordinate system. In this case Eq.(19) reduces to

$$c(n'_m) = \alpha[1 + \epsilon_x n'^4_1 + \delta_x n'^2_1(1 - n'^2_1) + \epsilon_z(1 - n'^2_1)^2]. \tag{25}$$

In Eq.(25), we can again see that the phase velocity depends only on the projection n'_1 of the phase normal into the axis of symmetry.

Relations for the components n'_i of the phase normal in the general coordinate system can be obtained by projecting the phase normal defined in Eq.(9) into the axes of the crystal coordinate system. The basis vectors of the crystal coordinate system can be defined as the vectors e_{1i}, e_{2i} and e_{3i} in Eqs.(9) and (10) with θ and φ substituted by θ_0 and φ_0. In the same way, in which θ and φ specify the phase normal, the angles θ_0 and φ_0 specify the axis of symmetry in the general coordinate system. For the components n'_i, we can then write

$$n'_1 = \cos(\varphi - \varphi_0)\cos\theta_0 \sin\theta - \cos\theta \sin\theta_0, \quad n'_2 = \sin\theta \sin(\varphi - \varphi_0),$$

$$n'_3 = \cos\theta \cos\theta_0 + \sin\theta \sin\theta_0 \cos(\varphi - \varphi_0). \tag{26}$$

VTI medium: Inserting n_3' from (26) into Eq.(22) and specifying the resulting equation for $\theta_0 = 0$, we get an approximate formula for the phase velocity in a VTI medium. As Thomsen (1986), who studied this case, we choose $\alpha^2 = A_{33}$. Then Eq.(22) yields

$$c(\theta) = \alpha(1 + \delta \cos^2\theta \sin^2\theta + \epsilon \sin^4\theta) \tag{27}$$

with

$$\delta = \frac{A_{13} + 2A_{55} - A_{33}}{A_{33}}, \quad \epsilon = \frac{A_{11} - A_{33}}{2A_{33}}. \tag{28}$$

We can see that Eq.(27) is identical to Thomsen's (1986) equation for the qP wave phase velocity. In a weak VTI medium, the Thomsen's parameter δ can be linearized, see Thomsen (1994), Sayers (1994), to the form used in Eq.(28) and the higher-order terms present in δ of Thomsen (1986) can be neglected.

The qP wave phase velocity in a weakly VTI medium specified by Eq.(27) is controlled by two parameters, δ and ϵ in addition to the above specified parameter α. Exhaustive discussion of these parameters can be found in Thomsen (1986).

For comparison with the case considered in the following section, it is convenient to rewrite (27) into an alternative form

$$c(\varphi, \theta) = \alpha[1 + \delta \sin^2\theta + (\epsilon - \delta)\sin^4\theta]. \tag{29}$$

It might be of interest to specify the Thomsen's formula for prevailingly horizontal propagation, as, e.g., in a crosshole experiment. In this case it is convenient to choose $\alpha^2 = A_{11}$. Then Eq.(22) together with (26) yields

$$c(\varphi, \theta) = \alpha(1 + \delta^* \cos^2\theta \sin^2\theta + \epsilon^* \cos^4\theta), \tag{30}$$

with

$$\delta^* = \frac{A_{13} + 2A_{44} - A_{11}}{A_{11}}, \quad \epsilon^* = \frac{A_{33} - A_{11}}{2A_{11}}. \tag{31}$$

We can see that for prevailingly horizontal propagation, the decisive role is played by the parameter δ^*.

HTI medium: Inserting n_1' from (26) into Eq.(25) and specifying the resulting equation for $\theta_0 = 0$ leads to the approximate formula for the phase velocity in the HTI medium. We consider again prevailingly vertical wave propagation and choose therefore $\alpha^2 = A_{33}$. We get

$$c(\varphi, \theta) = \alpha[1 + \delta \cos^2(\varphi - \varphi_0)\sin^2\theta + (\epsilon - \delta)\cos^4(\varphi - \varphi_0)\sin^4\theta] \tag{32}$$

with δ and ϵ given by (28). We can see that the phase velocity is controlled by the same WA parameters as in the case of VTI medium. Another parameter controlling the phase velocity is the angle φ_0 specifying orientation of the axis of symmetry. Note that Eq.(32) has the same structure as Eq.(29), only $\sin\theta$ is substituted by $\cos(\varphi - \varphi_0)\sin\theta$. For $\varphi = \varphi_0$, i.e. in the vertical plane containing the axis of symmetry, Eqs.(32) and (29) become the same and thus the "Thomsen's equation" (27) is applicable even in this special situation. It can be clearly seen from Eq.(32) that recovery of elastic parameters is difficult for $\varphi - \varphi_0 \approx \frac{\pi}{2}$, i.e. for directions close to the normal to the axis of symmetry.

Polarization

Formulae of this section offer a simple way of approximate estimation of polarization of qP waves in arbitrary weakly anisotropic media. We can see in Eq.(4) that a decisive role in the determination of the perturbation of the polarization vector is played by the factors B_{13} and B_{23}. These factors are measures of deviations of the qP wave polarization vector from the phase normal n_i in the directions of the unit vectors e_{1i} and e_{2i}. In the following sections, using Eqs.(7), we specify the factors B_{13} and B_{23} for various types of anisotropic symmetry.

General anisotropy.—Eq.(11) yields for B_{13} and B_{23}:

$$B_{13} = \alpha^2 D^{-1}\Big[2\epsilon_z n_3^5 + n_3^4(\epsilon_{34}n_2 + \epsilon_{35}n_1) + n_3^3(\delta_x n_1^2 + \delta_y n_2^2 + 2\chi_z n_1 n_2 - 2\epsilon_z)$$
$$+n_3^2[(4\chi_x - 3\epsilon_{34})n_1^2 n_2 + (4\chi_y - 3\epsilon_{35})n_1 n_2^2 + (4\epsilon_{15} - 3\epsilon_{35})n_1^3 + (4\epsilon_{24} - 3\epsilon_{34})n_2^3]$$
$$+n_3[(2\delta_z - \delta_x - \delta_y)n_1^2 n_2^2 + 2(2\epsilon_{16} - \chi_z)n_1^3 n_2 + 2(2\epsilon_{26} - \chi_z)n_1 n_2^3 + (2\epsilon_x - \delta_x)n_1^4 + (2\epsilon_y - \delta_y)n_2^4]$$
$$-\chi_x n_1^2 n_2 - \chi_y n_1 n_2^2 - \epsilon_{15} n_1^3 - \epsilon_{24} n_2^3\Big]. \qquad (33)$$

and

$$B_{23} = \alpha^2 D^{-1}\Big[n_3^3(\epsilon_{34}n_1 - \epsilon_{35}n_2) + n_3^2[(\delta_y - \delta_x)n_1 n_2 + \chi_z n_1^2 - \chi_z n_2^2]$$
$$+n_3[(2\chi_y - 3\epsilon_{15})n_1^2 n_2 - (2\chi_x - 3\epsilon_{24})n_1 n_2^2 + \chi_x n_1^3 - \chi_y n_2^3]$$
$$+(\delta_z - 2\epsilon_x)n_1^3 n_2 + (2\epsilon_y - \delta_z)n_1 n_2^3 + 3(\epsilon_{26} - \epsilon_{16})n_1^2 n_2^2 + \epsilon_{16} n_1^4 - \epsilon_{26} n_2^4\Big], \qquad (34)$$

where D is defined in Eq.(8) and the WA parameters in (17b).

We can see that both above expressions contain the same WA parameters as the expression for the phase velocity (17a). Thus from the polarization vector of the qP wave, no additional WA parameters to those obtained from the phase velocity formula can be obtained. The above formulae only contain an additional independent information on the parameters already appearing in the expression for the phase velocity.

Orthorhombic medium.—Let us consider the same orthorhombic medium as above and the phase normal specified in the crystal coordinate system. Then, Eqs.(33) and (34) reduce to

$$B_{13} = \alpha^2 D^{-1}\Big[2\epsilon_z n_3^{'5} + n_3^{'3}(\delta_x n_1^{'2} + \delta_y n_2^{'2} - 2\epsilon_z)$$
$$+n_3'[(2\delta_z - \delta_x - \delta_y)n_1^{'2}n_2^{'2} + (2\epsilon_x - \delta_x)n_1^{'4} + (2\epsilon_y - \delta_y)n_2^{'4}]\Big],$$

$$B_{23} = \alpha^2 D^{-1}[n_3^{'2}(\delta_y - \delta_x)n_1' n_2' + (\delta_z - 2\epsilon_x)n_1^{'3}n_2' - (\delta_z - 2\epsilon_y)n_1' n_2^{'3}]. \qquad (35)$$

To express (35) in a general coordinate system, Eq.(18) must be used. In the crystal coordinate system formulae (35) contain the same five WA parameters as the corresponding equation for the phase velocity, see Eq.(19).

Specification of Eqs.(35) in any symmetry plane leads again to equations with exactly same form as equations in symmetry planes of TI media, see the next section. This again confirms earlier observations of Cheadle et al. (1991).

TI medium.—Let us first consider the TI medium specified by (20) and (21). In this case Eqs.(35) reduce to

$$B_{13} = \alpha^2 D^{-1} n_3'(1 - n_3'^2)[(\delta_x - 2\epsilon_z)n_3'^2 - (\delta_x - 2\epsilon_x)(1 - n_3'^2)]. \quad B_{23} = 0 \tag{36}$$

We arrived to an expected result. For medium specified by (20), the vector e_{2i} is always perpendicular to the plane of propagation and it represents an eigenvector of the perturbed Christoffel matrix. Therefore, B_{23} must be zero and the perturbed qP wave polarization vector is obtained from the unperturbed one only by a correction in the plane of propagation.

Let us now consider the medium specified by (23) and (24). In this case, we get from (35) for B_{13} and B_{23}

$$B_{13} = \alpha^2 D^{-1} n_1'^2 n_3'[\delta_x + 2(\epsilon_x - \delta_x)n_1'^2 - 2\epsilon_z(1 - n_1'^2)],$$

$$B_{23} = -\alpha^2 D^{-1} n_1' n_2'[\delta_x + 2(\epsilon_x - \delta_x)n_1'^2 - 2\epsilon_z(1 - n_1'^2)]. \tag{37}$$

VTI medium: Let us assume that $\alpha^2 = A_{33}$. Inserting n_3' from (26) into (36) and specifying the result for $\theta_0 = 0$ leads to equations for B_{13} and B_{23} in the form

$$B_{13} = \alpha^2 \sin\theta \cos\theta[\delta + 2(\epsilon - \delta)\sin^2\theta], \quad B_{23} = 0, \tag{38}$$

where δ and ϵ are specified in (28). As expected, for vertical or horizontal directions B_{13} becomes zero. This is because both directions represent longitudinal directions along which the qP wave propagates as a P wave and for these directions B_{13} and B_{23} are zero.

HTI medium: Let us again assume that $\alpha^2 = A_{33}$. Inserting n_1', n_2' and n_3' from (26) into (37) and specifying the result for $\theta_0 = 0$, we get for B_{13} and B_{23}

$$B_{13} = \alpha^2 \sin\theta \cos\theta \cos^2(\varphi - \varphi_0)[\delta + 2(\epsilon - \delta)\sin^2\theta \cos^2(\varphi - \varphi_0)]$$

$$B_{23} = -\alpha^2 \sin\theta \cos(\varphi - \varphi_0)\sin(\varphi - \varphi_0)[\delta + 2(\epsilon - \delta)\sin^2\theta \cos^2(\varphi - \varphi_0)]. \tag{39}$$

We can see from Eqs.(38) and (39) that the polarization vectors in VTI and HTI media are controlled by the same WA parameters ϵ and δ, which control the phase velocities. We can also see that the structure of Eqs.(38) and (39) is identical, only $\sin\theta$ from (38) is substituted by $\sin\theta\cos(\varphi - \varphi_0)$ in (39). For $\varphi = \varphi_0$, i.e. in the vertical plane containing the axis of symmetry, Eqs.(38) and (39) become the same, see similar conclusion for the phase velocity.

MOVEOUT VELOCITY

It is shown in the Appendix A that the normal moveout velocity, V_{NMO}, in an arbitrary weakly anisotropic medium is given by the formula

$$V_{NMO}^{-2} = \alpha^{-2}(1 + 2\epsilon_z' - 2\delta_x'), \tag{40}$$

see (A.13). The primes indicate that the V_{NMO} is related to a vertical plane, which generally differs from the plane (x, z). We can see that in this plane, the departure of the V_{NMO} from the vertical velocity α is controlled by only two WA parameters ϵ'_z and δ'_x. If we specify α as the vertical velocity, $\alpha^2 = A_{33}$, i.e. $\epsilon'_z = 0$, the number of the involved WA parameters reduces to one

$$V_{NMO}^{-2} = \alpha^{-2}(1 - 2\bar{\delta}'_x). \tag{41}$$

The bar over δ'_x indicates that the specification $\alpha^2 = A_{33}$ is used in Eq.(41). We can see that Eq.(41) for arbitrary weakly anisotropic media is formally identical with a similar equation derived by Thomsen (1986) for VTI anisotropy. The only difference consists in the different dependence of the linearized Thomsen's parameter δ and of the parameter $\bar{\delta}'_x$ on elastic parameters of the medium. The dependence of $\bar{\delta}'_x$ varies for different vertical planes while the Thomsen's δ is invariant. Only in the coordinate plane (x, z), Thomsen's (1986) formula coincides completely with (41). For the VTI media, Eq.(41) reduces to Thomsen's formula.

In (40), the primed WA parameters correspond to a local coordinate system, see the Appendix A. If we wish to substitute the primed WA parameters by the WA parameters related to the general Cartesian coordinate system, we have to use the relations

$$\delta'_x = \delta_x \cos^2\Phi + 2\chi_z \sin\Phi\cos\Phi + \delta_y \sin^2\Phi, \quad \epsilon'_z = \epsilon_z. \tag{42}$$

They simply follow from the transformation relations for $A_{\alpha\beta}$ corresponding to the coordinate transformation (A.1). Using (42) in (40), leads to the formula

$$V_{NMO}^{-2} = \alpha^{-2}(1 + 2\epsilon_z - 2\delta_x \cos^2\Phi - 2\delta_y \sin^2\Phi - 4\chi_z \sin\Phi\cos\Phi). \tag{43}$$

For $\alpha^2 = A_{33}$, Eq.(43) coincides, in symmetry planes, with a similar formula of Tsvankin (1995). The Tsvankin's formula holds even for a stronger anisotropy and dipping interfaces but *only* in symmetry planes. Eq.(43) holds for any type of anisotropy even outside symmetry planes but *only* for weak anisotropy and horizontal reflectors.

NUMERICAL EXAMPLES

In this section, we illustrate accuracy of the approximate formulae for the qP wave phase velocity and polarization vector. We consider two models. The first model is a model of a HTI medium. Anisotropy of the model is the crack-induced anisotropy, see Hudson (1981). The second model is a triclinic model of Vosges sandstone considered previously by Mensch and Rasolofosaon (1997).

As HTI models, we consider "wet-cracks" models with very weak anisotropy and "dry-cracks" models with relatively strong anisotropy. If we use common measure of the strength of qP wave anisotropy in TI media, $\epsilon \times 100\%$, anisotropy of the dry-cracks models is about 20%, which is relatively strong anisotropy. Anisotropy of the wet-cracks model is small, less than 1%. In the following, we present tests of accuracy of the approximate formulae only for a dry-cracks model since they exhibit observable deviations from exact results. The deviations for the wet-cracks models are negligible.

The parameters of the HTI dry-cracks model are as follows. The P wave velocity of the host rock is 4 km/s, the P to S wave velocity ratio is $\sqrt{3}$ and the density is 2.5 g/cm^3. The crack density 0.1 and the aspect ratio 10^{-4} are used. The WA parameters are $\epsilon = -0.19$ and $\delta = -0.24$. The qP wave phase velocity section in the symmetry plane is shown in Figure 1. The matrix of density normalized elastic parameters, in GPa, has the form

$$\begin{pmatrix} 9.43 & 3.14 & 3.14 & 0.00 & 0.00 & 0.00 \\ & 15.27 & 4.60 & 0.00 & 0.00 & 0.00 \\ & & 15.27 & 0.00 & 0.00 & 0.00 \\ & & & 5.33 & 0.00 & 0.00 \\ & & & & 4.25 & 0.00 \\ & & & & & 4.25 \end{pmatrix}.$$

In the following, we display the HTI results in plots covering azimuthal angles $0 \leq \varphi \leq \frac{\pi}{2}$ and polar angles $0 \leq \theta \leq \frac{\pi}{2}$. In order to estimate better the accuracy of approximate formulae for the phase velocity, we present plots of relative errors of the results of the approximate formula (32) with respect to the values of the exact phase velocity. The relative errors are shown by isolines on the gray scaled background. The gray scale is used to indicate the variation of the errors between isolines.

Figure 2 shows the plot of relative errors of the equation (32) for the dry-crack model specified for $\varphi_0 = 0$. The constant α is chosen as $\alpha^2 = A_{33}$. Maximum relative error is 2.92% for nearly horizontal propagation. Remember, however, that with the above specification of α, the formula (32) is "tailored" for prevailingly vertical propagation. For polar angles θ less than 60^o, the error for the dry-cracks model is below 1% and for θ less than 30^o, it is less than 0.5%.

Figure 3 shows again relative errors of the phase velocity for the dry-crack model. This time the constant α is chosen according to Fedorov (1968) as the closest isotropic approximation of the anisotropic model, see Eq.(15a):

$$\alpha^2 = \frac{1}{15}(3A_{11} + 4A_{13} + 8A_{33} + 8A_{66}). \qquad (44)$$

We can see that for this specification, the maximum relative error of the formula (32) drops to slightly more than 1%. But the results for prevailingly vertical propagation are less accurate in Fig.3 than they are in Fig.2.

Figure 4 shows relative errors obtained with a formula alternative to (32), which yields approximation for c^2, see Eq.(14). In this case the approximate formula is independent of the choice of α. We can see that the relative errors are always negative as predicted. The formula gives exact results for any polar angle θ when azimuth $\varphi = \frac{\pi}{2}$, and for $\varphi = 0$, $\theta = \frac{\pi}{2}$ because these are specifications of longitudinal directions. The orientation of the axis of symmetry works in favor of the formula for c^2 so that for prevailingly vertical propagation, it yields results comparably accurate with results of Eq.(32) specified for $\alpha^2 = A_{33}$. This would change, in favor of Eq.(32) if the axis of symmetry is, for example, non-horizontal.

In the following we illustrate accuracy of the formulae (16) and (39) for approximate calculation of the polarization vectors in the HTI medium. In this case, we display the angular deviation of the normalized approximately calculated polarization vector from the exact polarization vector and call it a *polarization error*. The deviation is measured in degrees. In the following plots the white color indicates zero.

Figure 5 shows the polarization error for the dry-crack model. The constants α and β are chosen as $\alpha^2 = A_{33}$ and $\beta^2 = A_{66}$, respectively. The polarization errors do not exceed 4.3^o. Let us emphasize again that the approximate polarization formula has a local character. Thus the difference $\alpha^2 - \beta^2$ can be chosen locally in order to get the best local approximation. Zero polarization error for the vertical propagation and for the propagation along the axis of symmetry is a consequence of the fact that these directions are longitudinal.

In Figure 6, we choose the term $\alpha^2 - \beta^2$ equal to 8 $(\mathrm{km/s})^2$. This value corresponds to the difference of the squares of approximate qP and qS wave phase velocities at about 45^o from the axis of symmetry. Fig.6 clearly shows how well the approximate polarization vectors approximate the exact one in the mentioned region. Compare with Fig.5.

In order to have an idea how strong is the correction term in Eq.(16), we show the deviation of the exact polarization vector from the phase normal in Fig.7. For polar angles between 40^o and 60^o, it reaches maximum values of about 10^o. In the same region in Fig.5, the polarization error is approximately 3.4^o. In Fig.6, the polarization error in this region is even less, about 0.6^o.

In the last three figures, we show the same results as above but for the model of the dry Vosges sandstone, see Mensch and Rasolofosaon (1997). The matrix of density normalized elastic parameters, in GPa, has the form

$$\begin{pmatrix} 10.3 & 0.9 & 1.3 & 1.4 & 1.1 & 0.8 \\ & 10.6 & 2.1 & 0.2 & -0.2 & -0.6 \\ & & 14.1 & 0.0 & -0.5 & -1.0 \\ & & & 5.1 & 0.0 & 0.2 \\ & & & & 6.0 & 0.0 \\ & & & & & 4.9 \end{pmatrix} .$$

Since the model is triclinic, we have to display results in the extended interval of polar angles $0^o - 180^o$. For any azimuth, the polar angle of 90^o corresponds to the vertical direction. In this case we are testing applicability of the basic approximate formula for the phase velocity (17a). Otherwise, the display of results is the same as in the case of the HTI medium.

The considered material is again relatively strongly anisotropic. In the classification of Mensch and Rasolofosaon (1997), anisotropy slightly exceeds 21%. Figure 8 is a different display of Fig.6b of Mensch and Rasolofosaon (1997). In contrast to the mentioned authors, we choose, as usually, $\alpha^2 = A_{33}$ so that for the vertical direction, we have zero relative error. Maximum relative error is about 1.7%. It corresponds to nearly horizontal propagation. With the above specification of α, the formula (17a) is "tailored" for prevailingly vertical propagation and thus the relative error is below 1% upto nearly 40^o from the vertical and below 0.5% for upto 30^o from the vertical. If α were chosen as a horizontal phase velocity, the relative error in the horizontal direction would be substantially reduced. Due to the fast variation of relative errors for nearly horizontal propagation (see the intervals of polar angles $0^o - 30^o$ and $150^o - 180^o$ in Fig.8), however, the formula (17a) would yield less accurate approximation in this direction.

Figure 9 shows the deviation of the exact polarization vector from the phase normal as in Fig.7. The maximum deviations reach now values of about 16^o, in the interval of polar angles of upto 30^o from the vertical, the maximum deviations reach upto nearly 10^o. We test now formula (16) with (33) and (34). When we choose the constants α and β as $\alpha^2 = A_{33}$ and $\beta^2 = A_{66}$ again, the polarization errors in the interval of polar angles of upto 30^o from the vertical do not exceed 2^o, in most of the interval, they are below 1^o, see Fig.10. In the region of the maximum phase normal deviation, see Fig.9, to which our test is not tuned, the polarization error is of about 7^o, i.e. less than half of the phase normal deviation.

CONCLUSIONS

Approximate formulae for the qP wave phase velocity, polarization vector and normal moveout velocity in arbitrary weakly anisotropic media were given. The formulae depend on WA parameters, which represent a generalization of Thomsen's (1986) parameters and the formulae have properties of Thomsen's formulae. They considerably simplify the exact equations for the qP wave phase velocity, the polarization vector and the normal moveout velocity. The WA parameters are non-dimensional quantities. They represent normalized elastic parameters or their combinations. In an isotropic medium, representing a limiting case of weak anisotropy, all of the WA parameters reduce to zero and the corresponding equations reduce to equations describing wave quantities in isotropic media. For the case of VTI medium, equations for the phase and normal moveout velocity reduce to Thomsen's (1986) formulae with Thomsen's δ linearized.

In contrast to Thomsen's (1986) parameters, the WA parameters are linear functions of elastic parameters. This means that the formulae of this paper are designed for *weakly* anisotropic media only, i.e. for media, in which deviations of elastic parameters from isotropy represent the first order perturbations. Worse results of the approximate formulae should be expected when *stronger* anisotropy is considered.

For special applications, this limitation can be removed by the consideration of the δ parameter of Thomsen (1986), whose dependence on elastic parameters is nonlinear, see detailed discussion in Tsvankin (1997). In the framework of weak anisotropy, however, Thomsen's as well as WA formulae and parameters are equivalent. To increase the accuracy of the formulae presented in this paper, the second-order perturbations should be used. Let us mention that formulae for any higher-order terms of the perturbation theory were derived and tested by Druzhinin (1996). The linearity of the relation between the WA parameters and the elastic parameters is important for the study of sensitivity of derived formulae to elastic parameters and for solving inverse problems.

In the case of general anisotropy specified by 21 independent elastic parameters, the approximate formulae for the phase velocity and the polarization vector depend on 15 WA parameters. In media with orthorhombic symmetry, the number of WA parameters, which specify the phase velocity, reduces to 6 and in case of TI media to 3. This number can be reduced by one by a proper choice of the P wave velocity in the background isotropic medium. The WA parameters must be supplemented by the parameters specifying the transformation matrix from the general to the crystal coordinate system if these systems do not coincide, i.e. when, for example, axes of symmetry are not vertical and/or horizontal. The P and S wave velocities of the background medium can be used to control the accuracy of the approximate formulae for the phase velocity and the polarization vector for a specified direction in weakly anisotropic media.

Inspection of the derived formulae makes possible to determine how to optimize seismic measurements so that all recoverable WA parameters are determined at lowest effort. For example, measurements in 4 vertical planes and one horizontal plane or in 5 vertical planes are sufficient to find all the WA parameters of a general weakly anisotropic medium.

Since the integrand of travel time integral for weakly anisotropic media contains phase velocities, see e.g. Červený (1982), Hanyga (1982), Červený and Jech (1982), extension of the conclusions of this paper to inhomogeneous weakly anisotropic media is straightforward.

The approximate formula for the qP wave polarization vectors is local. In inverse problems, where we see its great potential, it can be used as a source of additional independent information to constraint the local distribution of parameters of the medium in the vicinity of the points, in which the polarization is observed.

The deviation of the normal moveout velocity from the vertical velocity is controlled by a formula, which is formally identical with the well-known formula for NMO velocity in VTI media derived by Thomsen (1986). The parameter playing the role of linearized Thomsen's δ and having the same structure can be expressed in terms of four WA parameters and azimuth in a general Cartesian coordinate system. For media with higher-order symmetry, the general formula reduces to formulae given by Sayers (1995), Tsvankin (1995, 1997). In the case of a VTI medium, the general formula reduces to that given by Thomsen (1986).

As performed tests indicate, the precision of the approximate formulae can be rather high even for relatively strong anisotropy. In presented examples with anisotropy of about 20%, the relative error of the approximate formula for the phase velocity does not exceed 3% in horizontal directions, for which the formula was not designed. For the prevailingly vertical propagation, upto 60^o from the vertical, the error is always less than 1%. Numerical experiments with the polarization error show similar results.

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APPENDIX A–NORMAL MOVEOUT VELOCITY

Here we derive the formula for the NMO velocity in a horizontal homogeneous layer of arbitrary but weak anisotropy, see also Pšenčík and Gajewski (1996).

Let us consider a homogeneous anisotropic layer of thickness h where the anisotropy of the layer is arbitrary but weak. On the surface of this layer, let us consider an offset ξ defined as

$$\xi = x\cos\Phi + y\sin\Phi, \tag{A.1}$$

where Φ is the azimuthal angle defined in the same way as φ in (9). According to Thomsen (1986), the normal-moveout velocity is defined as the initial slope of the curve $t^2 = t^2(\xi^2)$,

$$V_{NMO}^{-2} = \lim_{\xi\to 0}\frac{dt^2}{d\xi^2}. \tag{A.2}$$

Here t is the travel time of a wave reflected in the anisotropic layer.

The qP wave travel time in an arbitrary weakly anisotropic medium can be written as

$$t = t_{iso} - \frac{1}{2}\int_0^{t_{iso}} \alpha^{-2}B_{33}d\tau, \tag{A.3}$$

see e.g. Červený (1982), Hanyga (1982), Jech and Pšenčík (1989). The approximate travel time of the reflected wave in the homogeneous weakly anisotropic layer is then

$$t = t_{iso}[1 - \frac{1}{4\alpha^2}\Big(B_{33}(n_i^d) + B_{33}(n_i^u)\Big)]. \tag{A.4}$$

Here the symbols n_i^d and n_i^u denote the phase normal corresponding to down- and up-going parts of the reflected wave in the background isotropic medium, respectively. As before, α denotes the P wave velocity in the unperturbed isotropic medium and can be chosen arbitrarily. In a Cartesian coordinate system with ξ serving as an x_1 axis and x_3 axis coinciding with z axis of the general Cartesian coordinate system, we have

$$n_1^u = n_1^d, \quad n_2^u = n_2^d = 0, \quad n_3^u = -n_3^d. \tag{A.5}$$

Since Eq.(A.2) requires a derivative of the square of the travel time, we square the travel time t in (A.4) and, as always, neglect all terms of orders higher than first. We get

$$t^2 = t_{iso}^2[1 - \frac{1}{2\alpha^2}\Big(B_{33}(n_i^d) + B_{33}(n_i^u)\Big)]. \tag{A.6}$$

In the following, it is more convenient to work with the phase velocity $c(n_i)$ than with the term $B_{33}(n_i)$. We use therefore the relation

$$B_{33}(n_i) = 2\alpha(c(n_i) - \alpha), \tag{A.7}$$

which simply follows from Eqs.(12) and (13). If we also take into account the obvious relation

$$t_{iso}^2 = \frac{\xi^2 + 4h^2}{\alpha^2}, \tag{A.8}$$

we can write

$$t^2 = \frac{\xi^2 + 4h^2}{\alpha^2}\Big(3 - \frac{c(n_i^d) + c(n_i^u)}{\alpha}\Big). \tag{A.9}$$

We can now differentiate Eq.(A.9) with respect to ξ^2. We get

$$\frac{dt^2}{d\xi^2} = \alpha^{-2}\Big(3 - \frac{c(n_i^d) + c(n_i^u)}{\alpha}\Big) - \frac{\xi^2 + 4h^2}{\alpha^2}\frac{1}{\alpha}\Big(\frac{\partial c(n_i^d)}{\partial n_1} + \frac{\partial c(n_i^u)}{\partial n_1}\Big)\frac{dn_1}{d\xi^2}. \tag{A.10}$$

The derivative $dn_1/d\xi^2$ can be obtained from the relation

$$\xi^2 = n_1^2(\xi^2 + 4h^2), \tag{A.11}$$

which follows from a simple geometrical consideration. It remains to determine the partial derivatives $\partial c/\partial n_1$. They can be simply obtained by differentiating Eq.(17a), specified for $n_2 = 0$ with respect to n_1. In this way we get

$$\frac{1}{\alpha}\frac{dn_1}{d\xi^2}\Big(\frac{\partial c(n_i^d)}{\partial n_1} + \frac{\partial c(n_i^u)}{\partial n_1}\Big) = \frac{4h^2}{(\xi^2 + 4h^2)^2}(-4n_3^2\epsilon_z' - 2n_1^2\delta_x' + 2n_3^2\delta_x' + 4n_1^2\epsilon_x'). \tag{A.12}$$

The limiting process $\xi \to 0$, i.e. $n_1 \to 0$ and $n_3 \to 1$, yields, see (A.2)

$$V_{NMO}^{-2} = \alpha^{-2}(1 + 2\epsilon_z' - 2\delta_x'). \tag{A.13}$$

In (A.12) and (A.13), the primed WA parameters correspond to the coordinate system, in which ξ serves as an x_1 axis.

APPENDIX B–PROOF OF INEQUALITY $C^2 \le C_{EX}^2$ IN WEAKLY ANISOTROPIC MEDIA

We prove that the square of the phase velocity c^2 determined from the approximate equation (14) is, in weakly anisotropic media, always less than the square of the exact phase velocity c_{ex}^2.

The exact qP wave phase velocity c_{ex} can be found from the Christoffel equation

$$(\Gamma_{jk} - c_{ex}^2 \delta_{jk}) g_j^{ex} = 0, \tag{B.1}$$

where g_j^{ex} denotes the exact qP wave polarization vector. From (B.1) we simply get

$$c_{ex}^2 = a_{ijkl} n_i n_l g_j^{ex} g_k^{ex}. \tag{B.2}$$

Square of the approximately determined phase velocity is given in Eq.(14) as

$$c^2 = a_{ijkl} n_i n_l n_j n_k. \tag{B.3}$$

The polarization vector g_j^{ex} can be expressed in terms of the perturbation series

$$g_j^{ex} = n_j + A_i e_{ij} + C_i e_{ij} + ..., \tag{B.4}$$

where A_i denote the first-order coefficients, C_i second-order coefficients, etc. Inserting (B.4) into (B.2) and neglecting terms of the orders higher than second, we get

$$\begin{aligned} c_{ex}^2 = {} & a_{ijkl} n_i n_l n_j n_k + 2A_m \Delta a_{ijkl} n_i n_l n_j e_{mk} + 2A_m a_{ijkl}^0 n_i n_l e_{mj} n_k \\ & + a_{ijkl}^0 A_m A_n n_i n_l e_{mj} e_{nk} + 2C_m a_{ijkl}^0 n_i n_l n_j e_{mk}. \end{aligned} \tag{B.5}$$

The first term on the RHS is c^2, see (B.3). The third term on the RHS is zero due to (B.1). From the same reason, the fifth term is zero for $m = 1, 2$. It follows from Eq.(16) that the coefficients A_m have the form

$$A_M = \frac{B_{M3}}{\alpha^2 - \beta^2}, \quad A_3 = 0. \tag{B.6}$$

The coefficient C_3 is, according to Druzhinin (1996), given by formula

$$C_3 = -\frac{1}{2} \frac{B_{I3} B_{I3}}{(\alpha^2 - \beta^2)^2}. \tag{B.7}$$

With the accuracy upto the second-order terms, Eq.(B.5) yields

$$c_{ex}^2 = c^2 + \frac{B_{I3} B_{I3}}{\alpha^2 - \beta^2}, \tag{B.8}$$

Since the second term on the RHS of Eq.(B.8) is always nonnegative, Eq.(B.8) implies

$$c^2 \le c_{ex}^2. \tag{B.9}$$

The equality occurs when $B_{I3} = 0$ and this occurs when $g_i^{ex} = n_i$. This happens in longitudinal directions, of which the equality $B_{I3} = 0$ is an indication.

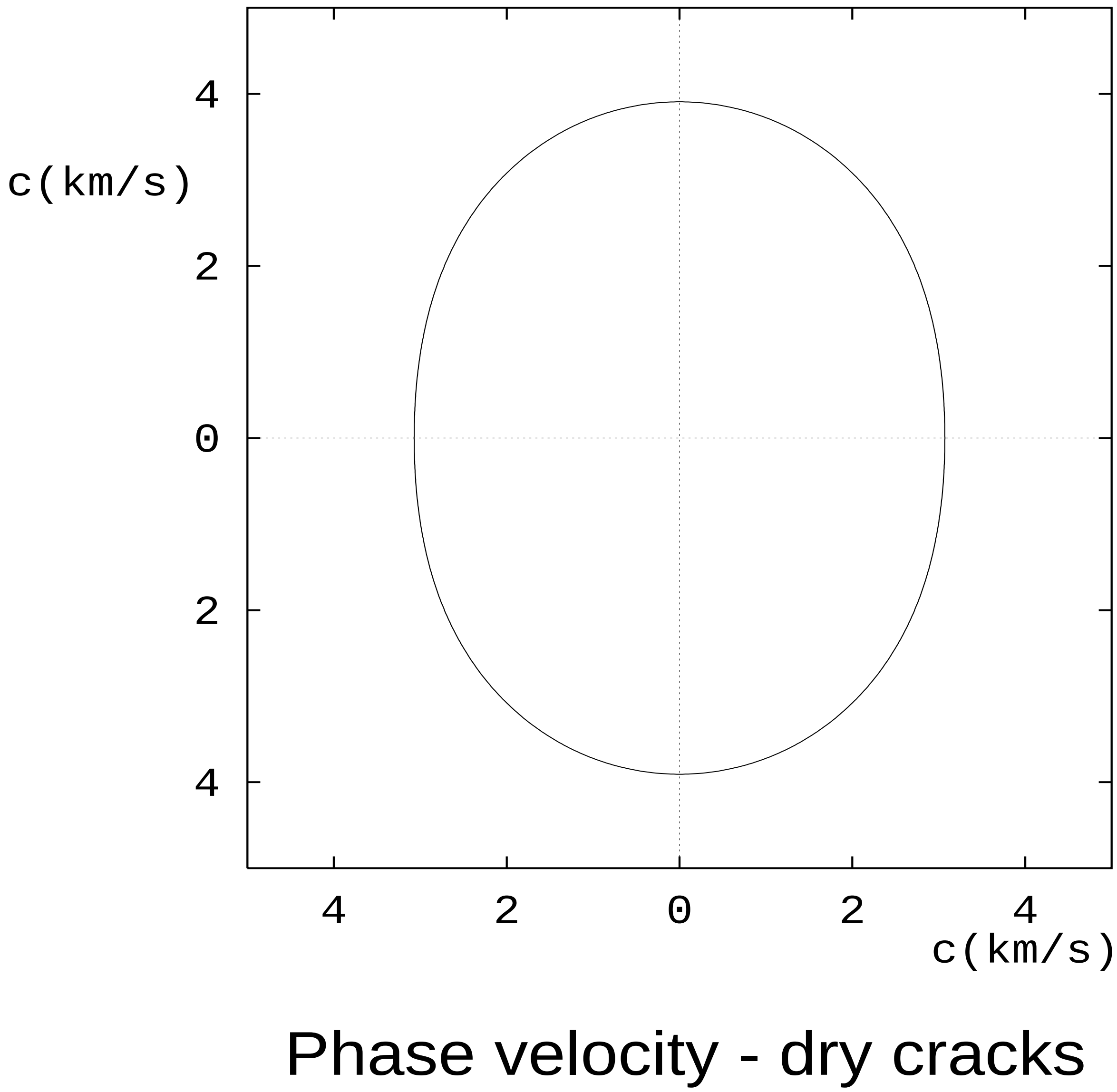

Figure 1: Phase velocity section in a symmetry plane of the "dry cracks" model.

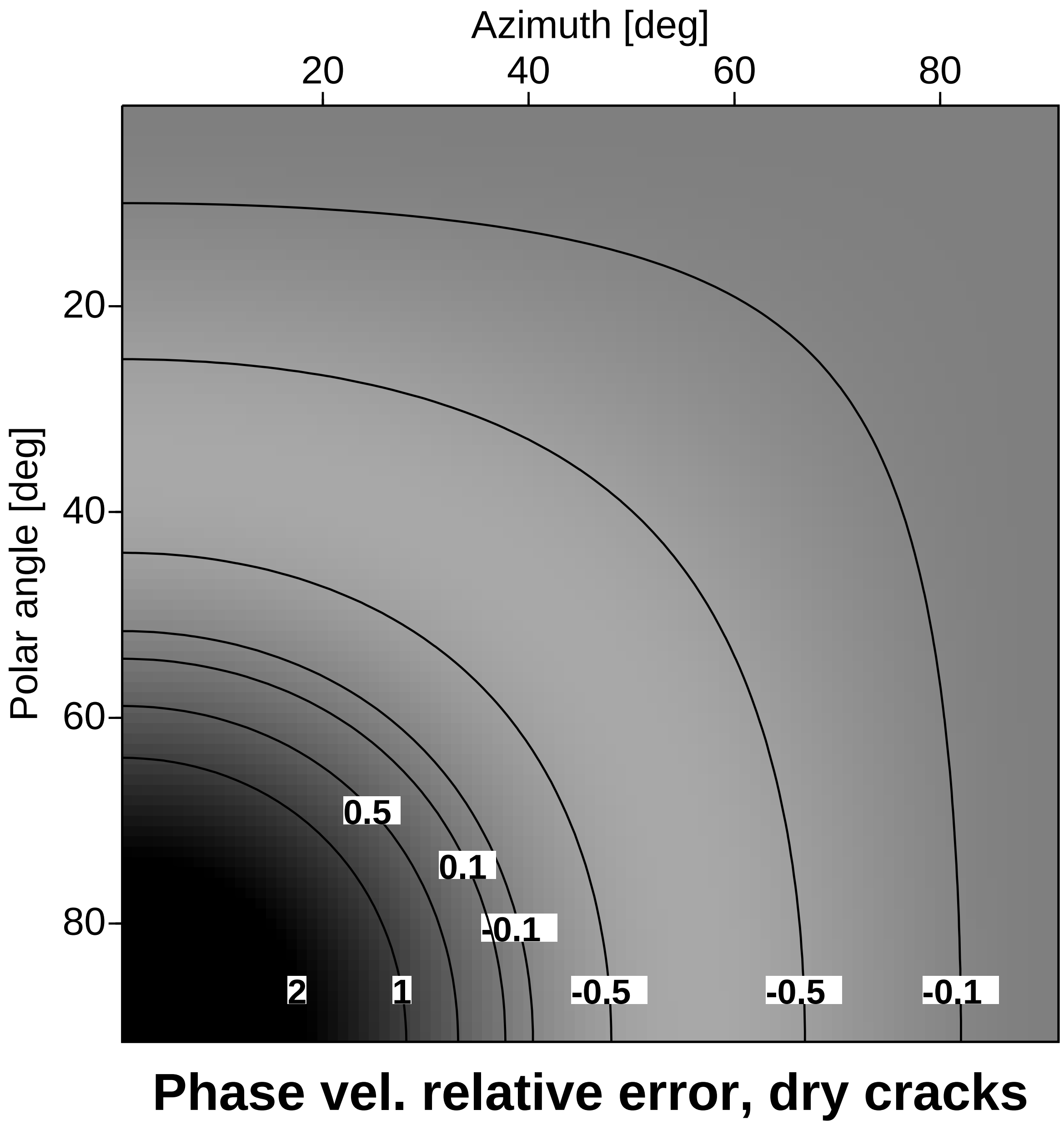

Figure 2: The map of relative errors, in %, of the approximate phase velocity formula (32) for $\varphi_0 = 0$ for the "dry cracks" model. The P wave velocity of the isotropic background is $\alpha = \sqrt{A_{33}}$.

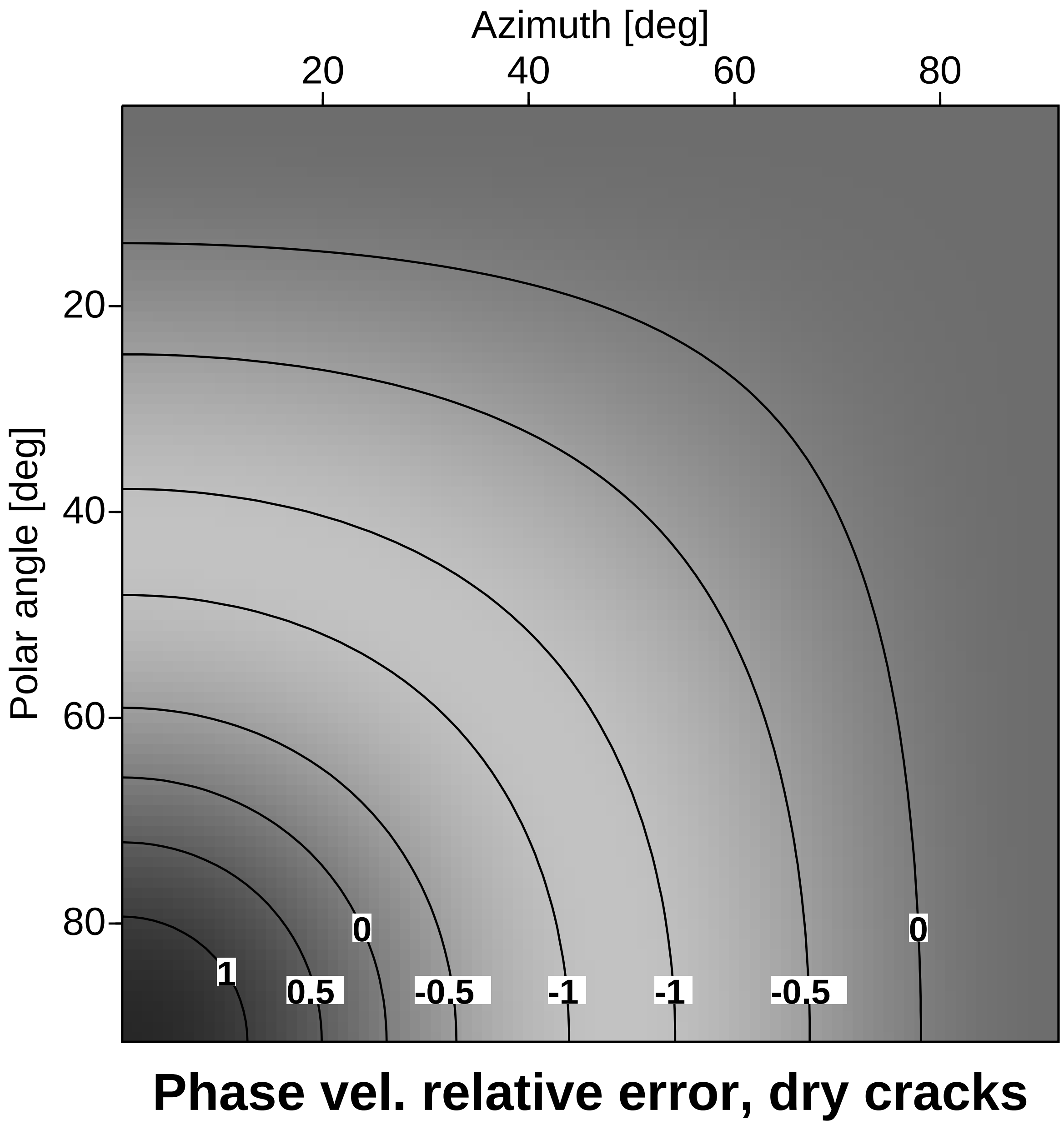

Figure 3: The map of relative errors, in %, of the approximate phase velocity formula (32) for $\varphi_0 = 0$ for the "dry cracks" model. The P wave velocity of the isotropic background is specified by Eq.(44) as the closest isotropic approximation of the exact model.

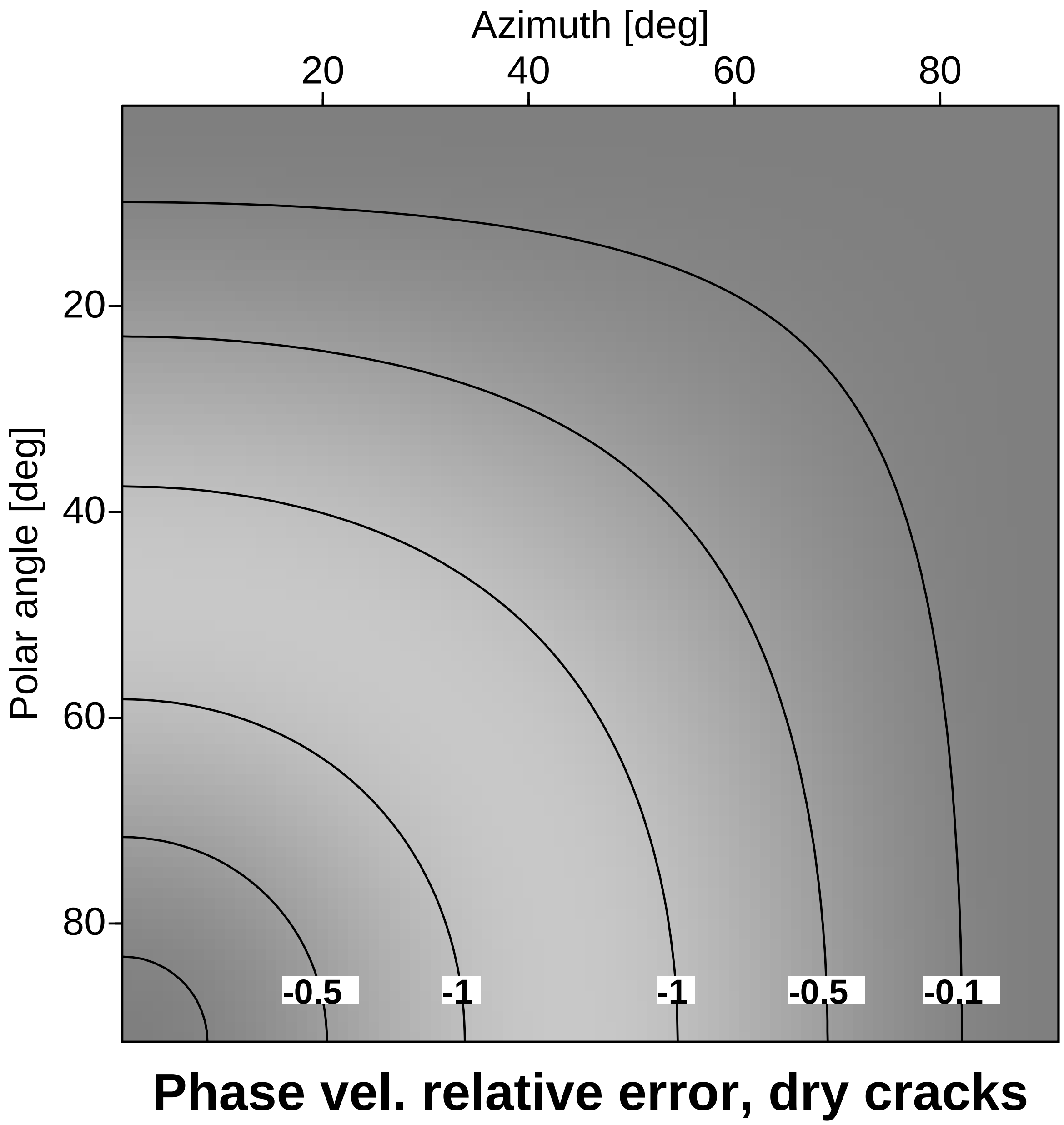

Figure 4: The map of relative errors, in %, of the phase velocity determined from Eq.(14) for the "dry cracks" model.

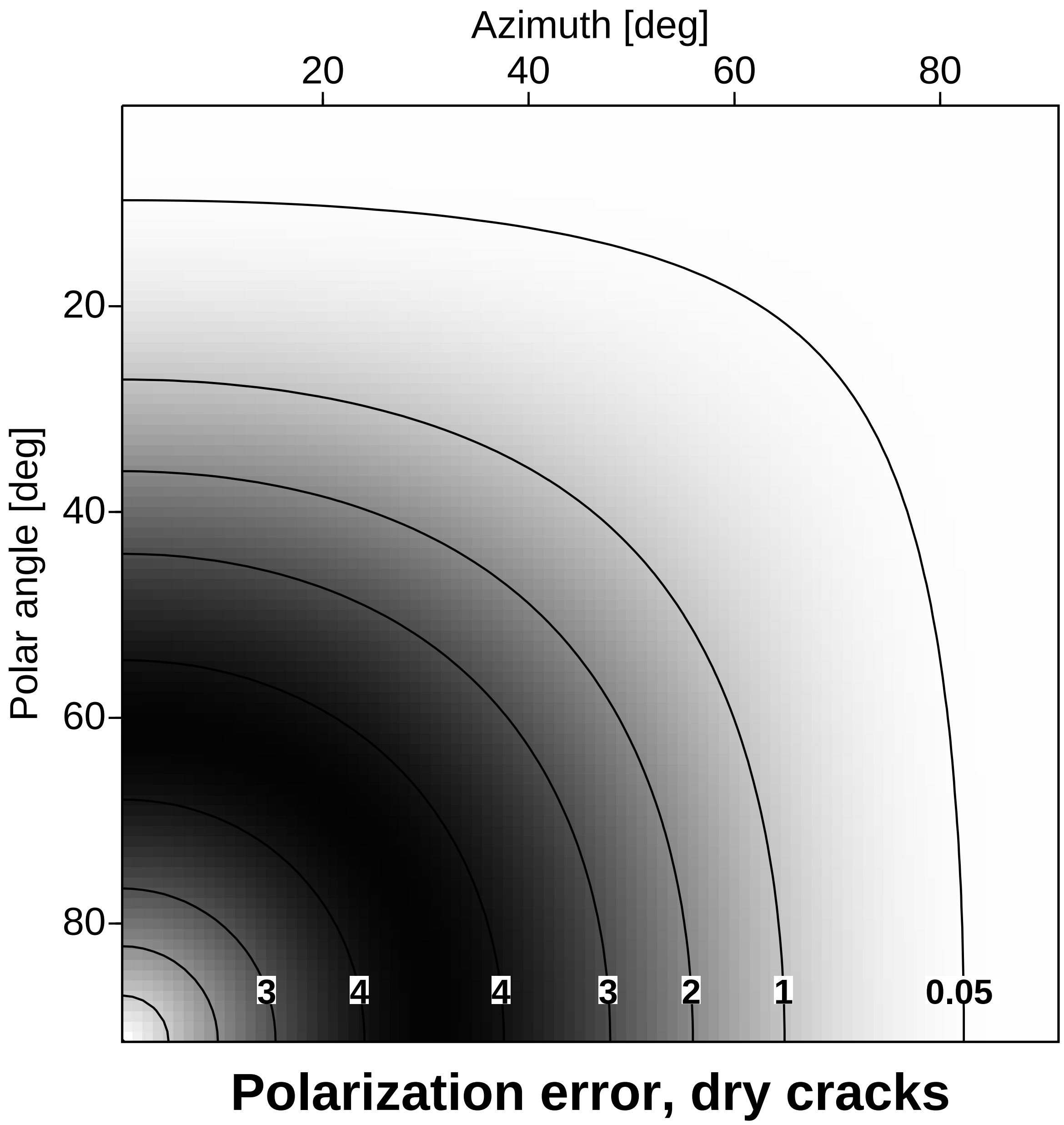

Figure 5: The map of polarization errors, in degrees, of Eqs.(16) and (40) for $\varphi_0 = 0$, for the dry cracks model. The P and S wave velocities of the isotropic background are $\alpha = \sqrt{A_{33}}$ and $\beta = \sqrt{A_{66}}$, respectively.

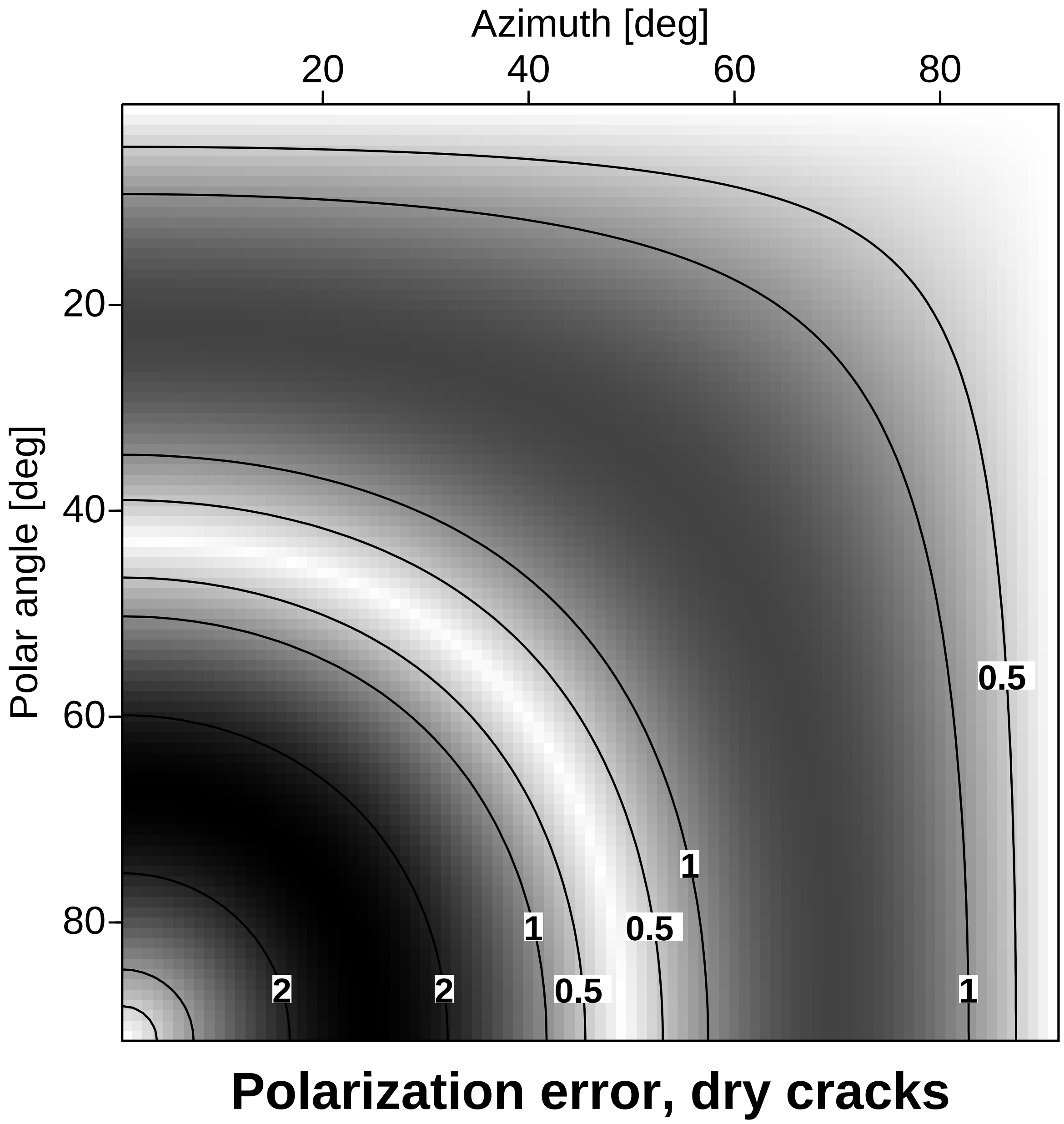

Figure 6: The map of polarization errors, in degrees, of Eqs.(16) and (40) for $\varphi_0 = 0$, for the "dry cracks" model. The difference $\alpha^2 - \beta^2$ equal to 8.

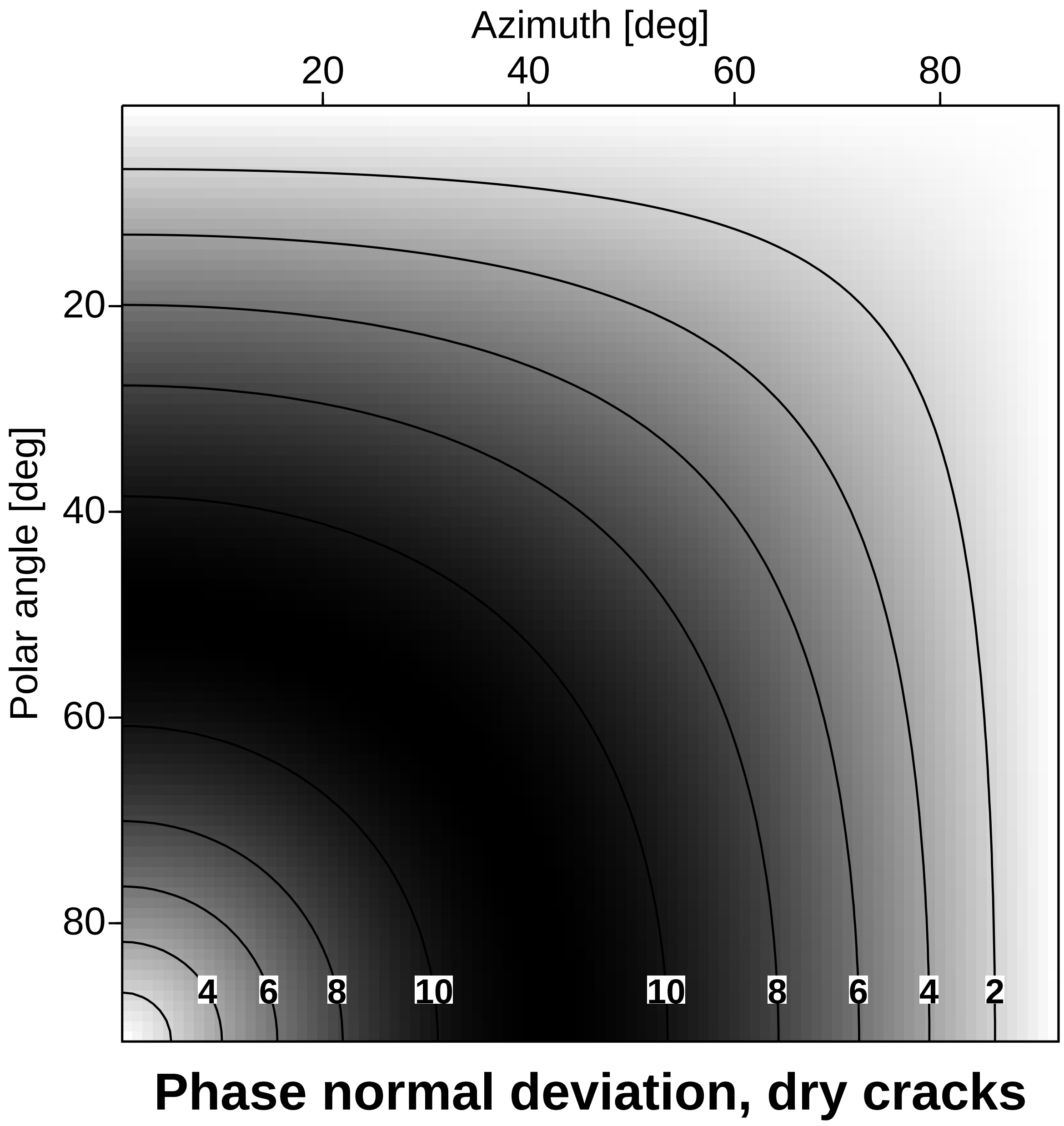

Figure 7: The map of deviations, in degrees, of the phase normals and the exact polarization vectors for the "dry cracks" model.

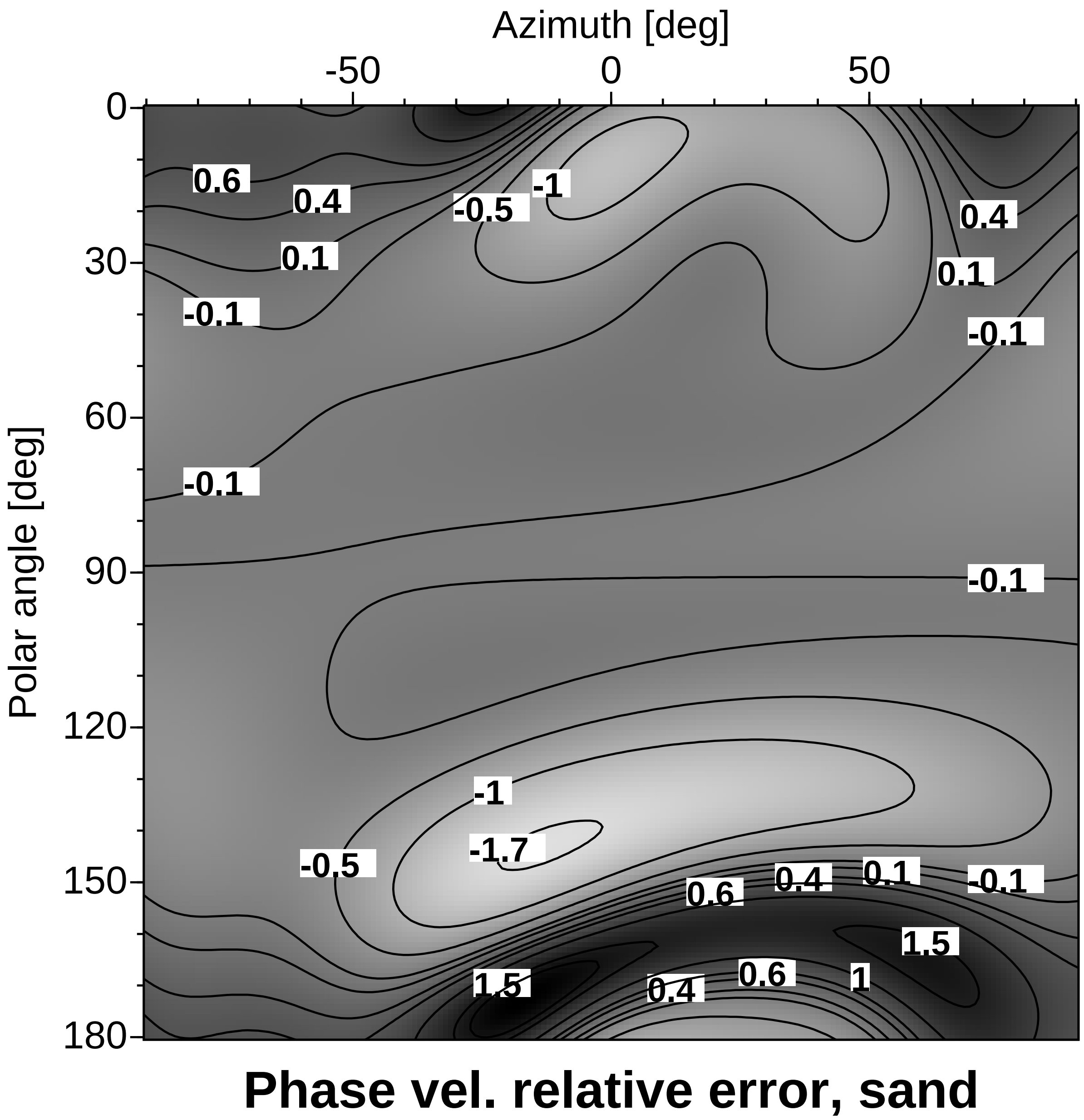

Figure 8: The map of relative errors, in %, of the approximate phase velocity formula (17) for the TRI model of dry Vosges sandstone. The P wave velocity of the isotropic background is $\alpha = \sqrt{A_{33}}$.

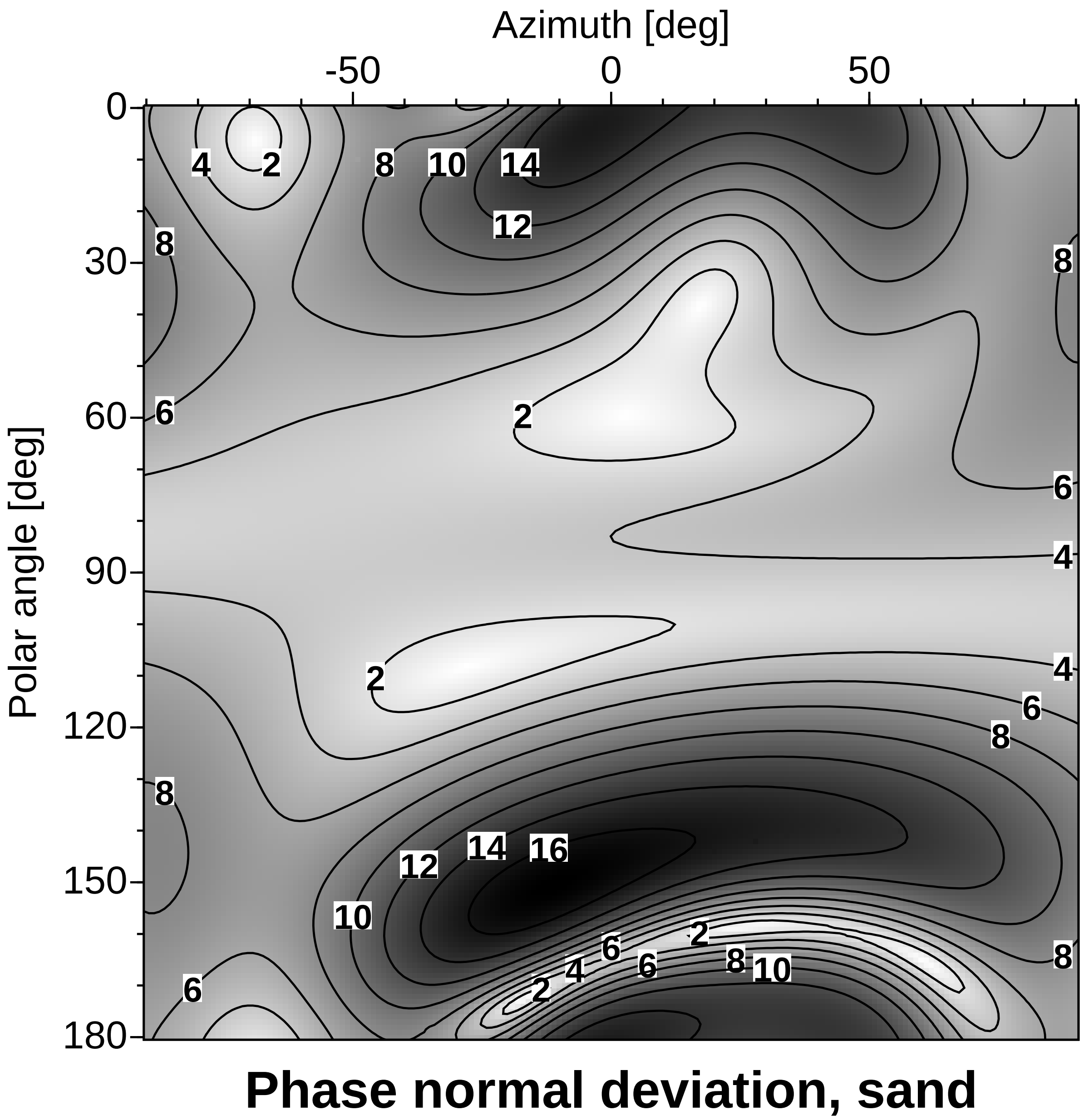

Figure 9: The map of deviations, in degrees, of the phase normals and the exact polarization vectors for the TRI model of dry Vosges sandstone.

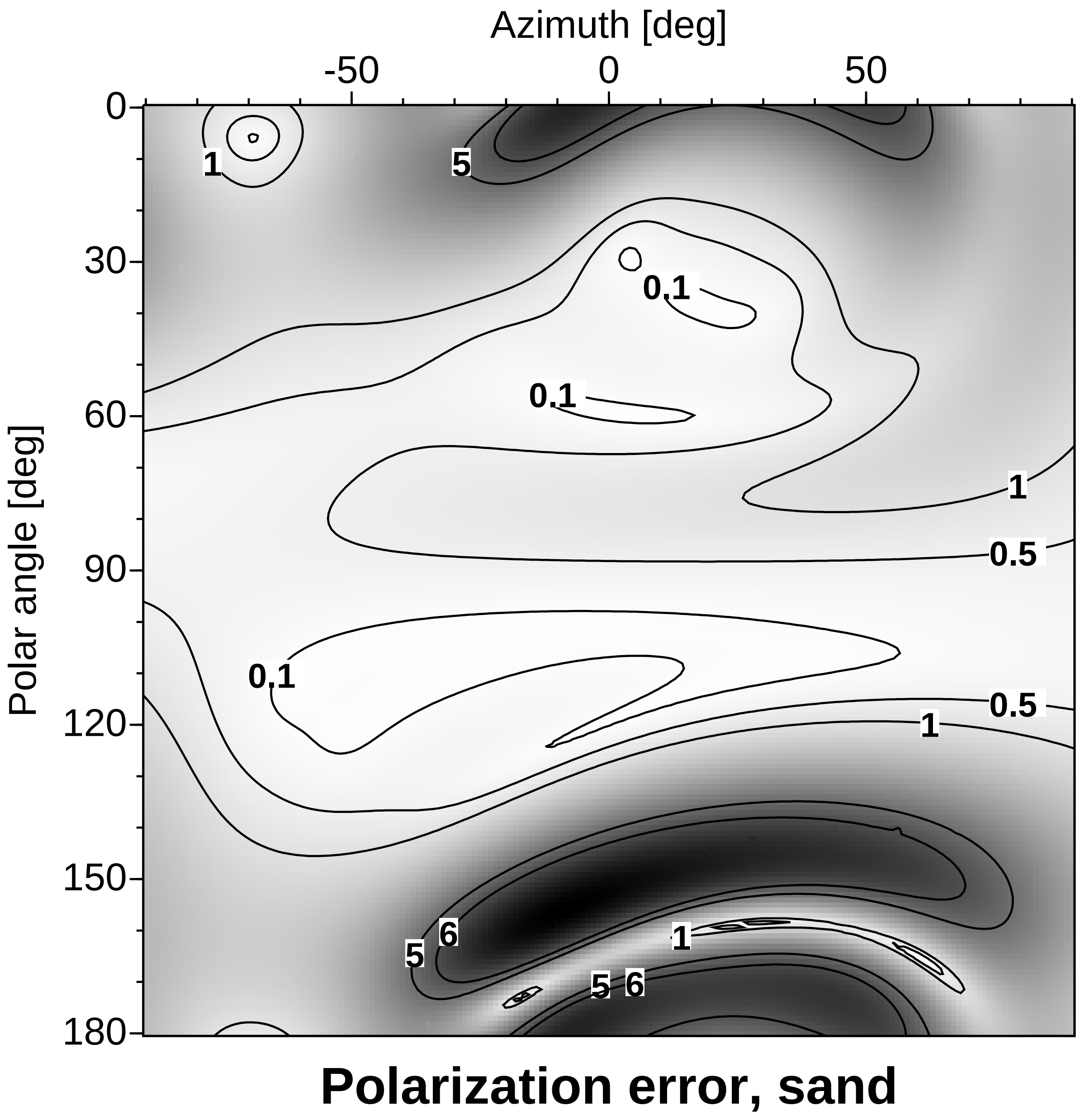

Figure 10: The map of polarization errors, in degrees, of Eqs.(16) and (34), (35) for the TRI model of dry Vosges sandstone. The P and S wave velocities of the isotropic background are $\alpha = \sqrt{A_{33}}$ and $\beta = \sqrt{A_{66}}$, respectively.